On Weight Distributions of Codes of Planes of Order 9

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Abstract

We study the weight distributions of the ternary codes of finite projective planes of order 9. The focus of this paper is on codewords of small Hamming weight. We show there are many weights for which there are no codewords.

1 Introduction

The p-ary code of a projective plane of order n, where p is a prime, is the GF(p)-span of the incidence matrix of the projective plane, with the rows indexed by lines and the columns indexed by points. As shown by Hamada [6], this code is only interesting when $p \mid n$.

In this paper, we will examine codewords of small Hamming weight in the (ternary) codes of the projective planes of order 9. Since there are four projective planes of order 9 (up to isomorphism) [4], there are four distinct codes to be studied. The weight distribution, which counts the number of codewords of each weight, is still unknown for all four of these codes. The weight distributions of the codes of the projective planes of orders 2, 3, 4, 5, and 8 have been calculated [3, 7, 8]. In each case, there are no words of weight w where $n + 1 < w < 2n$. It has also been shown that there are no codewords of weight w where $p + 1 < w < 2p$ in the p-ary code of the Desarguesian plane of prime order p [3]. Because of these gaps in the known weight distributions, it is conjectured that for any code of a projective plane of order n, there are no codewords of weight w where $n + 1 < w < 2n$. This paper proves the conjecture is true for the special case $n = 9$.

2 Planes, Codes, and Codewords

Let C be the p-ary code of a projective plane of order n. The set of coordinate positions where a codeword c has nonzero entries is called the *support* of c, denoted supp(c). The codeword of a line ℓ is the characteristic vector of the points incident with ℓ. The symbol ℓ will refer to both the line and its codeword. A *blocking set* is a set of points which is incident with every line but contains no line. The following

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facts are known for codes associated with planes ([1], Section 6.3):

1) The minimum weight of C is $n + 1$ and the minimum-weight vectors are precisely the nonzero scalar multiples of the lines.
2) The code $C \cap C^{\perp}$ is generated by codewords which are the differences of two lines.

The latter implies that for any codeword $c \in C$ and any two lines ℓ and m, the dot product $c \bullet (\ell - m) = 0$. It follows that $c \bullet \ell$ has the same value for every line ℓ of the plane. If $c \in C \backslash C^{\perp}$, these products are all nonzero and equal and these codewords have supports which intersect every line. If $c \in C \cap C^{\perp}$, these products are all necessarily zero.

We denote the *weight distribution* of a code C by A_0, A_1, A_2, etc..., where A_i refers to the number of codewords of C of weight i.

An affine plane can be created from a projective plane by removing one line and all of the points incident with it. We construct codes from affine planes in the same way in which we have constructed codes from projective planes. There is a natural projection from projective plane codes to affine plane codes (of the same order) which we get by deleting the coordinates which correspond to a line from the projective plane. The resulting affine plane code will be called a *residual code* and its codewords will be called *residual codewords*.

3 Words of Small Weight

3.1 Introduction

We begin by summarizing some known results. There are four projective planes of order 9 up to isomorphism [4]; the code of the Desarguesian plane of order 9 has dimension 37 and the three codes of the other planes have dimension 41 [8, 9]. The minimum weight of any of these codes is 10 and the minimum weight vectors are the non-zero scalar multiples of the incidence vectors of the lines [8, 9]. A previous result of Hall and Wilkinson [5] shows that the ternary code of a projective plane of order n $(3 \mid n)$ has no codewords of weight $\equiv 2 \pmod 3$. This implies that there are no codewords of weight 11, 14, 17, ... The result also shows that words of weight $\equiv 0 \pmod 3$ are codewords in $C \cap C^{\perp}$ and the words of weight $\equiv 1 \pmod 3$ are codewords in $C \backslash C^{\perp}$.

