A Direct Construction of Transversal Covers Using Group Divisible Designs

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Abstract

A transversal cover is a set of gk points in k disjoint groups of size g and, ideally, a minimal collection of transversal subsets, called blocks, such that any pair of points not contained in the same group appears in at least one block. In this article we present a direct construction method for transversal covers using group divisible designs. We also investigate a particular infinite family of group divisible designs that yield particularly good covers.

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1 Introduction

Transversal covers have received a lot of attention recently due in part to their applicability to software testing [4, 5, 16, 19] and compression problems [12]. Most discussion has focused on non-constructive and asymptotic bounds [7, 8, 9, 12, 13, 14], but recent work has focused on various recursive and direct constructions [3, 4, 5, 16, 17, 19] as well as lower bounds [18].

The study of transversal covers is challenging. Katona [10], Kleitman and Spencer [11] and Rényi [15] all used extremal set theory to completely solve the problem for group size 2. Gargano, Körner and Vaccaro [7, 8, 9] used binary entropy, probability theory, and Markov chains to produce asymptotic results, in particular that

$$\lim_{k \to \infty} \frac{tc(k,g)}{\log k} = \frac{g}{2}$$

where $tc(k,g)$ is the minimum number of blocks in a transversal cover with k groups of size g. This result is non-constructive. Sloane used recursive constructions and intersecting codes, and reported the use of integer programming by Applegate to find instances of transversal covers and the first known optimal transversal cover that is not a design for $g > 2$: $tc(5,3) = 11$ [16]. Cohen *et al.* and Williams and Probert implement recursive and direct constructions on computer to produce transversal covers for direct application [4, 5, 19] . The first and last author have used aspects of design theory to develop direct and recursive constructions [17] and together with L. Moura have derived a selection of lower bounds from extremal set theory, packings and design theory and found 11 new optimal transversal covers [18]. Each of these diverse methods seems to fill in separate pieces of the question.

Sloane reports the best known upper bounds for $tc(k,3)$ found by himself, Östergård, Cook and Mallows [16]. Examples of the best known upper bounds previous to the constructions discussed in this paper are shown in Table 1. Design theory and computer methods appear to be the most

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$tc(k,3)$	9	9	9	9	11	12	12	13	13	15	15	15	15	16	17	17	18	18	18	18	18	18	18	18	19
$tc(k,4)$	16	16	16	16	16	25	26	27	27	28	28	28	28	28	28	28	28	28	28	28	31	31	31	31	31

Table 1: Previously known best upper bounds on $tc(k,g)$.

successful for producing instances of transversal covers and generating concrete upper bounds. This paper discusses a direct construction using group divisible structures from design theory. We will derive a general construction method, improve a certain instance of it, produce an infinite family of transversal covers from the construction and improve the values shown in Table 1.

1.1 Definitions

Definition 1.1. Let k, g and $n \leq g$ be positive integers. A *transversal cover* ($TC(k, g : n)$) is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a set of kg points, $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ is a partition of X into k sets of size g, $\mathcal{B}$ is a collection of subsets of X, called blocks or transversals, each block has size k and intersects each G_i in exactly one point, and each pair of points of X not in the same G_i occurs in at least one block. Further, there is a set of at least n disjoint blocks in $\mathcal{B}$. The smallest number of blocks possible in a TC$(k, g : n)$ is denoted by $tc(k, g : n)$. When we are not concerned with the cardinality of a set of disjoint blocks we will omit the parameter n.

Example 1.2. Let $V = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be partitioned into groups $G_1 = \{0, 1\}$, $G_2 = \{2, 3\}$, $G_3 = \{4, 5\}$, $G_4 = \{6, 7\}$. Then the following blocks form a transversal cover:

$$\{0, 2, 4, 6\}, \{1, 3, 5, 6\}, \{1, 3, 4, 7\}, \{1, 2, 5, 7\}, \{0, 3, 5, 7\}.$$

A TC$(k, g : n)$ with g^2 blocks is a transversal design. We will call a TC$(k, g : n)$ with the n disjoint blocks removed an incomplete transversal cover or ITC$(k, g : n)$. It is clear that

$$tc(k, g : i) \leq tc(k, g : j) \leq tc(k, g : i) + j - i, \text{ for any } 1 \leq i < j \leq n.$$

