Competition Graphs of Semiorders and the Conditions $C(p)$ and $C^*(p)$

Suh-Ryung Kim*
Department of Mathematics
Kyung Hee University
Seoul, Korea

Fred S. Roberts†
Department of Mathematics and DIMACS
Rutgers University
Piscataway, NJ, USA

Abstract

Given a digraph D, its competition graph has the same vertex set and an edge between two vertices x and y if there is a vertex u so that (x, u) and (y, u) are arcs of D. Motivated by a problem of communications, we study the competition graphs of the special digraphs known as semiorders. This leads us to define a conditions on digraphs called $C(p)$ and $C^*(p)$ and to study the graphs arising as competition graphs of acyclic digraphs satisfying conditions $C(p)$ or $C^*(p)$.

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1 Introduction

The notion of competition graph arose from a problem in ecology and has since found application in problems of coding, channel assignment in communications, scheduling, and the modeling of complex systems arising in the study of energy and economic systems. (See [9] and [12] for details.) The notion of semiorder arose from problems in utility theory and psychophysics involving thresholds. (See [1], [6], [11], [13], [14].) Motivated by a problem in communications, we consider competition graphs of semiorders. This leads us to a general condition on digraphs that we call $C(p)$ and the study of competition graphs of digraphs satisfying condition $C(p)$. We also study a variant of this condition which we call condition $C^*(p)$.

Suppose $D = (V, A)$ is a digraph. Its *competition graph* $G = C(D)$ has the same vertex set and has an edge $\{x, y\}$ if for some vertex $u \in V$, the arcs (x, u) and (y, u) are in D. The long literature of competition graphs is summarized in several survey papers, [4], [7], [12]. In much of the literature, the study of competition graphs is restricted to acyclic digraphs D. We shall not make this restriction for the entire paper, but will in our main results. The literature of competition graphs sometimes allows loops in digraphs. However, we shall make the convention that all digraphs in this paper have no loops. In the communications application, the vertices represent transmitters or receivers and in the digraph D, there is an arc from a transmitter x to a receiver u if a message sent at x can be received at u. The competition graph $C(D)$ restricted to the set of transmitters then has the interpretation that there is an edge between two transmitters if and only if a message sent from them can be received at the same place. In this case, we think of the two transmitters as interfering. In channel assignment applications, interfering transmitters must be assigned different channels.

A digraph $D = (V, A)$ (on a finite set of vertices V) is a *semiorder* if there is a real-valued function f on V and a real number $\delta > 0$ so that for all $x, y \in V$,

$$(x, y) \in A \Leftrightarrow f(x) > f(y) + \delta. \tag{1}$$

Suppose the transmitters and receivers are lined up in a linear corridor and messages can only be transmitted from right to left. Suppose also that because of local interference (jamming?), a message sent at x can be received at y if and only if y is sufficiently far to the left of x, e.g., more than 10 miles to the left. Then the digraph D in the communications application is a semiorder (taking $\delta = 10$). This leads us to the question: What are the competition graphs of semiorders? This question has a very simple answer which we present in Section 2. In that section, we also consider the competition graphs of interval orders, natural relatives of semiorders, and then of the more general digraphs satisfying a condition we shall call condition $C(p)$. In Section 3, we consider a variant of condition $C(p)$ which we call condition $C^*(p)$ and the competition graphs of digraphs satisfying condition $C^*(p)$. There have been a number of papers about competition graphs of specific classes of digraphs. For instance, competition graphs of strongly connected digraphs have been studied in [2], of Hamiltonian digraphs in [2] and [3], of interval digraphs in [5], and for various classes of symmetric digraphs in [8] and [9].

