The Gluing Number of Ordered Sets

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ABSTRACT. For an ordered set A and B whose orders agree on its intersection, the *gluing* of A and B is defined to be the ordered set on the union of its underlying sets whose order is the transitive closure of the union of the orders of A and B. The *gluing number* of an ordered set P is the minimum number of induced semichains (suborders of dimension at most two) of P whose consecutive gluing is P. In this paper we investigate this parameter on some special ordered sets.

In this paper we introduce and study a new parameter, gluing number, for finite ordered sets related to their decompositions. We begin with some basic definitions and notations for ordered sets. Ordered sets are assumed to be finite throughout this article.

Let P be an ordered set. The elements of P are called *vertices* and an ordered pair (a, b) of elements of P is called an *edge* if b covers a, i.e., $a < b$ and there no vertex x such that $a < x < b$. In fact, vertices and edges are just the vertices and directed edges of the (Hasse) diagram of P as a directed graph. For an ordered set P, a *suborder* is a subset Q together with a subrelation of the ordering of P which is itself an ordering of Q. An *induced suborder* is a subset Q with the restriction of the ordering of P to the set Q as its ordering. A *linear extension* L of P is a linearly ordered set with the same underlying set as P such that $x < y$ in P implies $x < y$

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in L. The *dimension* of P is the minimum number of linear extensions the intersection of whose orderings is the ordering of P itself.

In 1950, Dilworth [1] proved the celebrated chain decomposition theorem that the width of an ordered set P is the minimum number of chains whose vertices cover P. Motivated by this result, Fishburn [2] introduced a new parameter of describing the structure and complexity of an ordered set. To do this he considered relatively simple ordered sets, called *semichains*, which are ordered sets of dimension at most two. Then he defined the *thickness* of an ordered set P to be the minimum number of induced semichains whose vertices cover all vertices of P. Unfortunately, in this decomposition we cannot reconstruct the original ordered set P from its semichain components. For example, any bipartite ordered set can be decomposed into two antichains, the maximal vertices and the minimal vertices, and so it has thickness at most two. But we cannot see its original ordering from the antichains. On the other hand, Lee [4] defined the *edge covering number* of an ordered set P to be the minimum number of (not necessarily induced) semichains so that every edge of P is included in one of the suborders. In this decomposition of edges we can reconstruct P from its components. Namely, the ordering of P is the transitive closure of the union of the orderings of the components.

We now define our new parameter of an ordered set. For an ordered set A and B whose orders agree on $A \cap B$, the *gluing* (or *amalgam*) $A * B$ of A and B is defined to be the ordered set on $A \cup B$ whose order is the transitive closure of, that is, the least order containing, the union of the orders of A and B (cf. Lee [3]). Observe that this operation is commutative but not associative and that both A and B are induced suborders of $A * B$ (see Lemma 1.1).

Lemma 1. *Let $P = A * B$. Then $x < y$ in P if and only if one of the following holds:*

(i) $x < y$ *in* A

(ii) $x < y$ *in* B

(iii) *There ezists $z \in A \cap B$ such that $x < z$ in A and $z < y$ in B*

(iv) *There exists $z \in A \cap B$ such that $x < z$ in B and $z < y$ in A.*

Now we define the *gluing number* $g(P)$ of an ordered set P to be the least number k of semichains $A_1, A_2, A_3, \dots, A_k$ such that $P = (\dots((A_1 * A_2) * A_3) * \dots) * A_k$ which is simply written as $P = A_1 * A_2 * \cdots * A_k$. In this case we can also reconstruct P from its components.

Let $[n] = \{1, 2, \dots, n\}$. Then $\mathbf{n}$ and $\underline{n}$ denote the chain and the antichain, respectively, on $[n]$. For ordered sets $P_1, \dots, P_n$, the *product* $P_1 \times \dots \times P_n$ is the ordered set defined by the condition that $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \Leftrightarrow a_i \leq b_i$ for each $i \in [n]$. Given an ordered set P, the notation P^n is used as shorthand for the n-fold product $P \times \ldots \times P$. If $\mathcal{P}([n])$ is the power set of $[n]$, then $(\mathcal{P}([n]), \subseteq) \cong \mathbf{2}^n$.

For a natural number $n \geq 3$, consider $S_n = (\{A \in \mathcal{P}([n]) \colon |A| \in \{1, n-1\}\}, \subseteq)$, which is usually called the *n-dimensional standard ordered set.* Let U_n be the ordered set obtained from the n-dimensional standard ordered set S_n by subdividing (inserting an element on) every edge and T_n the ordered set on the set $\{\{1\}, \{2\}, \ldots, \{n\}, \{1,2\}, \{2,3\}, \ldots, \{n,1\}, \ldots, [n]-\{n\}, [n]-\{1\}, \ldots, [n]-\{n-1\}\}$ ordered by set-inclusion. Then U_n and T_n are of gluing number 2, while they are n-dimensional (see Lee [3]).