Treating transversal covers as $b \times k$ arrays of elements from a g-ary alphabet, allows for an easy translation among the many ways that these objects have been viewed and approached in the literature. The array is formed by placing the same g-ary alphabet on each group and then listing the blocks explicitly as the rows of the array. The groups become the columns and a set of disjoint blocks becomes a set of rows with pairwise Hamming distance k. With this in mind, we define:

Definition 1.3. A *covering array* (CA$(k, g : n)$) is an array with k columns of values from a g-ary alphabet such that given any two columns, i and j, and for all ordered pairs of elements from a g-ary alphabet, (g_1, g_2), there exists a row, r, such that $a_{i,r} = g_1$ and $a_{j,r} = g_2$. Further, there is a set of at least n rows that pairwise differ in each column; they are disjoint.

Row and column permutations, as well as permuting symbols within each column, leave the covering conditions unchanged.

Example 1.4. The transversal cover from Example 1.2, with the binary alphabet $\{0,1\}$ placed on each group, yields the following covering array:

0	0	0	0
1	1	1	0
1	1	0	1
1	0	1	1
0	1	1	1

Definition 1.5. Let k, g and u be positive integers. A *group divisible design* of order ug (k-GDD of type g^u) is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a finite set of cardinality ug, $\mathcal{G}$ is a partition of V into u groups of size g, and $\mathcal{B}$ is a family of subsets, called blocks, of V which satisfy the properties:

1. If $B \in \mathcal{B}$ then $|B| = k$;
2. Every pair of distinct elements of V occurs in exactly one block or one group but not both; and
3. $u \geq 1$.

Definition 1.6. Let $\mathcal{B}$ be a set of blocks of some incidence structure. A *resolution class* is a collection of blocks which partitions the point set of the incidence structure.

A k-GDD of type g^u is called *resolvable* and denoted k-RGDD of type g^u if its blocks can be partitioned into resolution classes.

In any incidence structure k will denote the size of each block. We use log for $\log_2$. $\lfloor x \rfloor$ will denote the largest integer $\leq x$. $\lceil x \rceil$ will denote the smallest integer $\geq x$. A *g-ary alphabet* is a set of cardinality g; it will usually be $\{0, 1, \ldots, g-1\}$.

2 A Group Divisible Design Construction

2.1 The Basic Construction

Because all pairs of points of a group divisible design not on the same group are covered by blocks we can extend each block to a transversal and get

Theorem 2.1. *If a k-GDD of type g^u exists then*

$$tc(u, g : 1) \leq \frac{g^2 u(u-1)}{k(k-1)}.$$

Proof. Arbitrarily extend each block to a transversal. □

Using the block size recursive construction [17] we can show that

$$tc(k,g) \leq \left\lceil \frac{\log u}{\log(m+2)} \right\rceil (g^2 - g) + g$$

where m be the maximum number of idempotent mutually orthogonal latin squares of order g. Using Fisher's theorem on the number of blocks in a design [1], for the GDD construction to yield results better than already known, it is necessary that

$$(g-1)u < \frac{g^2 u(u-1)}{k(k-1)} \leq \left\lceil \frac{\log u}{\log(m+2)} \right\rceil (g^2 - g) + g. \tag{1}$$

If g is a prime power then $m = g - 2$.

Applying Inequality 1 shows that when g is a prime power, this construction only betters what we already know if $g + 2 \leq u \leq 2g$. Checking the divisibility conditions for $g \leq 7$ and comparing the results to the best covers known, this construction can only improve on what is known when $u = g + 2$. We deal with this case in Subsection 2.3. In Subsection 2.2, we improve the general group divisible design construction by adding more groups after the extension of each block to a transversal.

2.2 Adding More Groups

The restrictions on the GDDs enabling them to yield better covers, are quite strong. In all the cases with g a prime power except $g + 2 \leq n \leq 2g$ and for $g \leq 7$ except $n = g+2$, the number of blocks is far too large for this construction to provide more optimal covers than known. However, this construction has speed and simplicity advantages and, if we can extend the covers by a significant number of groups then we may be able to produce better covers than already known. If we are able to partition the blocks of the GDD in a particular way then we can add a number of groups to the final transversal cover.