We close this section by introducing some simple notation and recalling a useful concept from the theory of competition graphs. The graph K_r is the complete graph on r vertices and $V(K_r)$ is its vertex set. We also use the notaton $K_r - e$ for the complete graph on r vertices with one edge omitted, $K_r - P_3$ for K_r with a pair of adjacent edges omitted, and $K_r - K_3$ for K_r with all edges on a triangle omitted. Also, I_q is the graph with q vertices and no edges. We shall use the union notation $G \cup H$ only for the case where the vertex sets of graphs G and H are disjoint. It is easy to see that if $G = C(D)$ for D acyclic, then G must have an isolated vertex. If G is any graph, then adding sufficiently many isolated vertices produces a competition graph of an acyclic digraph ([10]). The smallest k so that $G \cup I_k$ is a competition graph of an acyclic digraph is called the *competition number* of G and is denoted $k(G)$. Clearly, then, $k(G) \geq 1$ whenever G is connected and has more than one vertex.

2 The Condition $C(p)$

The question of when a graph is the competition graph of a semiorder has a very simple answer:

Theorem 2.1 *A graph G is the competition graph of a semiorder if and only if $G = I_q$ for $q > 0$ or $G = K_r \cup I_q$ for $r > 1$, $q > 0$.*

This is easy to prove directly, but will follow from a more general result. A similar theorem holds for interval orders, which are closely related to semiorders. If J and J' are two real intervals, we say that $J \succ J'$ if $a > b$ for all $a \in J$ and $b \in J'$. $D = (V, A)$ is an *interval order* if there is an assignment of a (closed) real interval $J(x)$ to each vertex $x \in V$ so that for all $x, y \in V$,

$$(x, y) \in A \Leftrightarrow J(x) \succ J'(y). \tag{2}$$

Semiorders are a special case of interval orders where every interval has the same length. (For references on interval orders, see [1], [11], [14].)

Theorem 2.2 *A graph G is the competition graph of an interval order if and only if $G = I_q$ for $q > 0$ or $G = K_r \cup I_q$ for $r > 1$, $q > 0$.*

This theorem will also follow from the more general results below.

If $D = (V, A)$ is a digraph, we define a binary relation W on V as follows:

$$aWb \Leftrightarrow [(b,u) \in A \rightarrow (a,u) \in A].$$

If $p \geq 2$ is an integer, we say that D satisfies *condition* $C(p)$ if whenever S is a set of p vertices of D, then there is a vertex x in S so that yWx for all $y \in S - \{x\}$. The key condition that allows us to prove Theorem 2.1 and Theorem 2.2 is $C(2)$. Any such vertex x is called a *foot* of S and, by a somewhat ambiguous notation, will be denoted $F(S)$. (If there is more than one foot, the context will tell us which is denoted by $F(S)$.)

Proposition 2.3 *If $p < q$, then $C(p)$ implies $C(q)$.*

Proof. It suffices to show that if $C(p)$ holds, then $C(p+1)$ holds. Let $S = \{x_1, x_2, \ldots, x_{p+1}\}$. Consider $T = \{x_1, x_2, \ldots, x_p\}$ and $U = \{x_2, x_3, \ldots, x_{p+1}\}$. By $C(p)$, we may assume that $F(T) = x_p$ and that for some $i \in \{2, 3, \ldots, p+1\}, F(U) = x_i$. If $i = p$, then $F(S) = x_p$. If $i \neq p$, then transitivity of W implies that $F(S) = x_i$. Q.E.D.

It is straightforward to show that if $p < q$, then $C(q)$ does not imply $C(p)$. Simply define D on the vertex set $\{1, 2, \ldots, q\}$ by letting $A = \{(i, i+1) : i = 1, 2, \ldots, q-1\}$.

It is straightforward to show that if D is a semiorder, then $C(2)$ holds and therefore $C(p)$ for all $p \geq 2$. $C(2)$ follows by noting that $F(S)$ is the minimum element of S if f satisfies Equation (1). Similarly, by using the interval with the smaller left end point in a set of two intervals, we make use of Equation (2) to show that $C(2)$ and therefore $C(p)$ for all $p \geq 2$ hold for D an interval order.