3.2 Incidence equations

Suppose Π is a projective plane of order n. Let c be a codeword of C, where C

is the p-ary code of Π ($p \mid n$). An i-secant is a line of the projective plane meeting supp(c) in i points; we shall refer to i as the length of the secant line. Let s be the size of supp(c). For $i = 0, 1, ..., n+1$, let μ_i be the number of i-secants to supp(c). For a fixed point $y \notin$ supp(c), let y_i be the number of i-secants through y; for a fixed point $z \in$ supp(c), let z_i be the number of i-secants through z. Counting yields the following equations:

$$\sum_{i=0}^{n+1} \mu_i = n^2 + n + 1; \quad \sum_{i=1}^{n+1} i\mu_i = s(n+1); \quad \sum_{i=2}^{n+1} i(i-1)\mu_i = s(s-1) \qquad (1)$$

$$\sum_{i=0}^{n+1} y_i = n + 1; \quad \sum_{i=1}^{n+1} iy_i = s \qquad (2)$$

$$\sum_{i=1}^{n+1} z_i = n + 1; \quad \sum_{i=1}^{n+1} (i-1)z_i = s - 1; \quad \sum_{i=1}^{n+1} (i-2)z_i = s - n - 2 \qquad (3)$$

The final equation of (3) is the difference of the previous two equations.

3.3 Weight 12

Each codeword $c \in C$ of weight 12 lies in $C \cap C^{\perp}$. The following result is from Sachar [**8, 9**].

Proposition Let $C^{\perp}$ be the dual of the code of the projective plane of order n over $\boldsymbol{F}_p$ ($p \neq 2$, $p \mid n$). Then the minimum weight of $C^{\perp}$ is $\geq \frac{4}{3}n + 2$.

For $n = 9$, this bound is 14. Thus, $A_{12} = 0$.

3.4 Weight 13

As previously noted, all possible codewords of weight 13 lie in $C \backslash C^{\perp}$. Let c be such a codeword. The points of supp(c) form a blocking set of the projective plane. The argument presented here is to rule out each possible value of i for a maximum length i-secant.

Suppose ℓ is a maximum length i-secant, $5 \leq i \leq 10$. By removing this line and its points to construct an affine plane (for the non-Desarguesian planes this affine plane is not unique up to isomorphism), we get a natural projection of C onto a residual code for an affine plane. This projection maps c to a residual codeword with weight $13 - i$, which for the specified values of i is less than 9. Since the minimum weight of the (ternary) code of any of the seven affine planes of order 9 is 9, we have a contradiction [1, Theorem 6.3.3].

Suppose ℓ is a maximum length 4-secant. Removing this line and its points to get an affine plane leads to the residual codeword of c, call it c^*, of weight 9. We know that the code of an affine plane of order 9 has minimum weight 9, and that these minimum weight vectors are scalar multiples of incidence vectors of lines [2, p. 4]. This violates the maximality of the length of ℓ, since c must contain an 9- or 10-secant.

Suppose ℓ is a maximum length i-secant, $i \leq 3$. The incidence equations (1) yield the following:

$$\begin{aligned} \mu_1 + \mu_2 + \mu_3 &= 91 \\ \mu_1 + 2\mu_2 + 3\mu_3 &= 130 \\ \mu_2 + 3\mu_3 &= 78 \end{aligned}$$

which have no non-negative integral solution. So $A_{13} = 0$.

3.5 Weight 15

All possible codewords of weight 15 lie in $C \cap C^{\perp}$ by the result of Hall and Wilkinson referred to in Section 3.1. Let c be such a codeword. Again, the argument will be to rule out each maximum length i-secant. Below, a point of supp(c) with coefficient 1 is referred to as a '1', and a point with coefficient 2 is referred to as a '2'.