We will need to partition the blocks so that the blocks in each part, after they have been extended to transversals, cover every point of the GDD at least once. The following lemma is the necessary and sufficient condition on the blocks of a part for this to happen.

Lemma 2.2. *A collection of blocks of a GDD can be extended to transversals that cover every point at least once if and only if the number of blocks missing each group is at least the number of points missed on the group before the extension each block to a transversal.*

Proof. To see that the conditions are necessary we observe that each point on a group must be covered after the extension of the blocks to transversals. All the points on a group not covered before the extension must be covered by a block after extension. These blocks must be distinct and not go through the group before extension.

To see that the conditions are sufficient, for each point not covered on a group before extension, assign it to a block not going through this group so that no block will get assigned more than one point per group. No block will get assigned a point in a group where it already has a point and all points will be assigned. When extended, blocks will go through their assigned points, and through an arbitrary point on any group where they have not been assigned a point. □

This implies that each part must have at least g blocks and g disjoint blocks will suffice.

Theorem 2.3. *The transversal cover constructed from a k-GDD of type g^u with a partition of the blocks into p parts each satisfying Lemma 2.2 can be extended by at least e groups, where*

$$e = \left\lfloor \frac{p-1}{g-1} \right\rfloor .$$

Proof. There are p parts each with at least g blocks. Order them arbitrarily. Viewing the resulting transversal cover as a covering array on symbol set $\{0, 1, \ldots, g-1\}$, and defining e as above, add e zeros to the rows of the array corresponding to the blocks of the first part. In the first of the new e columns arbitrarily place the symbols $1, 2, \ldots, g-1$ on the rows of the next $g-1$ parts, all the rows within a part receiving the same symbol, a different symbol for each part. On the rows of each additional part, arbitrarily place the symbols $1, \ldots, g-1$, so every part has at least one row which gets each symbol. This can be done since each part has at least g blocks. On the second of the new e columns, arbitrarily place the symbols $1, 2, \ldots, g-1$ on the rows of the $g+1^{st}$ to $2g-1^{st}$ parts, a different symbol for each part, all the rows within a part receiving the same symbol. On all the other rows, arbitrarily place the symbols $0, 1, \ldots, g-1$, so every part has at least one

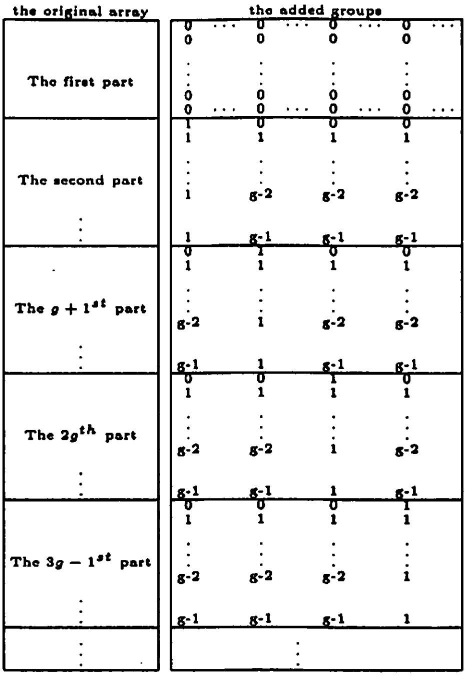

Figure 1: Method for Adding New Groups in Theorem 2.3.

row which gets each symbol. On the i^{th} new column place the symbols $1, 2, \ldots, g-1$ on the rows of the $(i-1)g - i + 3^{rd}$ to $ig - i + 1^{st}$ parts, a different symbol for each part, all the rows within a part receiving the same symbol. On all the other rows, again place the symbols $0, 1, \ldots, g-1$, so every part has at least one row which gets each symbol. See Figure 1. This is a covering array. □