Let us say that a connected component is *nontrivial* if it has at least two vertices.

Proposition 2.4 *If digraph $D = (V, A)$ satisfies condition $C(p)$ for some $p \geq 2$ and $K^1, K^2, \ldots, K^m$ are nontrivial connected components of $C(D)$ so that $T = K^1 \cup K^2 \cup \ldots \cup K^m$ has at least p vertices, then $m = 1$ and K^1 is a clique.*

Proof. Pick vertices $a \in K^i$ and $b \in K^j$ where i could equal j. Let S be a set of p vertices from T including vertices a and b. If $F(S) = x$, then $x \in K^t$ for some t and there is another $y \in K^t$. Hence, for some u, $(x,u), (y,u) \in A$. It follows from $(x,u) \in A$ that $(a,u), (b,u) \in A$ and so a, b are adjacent in $C(D)$. Q.E.D.

Corollary 2.5 *Suppose that $p \geq 2$ and the union of all nontrivial connected components of graph G has at least p vertices. Then G is $C(D)$ for some digraph D satisfying condition $C(p)$ if and only if $G = K_r \cup I_q$ for $r > 1, q > 0$.*

Proof. The proposition shows that $G = K_r \cup I_q$ for $r > 1, q \geq 0$. But $q = 0$ is impossible. For then $r \geq p$ so by Proposition 2.3, $C(r)$ holds. Let $x = F(S)$ for S the set of all vertices. Then (x, y) is in D for some y in S and hence (y, y) is in D, which is a contradiction of our convention that digraphs have no loops. To prove the other direction, we construct D as follows. Take the vertex set of D to be the vertex set of G and include arc (x, a) in A for all $x \in K_r, a \in I_q$. $C(p)$ is straightforward. Q.E.D.

In the next theorem, we use the concept of competition number $k(G)$ defined in Section 1.

Theorem 2.6 *Suppose that $p \geq 2$ and G is a graph. Then G is the competition graph of an acyclic digraph D satisfying condition $C(p)$ if and only if G is one of the following graphs:*
(a). $I_q, q > 0$
(b). $K_r \cup I_q, r > 1, q > 0$
(c). $L \cup I_q$ where L has fewer than p vertices, $q > 0$ and $q \geq k(L)$.

Proof. Suppose that $G = C(D)$ for D acyclic and satisfying $C(p)$. If there is no nontrivial connected component, then (a) holds. If the union of all nontrivial connected components of graph G has at least p vertices, then by Corollary 2.5, (b) holds. Suppose that L is the union of all nontrivial connected components of G and L has fewer than p vertices. Then $G = L \cup I_q$ and by acyclicity, $q > 0$. Since G is a competition graph, $k(G) = 0$. Hence, there must be at least as many vertices in I_q as $k(L)$. Thus, (c) follows.

Conversely, if (a), then define D on the vertex set of G by putting no arcs in. If (b), the result follows from Corollary 2.5. Finally, suppose (c) holds. Since $k(L) \leq q$, there is an acyclic D' so that $G = C(D')$. Let D be such an acyclic digraph that is minimal in terms of arcs. To see that D satisfies $C(p)$, let S have p vertices. Then there is a vertex x in $S - L$. By minimality of D, there are no arcs from x in D and therefore $x = F(S)$. Q.E.D.

Theorems 2.1 and 2.2 follow as simple corollaries given our earlier observation that if D is a semiorder or interval order, then D satisfies $C(p)$ for all $p \geq 2$. The next four corollaries follow in a straightforward way by considering the possible graphs L.

We use the notation P_k for the path of k vertices, C_k for the cycle of k vertices, and $K_{1,3}$ for the complete bipartite graph consisting of one vertex adjacent to three others.