It is not always true that if Q is an induced suborder of P then $g(Q) \leq g(P)$. For instance, even though S_n is an induced suborder of both T_n and U_n of gluing number 2, we will see in Theorem 3 that $g(S_n) = \lceil n/2 \rceil$. Now it is natural to ask when $g(Q) \leq g(P)$. We consider a special kind of induced suborders for our purpose. An induced order Q of an ordered set P is said to be *convex* provided that $x \in Q$ whenever $a < x < b$ in P for $a, b \in Q$.

Lemma 2. *If Q is a convex induced suborder of an ordered set $P = A * B$, then $Q = (A \cap Q) * (B \cap Q)$.*

Proof: Let $a < b$ in Q. Since the other cases can be treated similarly, we only consider the case that $a \in A - B$ and $b \in B - A$ for $a, b \in Q$. Then there exists $x \in A \cap B$ with $a < x < b$ and, by the convexity of Q, $x \in Q$, whence $a < b$ in $(A \cap Q) * (B \cap Q)$, as desired. □

Theorem 1. *If Q is a convex induced suborder of an ordered set P then $g(Q) \leq g(P)$.*

Proof: Let $g(P) = n$ and $P = A_1 * A_2 * \cdots * A_n$ with semichains $A_1, A_2, \ldots, A_n$. By Lemma 2,

$$Q = ((A_1 * \cdots * A_{n-1}) \cap Q) * (A_n \cap Q).$$

Since $(A_1 * \cdots * A_{n-1}) \cap Q$ is also a convex induced order of $A_1 * \cdots * A_{n-1}$, we have

$$(A_1 * \cdots * A_{n-1}) \cap Q = ((A_1 * \cdots * A_{n-2}) \cap Q) * (A_{n-1} \cap Q)$$

Continuing this we conclude that

$$Q = (A_1 \cap Q) * (A_2 \cap Q) * \ldots (A_n \cap Q),$$

whence $g(Q) \leq n$, as desired. □

Let P and Q be two disjoint ordered sets. The disjoint sum $P + Q$ of P and Q is the ordered set on $P \cup Q$ such that $x < y$ if and only if $x, y \in P$

and $x < y$ in P or $x, y \in Q$ and $x < y$ in Q. The *linear sum* $P \oplus Q$ of P and Q is obtained from $P + Q$ by adding the new relations $x < y$ for all $x \in P$ and $y \in Q$. An ordered set P is called *series-parallel* if it can be constructed from singletons using the operations of $+$ and $\oplus$ only. The *lexicographic sum* $\sum_{x\in P} Q_x$ of ordered sets Q_x over an ordered set P is defined to be the ordered set on $\bigcup_{x\in P} Q_x$ such that $a < b$ if and only if $a < b$ in Q_z for some $z \in P$ or $x < y$ in P when $a \in Q_x$ and $b \in Q_y$.

Theorem 2. *Let P be a series-parallel ordered set. If $g(Q_x) = 2$ for $x \in P$, then $g(\sum_{x\in P} Q_x) = 2$.*

Proof: Let $P_1 = A_1 * B_1$ and $P_2 = A_2 * B_2$ with semichains A_1, B_1, A_2, B_2. Clearly, $P_1 + P_2 = (A_1 + A_2) * (B_1 + B_2)$, whence $g(P_1 + P_2) = 2$. Now consider $P_1 \oplus P_2$. Let $Q = \max P_1 \oplus \min P_2$. Then $P_1 \oplus P_2 = (A_1 \cup Q \cup A_2) * (B_1 \cup Q \cup B_2)$. However we can see easily that $A_1 \cup Q \cup A_2$ and $B_1 \cup Q \cup B_2$ are semichains. Hence $g(P_1 \oplus P_2) = 2$. Now the result follows. □

Now we determine the gluing number of the standard ordered set S_n. To do this we give a simple but useful lemma.

Lemma 3. *Let $P = A * B$. If y covers x in P, then y covers x in A or B.*

Theorem 3. *For a natural number $n \geq 3$,*

$$g(S_n) = \lceil n/2 \rceil.$$

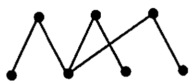
Proof: For $i \in [n]$, let $a_i = \{i\}$ and $b_i = [n] - \{i\}$, the complement of a_i in S_n. Then we can divide S_n into $\lceil n/2 \rceil$ semchains each of which consists of two (or one) maximal elements and all minimal elements. Hence, $g(S_n) \leq \lceil n/2 \rceil$. So, it can be seen that $g(S_3) = g(S_4) = 2$.