Theorem 2.4. *The transversal cover constructed from a k-GDD of type g^u with a partition of the blocks into p parts each satisfying Lemma 2.2, where the g smallest parts have cardinality c_i, $1 \leq i \leq g$, can be extended by e groups where e is the maximum integer such that*

$$tc(e, g : g) \leq \frac{g^2 u(u-1)}{k(k-1)} + g - \sum_{i=1}^{g} c_i.$$

Proof. Again we will think of the covers as covering arrays and extend by e columns. On the rows corresponding to the smallest g parts (assume that these make up the top set of rows in the array) we put the symbols $0, 1, \ldots, g-1$ in each of the e columns, one symbol per part. We have covered all pairs of columns, one from the original set and one from the new set, and because we started with a covering array we have covered all pairs of columns from the original set. All we must do is now cover all pairs of columns from the new set. We have

$$\frac{g^2u(u-1)}{k(k-1)} - \sum_{i=1}^{g} c_i$$

rows empty in the new set of columns. In each pair of new columns, we have covered the pairs of identical symbols (i, i) and so we place in the set of unfilled rows (the empty lower right hand subarray of dimension $\left(\frac{g^2u(u-1)}{k(k-1)} - \sum_{i=1}^{g} c_i\right) \times e$), the largest ITC$(e, g : g)$ that will fit. See Figure 2. □

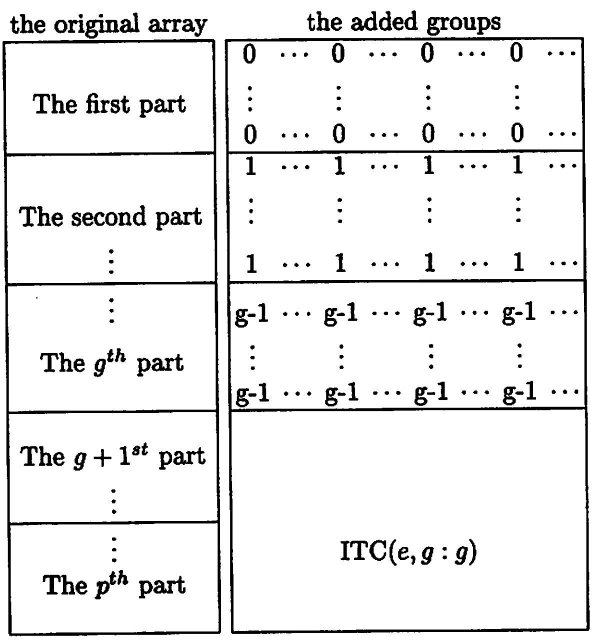

Figure 2: Method for Adding New Groups in Theorem 2.4.

Theorem 2.5. *The transversal cover constructed from a k-GDD of type g^u with a partition of the blocks into p parts each satisfying Lemma 2.2 can be extended by e groups where e is the maximum integer such that*

$$tc(e, g : g) \leq p.$$

Proof. On the rows from each part, we will always place the same symbol. This will guarantee coverage of all pairs of columns, one from the original array and one from the new set of columns. Then to cover all pairs of columns, both from the new set, we put the largest covering array with fewer than $\frac{g(u-1)}{k-1}$ rows, on the new columns treating each part as a single row, and arbitrarily completing any empty cells. □

When the GDD is resolvable we can obtain the desired partition by removing sets of g disjoint blocks from each resolution class. When there are less than g blocks left in the class we can add these blocks to any of the parts already obtained. This gives us

$$p = \frac{g(u-1)}{k-1}\left\lfloor\frac{u}{k}\right\rfloor$$

and since we can add the unused blocks to any part we like and $\frac{u-1}{k-1} > 1$ (resolvable transversal designs cannot be extended by Theorem 2.4 and by only one column in Theorems 2.3 and 2.5) the smallest p parts all have cardinality g. Similarly when the GDD is a k-frame we get

$$p = \frac{gu}{k-1}\left\lfloor\frac{u-1}{k}\right\rfloor$$

and again we can assume that the g smallest parts all have cardinality g.

Unfortunately there are not many families of RGDDs or frames known yet with large block sizes. For all the families known [6] we have checked to see what Theorems 2.3, 2.4 and 2.5 yield and none of the transversal covers constructed are better than those already known for $g \leq 7$ and extended block size less than 50. These constructions produce many transversal covers with enormous parameters. No GDD with group size less than 8, block size less than 5 and fewer than 50 groups produces better covers than by other methods, even for large parameters [17]. It is unclear how to obtain the desired partitions for other GDDs. The next family of GDDs, although not resolvable, have large block size and at least one new group can always be added. We will not use Theorems 2.4 or 2.5 to add the additional groups. The method in Subsection 2.3 is superior in this particular case.