Corollary 2.7 *Let G be a graph. Then G is the competition graph of an acyclic digraph satisfying condition $C(2)$ if and only if $G = I_q$ for $q > 0$ or $G = K_r \cup I_q$ for $r > 1, q > 0$.*

Corollary 2.8 *Let G be a graph. Then G is the competition graph of an acyclic digraph satisfying condition $C(3)$ if and only if $G = I_q$ for $q > 0$ or $G = K_r \cup I_q$ for $r > 1, q > 0$.*

Corollary 2.9 *Let G be a graph. Then G is the competition graph of an acyclic digraph satisfying condition $C(4)$ if and only if $G = I_q$ for $q > 0$ or $G = K_r \cup I_q$ for $r > 1, q > 0$ or $G = P_3 \cup I_q$ for $q > 0$, where P_3 is the path of 3 vertices.*

Corollary 2.10 *Let G be a graph. Then G is the competition graph of an acyclic digraph satisfying condition $C(5)$ if and only if G is one of the following graphs: (1) $I_q, q > 0$; (2) $K_r \cup I_q, r > 1, q > 0$; (3) $P_3 \cup I_q, q > 0$; (4) $P_4 \cup I_q, q > 0$; (5) $K_{1,3} \cup I_q = K_4 - K_3 \cup I_q, q > 0$; (6) $K_2 \cup K_2 \cup I_q, q > 0$; (7) $C_4 \cup I_q, q > 1$; (8) $K_4 - e \cup I_q, q > 0$; (9) $K_4 - P_3 \cup I_q, q > 0$.*

Note that $q > 1$ in (7) of Corollary 2.10.

3 The Condition $C^*(p)$

The *converse* D^c of a digraph $D = (V, A)$ has the same set of vertices and an arc (x, y) whenever (y, x) is an arc of D. It is clear that the converse of a semiorder or interval order is a semiorder or interval order, respectively. The *common enemy graph* $CE(D)$ of a digraph $D = (V, A)$ has vertex set V and an edge between x and y if there is a vertex $u \in V$ so that $(u, x), (u, y)$ are in A. Clearly, $C(D) = CE(D^c)$.

Given $D = (V, A)$, let W' be the binary relation on V defined by

$$aW'b \Leftrightarrow [(u, a) \in A \rightarrow (u, b) \in A].$$

Condition $C(p)$ has a natural dual version: If $p \geq 2$ is an integer, we say that D satisfies *condition* $C'(p)$ if whenever S is a set of p vertices of D, then there is a vertex x in S so that $xW'y$ for all $y \in S - \{x\}$. By using the converse D^c, we see that given an integer $p \geq 2$ and a graph G, there is a (acyclic) digraph D so that $C(D) = G$ and D satisfies condition $C(p)$ if and only if there is a (acyclic) digraph D' so that $CE(D') = G$ and D' satisfies condition $C'(p)$.

A more interesting variant on condition $C(p)$ is the following. If $p \geq 2$ is an integer, we say that D satisfies *condition* $C^*(p)$ if whenever S is a set of p vertices of D, then there is a vertex x in S so that xWy for all $y \in S - \{x\}$. If there is such a vertex x in the set S, we call x a *head* of S and denote any such vertex by $H(S)$.

The following proposition is analogous to Proposition 2.3 and has a completely analogous proof.

Proposition 3.1 *If $p < q$, then $C^*(p)$ implies $C^*(q)$.*

There are some simple properties of head that we will use in the following. We summarize them in the following proposition which has a straightforward proof.

Proposition 3.2 *If $G = C(D)$ for some digraph D, then:*
1). $H(S)$ is adjacent in G to any vertices in S that have neighbors in G and to the neighbors of such vertices.
2). If S has any vertices not isolated in G, then $H(S)$ cannot be isolated in G.