Now we proceed by induction on n. Suppose that $S_n = A * B$ ($n \geq 5$) and B is a semichain. If $b_1 \in B - A$, then, by Lemma 3, $a_2, a_3, \ldots, a_n$ belong to B. If $a_2 \in B - A$, then b_3, b_4, b_5 belong to B, whence B contains S_3 of dimension 3 as an induced order. This contradiction implies that $a_2 \in A \cup B$. By similar arguments we can see that $a_3, \ldots, a_n$ belong to $A \cup B$. If $a_1 \in B - A$, then B again contains S_3. If $a_1 \in A - B$, then $b_2, \ldots, b_n$ belong to $A \cup B$ as above and so B again contains S_3. Consequently, all of $a_1, \ldots, a_n$ belong to $A \cup B$, whence B contains at most one more element among $b_2, \ldots, b_n$, say b_2. Now $B \supseteq S_n - \{b_1, b_2\} \supseteq S_{n-2}$ as induced suborders. Since $g(S_{n-2}) = \lceil n/2 \rceil - 1$, we conclude that $g(S_n) = \lceil n/2 \rceil$. □

It is well known that every tree has dimension at most 3. But we show in the following theorem that even a bipartite tree may have an arbitrarily large gluing number. For a natural number n, let $F_n = \{\emptyset\} \cup \{u \colon u = (i_1 \ldots i_k),\ 1 \leq k \leq n,\ 1 \leq i_m \leq 2n - 1 \text{ for } m = 1, 2, \ldots, k\}$ be an ordered set such that only comparabilities are

$$\emptyset < (i_1) > (i_1 i_2) < (i_1 i_2 i_3) > (i_1 i_2 i_3 i_4) \ldots$$

for $i_1, i_2, i_3, \dots \in \{1, 2, \dots, 2n-1\}$. For example, $\emptyset < (1) > (12) < (121) > (1214)$ but (121) and (1121) are incomparable. Then each F_n ($n > 1$) is a bipartite tree and contains a lot of induced ordered sets isomorphic to the ordered set (spider) SP (Figure 1) or its dual which is not a semichain.



SP

Figure 1

Theorem 4. *For a natural number n,*

$$g(F_n) = n.$$

Proof: We shall proceed by induction on n. If $n = 1$, then $F_1 \cong \mathbf{2}$ so that $g(F_1) = 1$. Assume that $g(F_n) = n$. Let $F_{n+1} = A * B$ with $g(B) = 1$. Observe that no more than two elements of the form $(i_1 \dots i_k)$ for a fixed $(i_1 \dots i_{k-1})$ belong to $B - A$, since otherwise B contains the spider SP or its dual as an induced suborder. For, if $(i_1 \dots i_k) \in B - A$, then by Lemma 3 $(i_1 \dots i_{k-1}) \in B$ and $(i_1 \dots i_k i) \in B$ for any $i \in \{1, 2, \dots, 2n+1\}$. Consequently, for each $k \geq 1$ and each $(i_1 \dots i_{k-1})$, at least $2n-1$ elements of the form $(i_1 \dots i_{k-1} i_k)$ belong to A, whence A contains F_n as an induced suborder. Hence, $g(F_{n+1}) \geq n+1$. But we can easily see that $g(F_{n+1}) \leq n+1$. In fact, for each k, the set of all elements of the form $(i_1 \dots i_k)$ or $(i_1 \dots i_{k+1})$ induces a semichain. □

Finally we consider the gluing number of some products of chains. It is well known that the dimension of $\mathbf{n}^k$ is k. We can see easily that $g(\mathbf{2}^k) \leq 2$ for $k \leq 3$. Now we can verify $g(\mathbf{2}^4) = 2$ by the following two semichains:

$$\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, [4]\};$$
$$\{\emptyset, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, [4]\}.$$

Further we can only show at this moment that $g(\mathbf{2}^5) > 2$. Let $X = \{1, 2, 3, 4, 5\}$. Suppose that the power set $\mathcal{P}(X)$ be the gluing of two semichains A and B. If more than two of one-element subsets of X belong to either $A - B$ or $B - A$, say $\{1\}, \{2\}, \{3\} \in A - B$, then, by Lemma 3, A contains $\{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\}\}$ which is isomorphic to S_3. Now suppose that at most two of one-element subsets of X belong to either $A - B$ or $B - A$. If $\{1\}, \{2\} \in A - B$, then one of $\{3\}$, $\{4\}$, $\{5\}$ belongs to $A \cap B$, say $\{3\} \in A \cap B$, then A again contains $\{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\}\}$ which is isomorphic to S_3. Finally, if more than three of one-element subsets of X belong to A or B, say $\{1\}, \{2\}, \{3\}, \{4\} \in A$, then at least two of their complements in X belong to $B - A$ because otherwise A contains S_3. But this case also leads to a contradiction as the preceding argument applies dually.

Although the dimension of $\mathbf{n}^3$ is 3, its thickness can be arbitrarily large as n gets large. Since clearly the thickness is less than or equal to the gluing number for any ordered set, the gluing number of $\mathbf{n}^3$ can also be arbitrarily large as n gets large. Even though the thickness of $\mathbf{3}^3$ is 2, it may not be easy to determine $g(\mathbf{3}^3)$.

References

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