2.3 An Infinite Family from the Construction

As mentioned in Subsection 2.1 when $u = g + 2$, the divisibility conditions are met and the number of blocks is reasonably small. An affine plane of order q with one point removed is a q-GDD of type $(q-1)^{q+1}$. From this we can construct a $\mathrm{TC}(q+1, q-1 : 1)$ with q^2-1 blocks. The blocks of this GDD can be partitioned into $q+1$ sets of $q-1$ blocks which are mutually disjoint. Each of these sets of blocks misses one group entirely, so by Lemma 2.2, when we choose the new points of these blocks, extending the GDD to a TC they remain a set of disjoint blocks. Theorem 2.3 allows us to extend this TC by one group to get

Theorem 2.6. *If g is one less than a prime power then $tc(g+3, g : g) \leq g^2+2g$.* □

This gives

$$\begin{aligned} tc(7,4:4) &\leq 24 \\ tc(9,6:6) &\leq 48 \\ tc(10,7:7) &\leq 63 \end{aligned} \tag{2}$$

where the previous constructions [17] only give

$$\begin{aligned} tc(7,4:4) &\leq 28 \\ tc(9,6:6) &\leq 64 \\ tc(10,7:7) &\leq 91. \end{aligned} \tag{3}$$

In a few cases we can take advantage of the structure of these GDDs to add more than one group. These GDDs come from the 1-rotational presentation of the affine plane [2, 6]: the q^2 points of the affine plane are ∞ and the points of $\mathbb{Z}_{q^2-1}$; and the blocks are generated additively from two base blocks. The first is the short block $G_0 = \{\infty\} \cup \{a(q+1) : 0 \leq a \leq q-2\}$ and the second is $B_0 = \{d_1, \ldots, d_q\}$ where $d_1 = 0$ and $\mu^{d_i} = 1 + \mu^{u+(i-2)(q-1)}$ for $i = 2, \ldots, q$, μ a primitive element of $\mathbb{F}_{q^2}$ and u an integer not a multiple of $q+1$. These GDDs are difference packings which can be used to construct difference triangle sets [6]. The GDD is just this design with ∞ removed, the groups generated by the short base block, G_0, and the blocks generated additively from B_0. The points are labeled as shown, where columns represent groups.

$$\begin{array}{cccc} 0 & 1 & \cdots & q \\ q+1 & q+2 & \cdots & 2q+1 \\ \vdots & \vdots & & \vdots \\ q^2-q-2 & q^2-q-1 & \cdots & q^2-2 \end{array}.$$

To make the blocks transversals, we will add a point to B_0. We will choose this point later. With this presentation, the sets of disjoint blocks are $P_i = \{B_0 + k(q+1) + i\}_{k=0}^{q-2}$ where $j = 0, 1, \ldots, q$. But consider also the sets of blocks $Q_j = \{B_0 + k(q-1) + j\}_{k=0}^{q}$ where $j = 0, 1, \ldots, q-2$.

If q is a power of 2 then $q-1$ and $q+1$ are relatively prime. We define the extension by two new groups. For any block B add symbol j in the first new column if $B \in Q_j$ and i in the second new column if $B \in P_i$. Since $q-1$ and $q+1$ are relatively prime, the last two columns are covered. This method will cover all pairs of columns, one from the original set and one from the new set of two columns as long as every point in $\mathbb{Z}_{q^2-1}$ appears in each of the Q_js. We have not yet extended the blocks, nor added symbols in the second column if $B \in P_i$ for $i \geq q-1$ (we did not need to put the symbols in the second new column from the first $q-1$ P_is, but we could have chosen any $q-1$ of the P_is in any order). These flexibilities may allow us to guarantee coverage and possibly also to extend by more than two columns.