Proposition 3.3 *Suppose that digraph $D = (V, A)$ satisfies condition $C^*(p)$ for some $p \geq 2$ and $K^1, K^2, \ldots, K^m$ are $m \geq 2$ nontrivial connected components of $C(D)$ so that $T = K^1 \cup K^2 \cup \ldots \cup K^m$ has r vertices. If q is the number of isolated vertices of $C(D)$, then $r + q < p$.*

Proof. Suppose $r + q \geq p$. By definition of r, $r \geq 4$. Form p-element subset S of V by taking $s = \min\{p, r\}$ vertices from T with at least two from different components K^i and K^j. Add $p - s$ isolated vertices to get S. By the second part of Proposition 3.2, $H(S) = x$ cannot be isolated in G. There is a vertex y in S belonging to a nontrivial component of G not containing x, which violates the first part of Proposition 3.2. Hence, $C^*(p)$ fails. Q.E.D.

It follows that if $p \leq 5$ and $G = C(D)$ for D acyclic, then G cannot have more than one nontrivial component, since $q \geq 1$ for D acyclic. This observation will also follow from Propositions 3.7 to 3.9 and Theorem 3.10. However, we note that $K_2 \cup K_2 \cup I_1$ is $C(D)$ for an acyclic D satisfying $C^*(6)$ (vacuously). By the proposition, if D is an acyclic digraph satisfying $C^*(p)$ non-vacuously, i.e., if the number of vertices of D is at least p, then $C(D)$ has at most one nontrivial component.

Proposition 3.4 *If digraph $D = (V, A)$ satisfies condition $C^*(p)$ for some $p \geq 2$, then D does not have a directed cycle of length at least p.*

Proof. Suppose C is a directed cycle of length p, with vertices $v_1, v_2, \ldots, v_p$ in order around the cycle. The set S of vertices of C violates condition $C^*(p)$. That is because if $H(S) = v_i$, then $(v_{i-1}, v_i) \in A$ implies that $(v_i, v_i) \in A$, contradicting our convention that D is loopless. (If $i = 1, v_{i-1}$ is v_p.)

We show next that there are no cycles of length q such that $p < q \leq 2p - 2$. Suppose $v_1, v_2, \ldots, v_q$ are the vertices in order around such a cycle. Let $t = 2p - q - 1$. Then $t \geq 1$. Consider the p-element set $S = \{v_p, v_{p+1}, \ldots, v_q, v_1, v_2, \ldots, v_t\}$. This has a head. Since there are no loops, v_p is the only possible head and so $(v_q, v_1) \in A(D)$ implies that $(v_p, v_1) \in A(D)$. Thus, $v_1, v_2, \ldots, v_p, v_1$ is a cycle of D of length p, which is a contradiction.

We now argue by induction on s that D has no cycles of length greater than $(s-2)(p-2)+p$ and at most $(s-1)(p-2)+p$, $s \geq 2$. We have already established the case $s = 2$. Suppose that G has no cycles of length greater than $(s-2)(p-2)+p$ and at most $(s-1)(p-2)+p$, $s \geq 2$ and suppose that $(s-1)(p-2)+p < q \leq s(p-2)+p$ where $s \geq 2$. Let $v_1, v_2, \ldots, v_q$ be the vertices of a cycle of length q in order around the cycle. Let $r = q - p + 2$. Note that $r \leq q$ since $p \geq 2$. Let $S = \{v_r, v_{r+1}, \ldots, v_q, v_1\}$. Since this has p elements, there is a head. Since D has no loops, v_r is the only candidate for a head. It follows that since $(v_q, v_1) \in A(D)$ that $(v_r, v_1) \in A(D)$ and so $v_1, v_2, \ldots, v_r, v_1$ is a cycle. We show that r violates the inductive hypothesis.

Note that since $q \leq s(p-2)+p$, we have $r = q-p+2 \leq s(p-2)+2 = (s-1)(p-2)+p$. Since $q > (s-1)(p-2)+p$, we have $r = q-p+2 > (s-1)(p-2)+2 = (s-2)(p-2)+p$. This violates the induction hypothesis. Q.E.D.