Note that $\mu_1 = y_1 = z_1 = 0$ for $c \in C \cap C^{\perp}$. There are no i-secants for $i \geq 7$, because equation (3) becomes

$$z_3 + 2z_4 + 3z_5 + 4z_6 + 5z_7 + \ldots = 4$$

and for $i \leq 6$, the incidence equations (1) yield

$$\begin{aligned} \mu_0 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 &= 91 \\ 2\mu_2 + 3\mu_3 + 4\mu_4 + 5\mu_5 + 6\mu_6 &= 150 \\ \mu_2 + 3\mu_3 + 6\mu_4 + 10\mu_5 + 15\mu_6 &= 105 \end{aligned}$$

and after subtracting the third equation from the second, we get

$$\mu_2 - 2\mu_4 - 5\mu_5 - 9\mu_6 = 45$$

which implies that $\mu_2 \geq 45$.

Remark For any codeword c, let $n_i(c)$ (or n_i) be the number of entries in c that are equal to i.

There is also an upper limit on μ_2. Since $c \in C \cap C^{\perp}$, we must have that $(c, \ell) = 0$ for each line ℓ. This implies that each 2-secant intersects c at a 1 and a 2. Since we have $n_1 + n_2 = 15 \equiv 0 \pmod 3$ for any weight 15 codeword, the upper bound for the number of 2-secants is

$$\mu_2 \leq n_1 n_2 \leq 56.$$

We also know that the vector product of c with the all-ones vector is 0, which implies that $n_1 + 2n_2 \equiv 0 \pmod 3$. Subtracting equations, we get $n_2 \equiv 0 \pmod 3$ and so $n_1 \equiv 0 \pmod 3$. Because $45 \leq n_1 n_2 \leq 56$, we are limited to two cases:

$$n_1 = 9 \text{ and } n_2 = 6$$
$$n_1 = 6 \text{ and } n_2 = 9$$

Thus, the number of 1's and 2's for the codeword c is six of one value and nine of the other value.

Suppose ℓ is a maximum length 6-secant. By removing this line and its points, c would project onto a codeword of an affine plane, call it c^*. The weight of c^* is 9, forcing this codeword to be a scalar multiple of a line, a contradiction of the maximal length of ℓ.

Suppose ℓ is a maximum length 5-secant. Because $(c, \ell) = 0$, the two possibilities for the nonzero coefficients of the 5-secant are 2-2-2-2-1 and 2-1-1-1-1. Without loss of generality, multiply c by the appropriate scalar so that the 5-secant has nonzero coefficients 2-2-2-2-1.

Take the pencil of ten lines through the point z with coefficient 1 on the 5-secant. Since $z_5 = 1$, equation (3) gives $z_3 + 2z_4 = 1$, which implies $z_3 = 1$ and $z_2 = 8$. However, this is a contradiction, as there are at most nine 2's of supp(c).

Suppose ℓ is a maximum length line with length ≤ 4. Because $(c, \ell) = 0$, there are only two possibilities for the nonzero coefficients of any 3-secant: 2-2-2 and 1-1-1. Also, there is only one possibility for the nonzero coefficients of any 4-secant: 2-2-1-1. Thus, any line through a 1 and a 2 must be either a 2-secant or a 4-secant. By counting the number of pairings of each 1 with each 2, we get

$$\mu_2 + 4\mu_4 = n_1 n_2 = 54.$$

Adding this equation to the derived incidence equation $\mu_2 - 2\mu_4 = 45$, we get

$$2\mu_2 + 2\mu_4 = 99$$

which has no integral solution. So $A_{15} = 0$.

3.6 Weight 16

All possible codewords of weight 16 lie in $C \backslash C^{\perp}$. Let c be such a codeword. The points of supp(c) form a blocking set of the projective plane. Assume

$(c,m) = 1$ for all lines m w.l.o.g. For any fixed point $y \notin \text{supp}(c)$, let y_i be the number of i-secants through y. The incidence equations (2) become (note $y_0 = 0$)

$$y_1 + y_2 + y_3 + \ldots + y_{10} = 10$$
$$y_1 + 2y_2 + 3y_3 + \ldots 10y_{10} = 16$$

and after subtracting the two equations, we have

$$y_2 + 2y_3 + 3y_4 + 4y_5 + 5y_6 + 6y_7 + 7y_8 + \ldots = 6.$$

This equation implies that $y_i = 0$ for $i \geq 8$. If we assume $i \geq 4$ for the maximum i-length, then the non-negative integral solutions to these incidence equations are:

$$y_7 = 1;\ y_1 = 9$$
$$y_6 = 1;\ y_2 = 1;\ y_1 = 8$$
$$y_5 = 1;\ y_3 = 1;\ y_1 = 8$$
$$y_5 = 1;\ y_2 = 2;\ y_1 = 7$$
$$y_4 = 2;\ y_1 = 8$$
$$y_4 = 1;\ y_3 = 1;\ y_2 = 1;\ y_1 = 7$$
$$y_4 = 1;\ y_2 = 3;\ y_1 = 6$$

Suppose ℓ is a maximum length 7-secant (the case $y_7 = 1$ and $y_1 = 9$). By removing this line and its points, c would project onto a codeword of an affine plane, call it c^*. The weight of c^* is 9, forcing this codeword to be a scalar multiple of a line, a contradiction of the maximal length of ℓ.

Suppose ℓ is a maximum length 6-secant. Pick a point of ℓ which is not in supp(c); call it y. We have $y_6 = 1, y_2 = 1$ and $y_1 = 8$. With $(c,m) = 1$ for all lines m, the coefficients must be as shown in Figure 1.

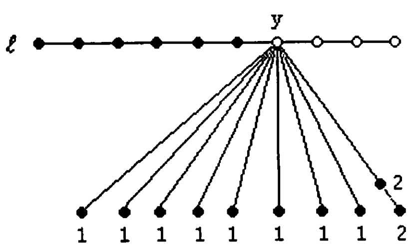

Figure 1

The points of $\ell \setminus \text{supp}(c)$ must have the same configuration as y; however, this forces the 2 points of $\text{supp}(c) \setminus \ell$ with codeword value 2 to be together on more than one 2-secant, a contradiction of the axioms for a projective plane.

Suppose ℓ is a maximum length 5-secant. Pick a point of the 5-secant ℓ which is not in supp(c), call it y. Because $(c, m) = 1$ for all lines m, we have one of the following two types of configurations:

I : $y_5 = 1;\ y_3 = 1;\ y_1 = 8$ (Figure 2)

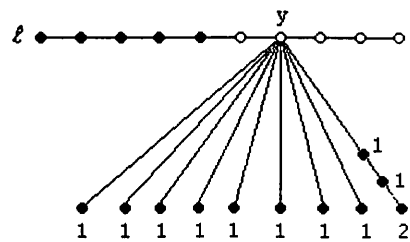

Figure 2

II : $y_5 = 1;\ y_2 = 2;\ y_1 = 7$ (Figure 3)

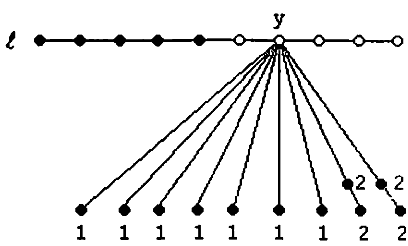

Figure 3

Because of the different coefficients, the five points of the 5-line *not* in supp(c) are collectively all of type I or type II. The type II case can be ruled out because the two points of supp(c) \ ℓ with codeword value 2 cannot be together on five different 2-secants. Assume the type I case. Then supp(c) \ ℓ has only one 2. Let P be a point of the 5-secant which is a 2 (there must be such a point, because if none exist, then $(c,\ell) = 2$). Since $(c,m) = 1$ for all lines m, each line through P must contain either another point with coefficient 2, or two more points with coefficient 1, requiring that there be at least $5 + 1 + 2(8) = 22$ points of supp(c), a contradiction, ruling out the type I case and the possibility of a maximum length 5-secant.