The base block, B_0, from the GDD contains q points and the pairwise differences cover every element of $\mathbb{Z}_{q^2-1}$ which is not a multiple of $q+1$. The q differences which are multiples of $q-1$ will be covered, and the pairs of elements of B_0 whose differences are multiples of $q-1$ will generate the same point sets under development in the Q_j. In each Q_j, we will generate one of these point sets twice if $q=4$ and at least three times if $q>4$. Since we can still add a point to the B_0 to cover all the points by each Q_j, we need at least $q-2$ different point sets generated which is impossible when $q>4$. Hence, this construction can only work if $q=4$ and indeed, it does: use base block $B_0 = \{0,2,3,11\}$ and add the point 1 to it, then each Q_j will cover each point at least once. Each P_i also covers each point exactly once and so we can extend by two columns.

However, in this case, we can do even better by taking all possible sets of three P_is (we can pick any $q-1$ of the P_is) and adding the columns given in Figure 3. The first column is from the Q_js and the remaining ten columns

0	0	0	0	0	0	0	0	1	1	2
1	1	1	1	0	2	1	0	0	0	2
2	2	0	1	1	2	1	1	1	2	0
0	0	2	1	2	2	1	2	2	1	1
1	0	1	2	1	1	2	2	2	2	2
2	0	0	0	0	0	0	1	0	2	1
0	1	1	1	1	0	2	0	0	0	1
1	2	1	0	1	2	2	1	1	1	0
2	1	2	0	2	1	1	2	1	1	1
0	1	0	2	2	1	2	1	2	2	2
1	0	0	0	0	0	0	2	2	0	0
2	1	1	1	2	1	0	0	0	0	0
0	2	2	2	1	2	0	1	1	0	0
1	2	2	2	2	0	1	2	0	1	1
2	2	2	2	0	1	2	0	2	2	2

Figure 3: The extension of TC(5, 3 : 1) by eleven additional columns.

are from the ten possible triples of P_is which are ordered lexicographically as the ten possible triples from a 5-set. These additional columns gives us $tc(16,3:1) \leq 15$ which is better than 17, the value obtained from other methods.

When q is an odd prime power, we can do the same sort of construction to try to extend by two groups. Because $q-1$ and $q+1$ are not relatively prime, we will not automatically get that the two new columns (from the P_is and Q_js) are covered. We won't have filled in all of the columns determined by the P_is (the second new column). This flexibility may allow us still to succeed. Because we need all the points to be in each Q_j, a similar argument shows that $q=3$ and $q=5$ are the only possibilities. When $q=3$, we are constructing a cover with $g=2$. Since we know what optimal covers for $g=2$ look like, this is uninteresting. It is, however, worth mentioning that a total of 31 groups can be added which is the most possible and achieves the optimal cover with block size 35.

When $q=5$, we have $B_0 = \{0,2,15,16,19\}$ to which we add the point 1. Each Q_j covers all the points and so we must only worry about covering all the pairs on the two new columns. This can be done with these columns:

$$\begin{pmatrix} 0\,1\,2\,3\,0\,1\,2\,3\,0\,1\,2\,3\,0\,1\,2\,3\,0\,1\,2\,3\,0\,1\,2\,3 \\ 0\,1\,2\,3\,1\,0\,0\,1\,2\,3\,1\,0\,0\,1\,2\,3\,3\,2\,0\,1\,2\,3\,3\,2 \end{pmatrix}^T$$

so we get $tc(8,4:2) \leq 24$ which is better than 27, the value from other methods. By this method, we can only extend by two groups because we cannot add another column from a different set of four P_is. Some variation of these methods may generate additional new columns for these particular values of q. We have now improved the best known values to those shown in Table 2. Many of these values have since been further improved [17].

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$tc(k,3)$	9	9	9	9	11	12	12	13	13	15	15	15	15	15	15	15	18	18	18	18	18	18	18	18	19
$tc(k,4)$	16	16	16	16	16	24	24	24	27	28	28	28	28	28	28	28	28	28	28	28	31	31	31	31	31

Table 2: Improved upper bounds for $tc(k,g)$.

3 Conclusion

Group divisible designs offer a convenient and fast way of constructing transversal covers. Simply extending the blocks of a group divisible design to transversals improves some previously known transversal covers. However, if the group divisible design is resolvable, a frame, or we can suitably partition the blocks, then we can use this additional structure to extend the transversal cover obtained by several groups and improve a larger range of parameters. None of the few resolvable group divisible designs or frames known construct transversal covers better than those already found. The constructions applied to resolvable group divisible designs and frames do construct many transversal covers larger than any known and may well be better than those we can construct by other methods.

We have found one infinite family of group divisible designs yielding transversal covers better than those previously known. Each of these constructions can be extended by at least one group. Three of them are known to be extendible by more.

More work remains to be done for transversal covers in general. The group divisible design constructions seem to offer a powerful and efficient method for constructing transversal covers. This method will become more and more useful as more is discovered about group divisible designs, resolvable GDDs and frames. As we construct more transversal covers with large parameters we may find that the constructions herein are the best. These GDD constructions fit into broader ongoing work on transversal covers. The first author is keeping a table of the best known transversal covers. So far the table is kept for $g \leq 7$ and $k \leq 50$. For the most recent published table see [17]. Please send any new covers, even those beyond the scope of these bounds, and the method used to obtain them to the address given; they will be added to the table with the method used and credit given to the contributer.

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References

[1] T. Beth, D. Jungnickel, and H. Lenz. *Design Theory*. Cambridge University Press, Cambridge, 1985.

[2] R.C. Bose. An affine analogue of Singer's theorem. *J. Ind. Math. Soc.*, 6:1–15, 1942.

[3] M. A. Chateauneuf, C. J. Colbourn, and D. L. Kreher. Covering arrays of strength three. *Des. Codes Cryptogr.*, 16(3):235–242, 1999.

[4] D. M. Cohen, S. R. Dalal, M. L. Fredman, and G. C. Patton. The AETG system: An approach to testing based on combinatorial design. *IEEE Trans. Soft. Eng.*, 23:437–444, 1997.

[5] D. M. Cohen, S. R. Dalal, J. Parelius, and G. C. Patton. The combinatorial design approach to automatic test generation. *IEEE Software*, 13(5):83–88, Sep. 1996.

[6] C. J. Colbourn and J. H. Dinitz, editors. *The CRC Handbook of Combinatorial Designs*. CRC Press, Boca Raton, 1996.

[7] L. Gargano, J. Körner, and U. Vaccaro. Qualitative independence and Sperner problems for directed graphs. *J. Combin. Theory. Ser. A*, 61:173–192, 1992.

[8] L. Gargano, J. Körner, and U. Vaccaro. Sperner capacities. *Graphs Combin.*, 9:31–46, 1993.

[9] L. Gargano, J. Körner, and U. Vaccaro. Capacities: From information theory to extremal set theory. *J. Combin. Theory. Ser. A*, 68:296–316, 1994.

[10] G. O. H. Katona. Two applications (for search theory and truth functions) of Sperner type theorems. *Period. Math. Hungar.*, 3:19–26, 1973.

[11] D. J. Kleitman and J. Spencer. Families of k-independent sets. *Discrete Math.*, 6:255–262, 1973.

[12] J. Körner and M. Lucertini. Compressing inconsistent data. *IEEE Trans. Inform. Theory*, 40(3):706–715, May 1994.

[13] S. Poljak, A. Pultr, and V. Rödl. On qualitatively independent partitions and related problems. *Discrete Applied Mathematics*, 6:193–205, 1983. .

[14] S. Poljak and Z. Tuza. On the maximum number of qualitatively independent partitions. *Journal of Combinatorial Theory*, 51:111–116, 1989.

[15] A. Rényi. *Foundations of Probability.* Wiley, New York, 1971.

[16] N. J. A. Sloane. Covering arrays and intersecting codes. *J. of Combin. Des.*, 1:51–63, 1993.

[17] B. Stevens and E. Mendelsohn. New recursive methods for transversal covers. *J. Combin. Des.*, 7(3):185–203, 1999.

[18] Brett Stevens, Lucia Moura, and Eric Mendelsohn. Lower bounds for transversal covers. *Des. Codes Cryptogr.*, 15(3):279–299, 1998.

[19] A.W. Williams and R.L. Probert. A practical strategy for testing pair-wise convergence of network interfaces. *Proc. of the 7th International Symposium on Software Reliability Engineering*, pages 246–254, 1996.