Suppose ℓ is a maximum length 4-secant. Pick a point of the 4-secant ℓ which is not in supp(c); call it y. Because $(c,m) = 1$ for all lines m, we have one of the following four types of configurations:

I : $y_4 = 2;\ y_1 = 8$ (Figure 4)

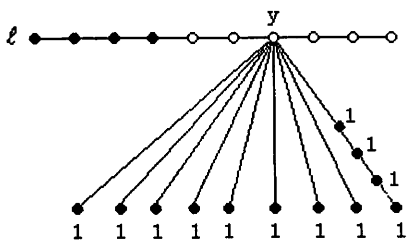

Figure 4

II: $y_4 = 2;\ y_1 = 8$ (Figure 5)

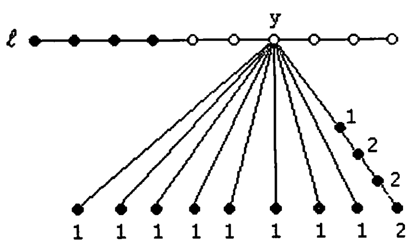

Figure 5

III: $y_4 = 1$; $y_3 = 1$; $y_2 = 1$; $y_1 = 7$ (Figure 6)

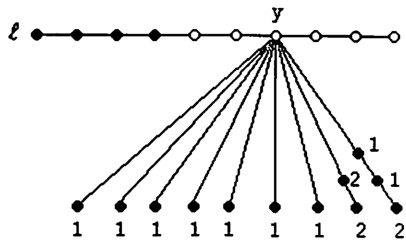

Figure 6

IV : $y_4 = 1$; $y_2 = 3$; $y_1 = 6$ (Figure 7)

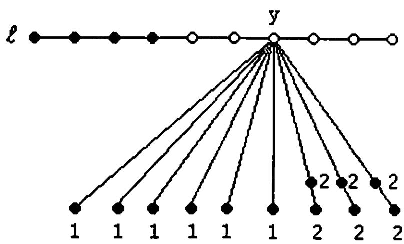

Figure 7

Examine the 2's from each case. The type I case has none. The type II case has three and they are **collinear**. The type III case has three and they are **noncollinear**. Finally, the type IV case has six. By the counts and arrangements of these 2's, all 6 points of $\ell \setminus \mathrm{supp}(c)$ must have the same type. Type II is impossible because the same three points would be collinear on six different lines. Types III and IV are impossible because there are too few 'partners' for points with coefficient 2 on 2-secants. So the 6 points of ℓ not belonging to supp(c) must all be of type I. By reversing the roles of ℓ and a different 4-secant, we can conclude that all coefficients are 1's, and that every line is a 4-secant or a 1-line. The incidence equations (1)

$$\begin{aligned} \mu_1 + \mu_4 &= 91 \\ \mu_1 + 4\mu_4 &= 160 \\ 6\mu_4 &= 120 \end{aligned}$$

have no solution, ruling out the possibility of a maximum length 4-secant.

Suppose ℓ is a maximum length i-secant, $i \leq 3$. The incidence equations (1) yield the following equations:

$$\begin{aligned} \mu_1 + \mu_2 + \mu_3 &= 91 \\ \mu_1 + 2\mu_2 + 3\mu_3 &= 160 \\ \mu_2 + 3\mu_3 &= 120 \end{aligned}$$

which have no non-negative integral solution. So $A_{16} = 0$.

4 Summary

Let C be the code of a projective plane of order 9. The first 18 terms of the weight distribution for C are $A_0 = 1$, $A_{10} = 182$, and , $A_i = 0$ for $1 \leq i \leq 17$, $i \neq 10$. Further, A_i is unknown for larger values of i. The words of weight 10 are the scalar multiples of the lines. A_{18} and A_{19} are non-zero; there exist words of weight 18 and 19 which are linear combinations of lines. Assmus and Key have shown that the number of words of weight 18 varies amongst the codes of the four planes. The gap between the smallest and second smallest nonzero weights is interesting. Does such a gap from weights $n + 1$ to $2n$ exist for all codes of projective planes of order n? The answer is yes in all of the presently known cases ($n = 2, 3, 4, 5, 8, 9$, and all Desarguesian planes of prime order p).

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