Corollary 3.5 *A digraph satisfying condition $C^*(2)$ is acyclic.*

Lemma 3.6 *Let D be a digraph satisfying condition $C^*(p)$, $G = C(D)$, and q be the number of isolated vertices in G. Then:*
1). The size of an independent set T of vertices none of which is isolated in G is at most $\max\{1, p-q-1\}$.
2). If G has an independent set T of exactly $p-q-1 > 1$ vertices that are not isolated in G, then every vertex outside of T not isolated in G is adjacent to every vertex of T and every pair of vertices not isolated in G other than vertices of T are adjacent.

Proof.
1). Let T be an independent set of vertices none of which is isolated in G, suppose T has more than one vertex, and suppose that T has at least $p-q$ vertices. Let a set S of p vertices consist of T plus up to q isolated vertices.

By Proposition 3.2, $H(S) = x$ cannot be isolated in G. However, there is a second vertex y in S that is not isolated in G and is not adjacent to x, which violates the first part of Proposition 3.2. Thus, $C^*(p)$ fails.

2). Let T be an independent set of $p - q - 1$ vertices not isolated in G and z be any vertex not in T that is not isolated in G. Then let S be $T \cup \{z\}$ together with q isolated vertices. Again by Proposition 3.2, the only candidate for $H(S)$ is z and therefore z is adjacent to all vertices in T. If w is another vertex not in T that is not isolated in G, then let S' be $T \cup \{z, w\}$ together with $q - 1$ isolated vertices. It follows that either z or w is $H(S')$ and thus z and w are adjacent. Q.E.D.

The lemma allows us to draw some simple conclusions in the next three propositions.

Proposition 3.7 *Let G be a graph. Then G is the competition graph of a digraph satisfying condition $C^*(2)$ if and only if $G = I_q$ for $q > 0$ or $G = K_r \cup I_q$ for $r > 1, q > 0$.*

Proof. By Corollary 3.5, if D satisfies $C^*(2)$, then D is acyclic. By acyclicity, there is an isolated vertex. If G has any non-isolated vertices, then by Lemma 3.6, part 1), the maximum-sized independent set of vertices non-isolated in G has one element. Hence, every connected component of G is a clique and there cannot be two nontrivial cliques. Conversely, suppose $G = K_r \cup I_q, q > 0$. Since $q > 0$, we can define D on V by including all arcs (x, a) for $x \in K_r$ and $a \in I_q$. Q.E.D.

Proposition 3.8 *Let G be a graph. Then G is the competition graph of an acyclic digraph satisfying condition $C^*(3)$ if and only if $G = I_q$ or $G = K_r \cup I_q$ for $r > 1, q > 0$.*

Proof. By acyclicity, there is an isolated vertex. By Lemma 3.6, part 1), with $p = 3, q \geq 1$, the maximum-sized independent set of non-isolated vertices of G has one element. Thus, the argument is as in the previous proof. Q.E.D.

We note that acyclicity is needed as a hypothesis in Proposition 3.8. To see why, define D as follows: $V(D) = \{x, y, z\}, A(D) = \{(x, z), (z, x), (y, x), (y, z)\}$. Then $\{x, y, z\}$ has a head y, so $C^*(3)$ holds. However, $C(D) = P_3$.

Proposition 3.9 *Let G be a graph. Then G is the competition graph of an acyclic digraph satisfying condition $C^*(4)$ if and only if $G = I_q$ or $G = K_r \cup I_q$ for $r > 1, q > 0$ or $G = K_r - e \cup I_1$ for $r > 2$.*

Proof. By acyclicity of D, there must be an isolated vertex in G. Let t be the size of a maximum independent set of non-isolated vertices of G. Since $p = 4, q \geq 1$, Lemma 3.6, part 1), shows that $t \leq 2$. If $t = 1$, then argue as in the proof of Proposition 3.7. If $t = 2$, then $q = 1$ and by Lemma 3.6, part 2), there are two nonadjacent vertices x and y that are not isolated in G and so that every non-isolated vertex in G other than x and y is adjacent to x and y and every pair of non-isolated vertices of G other than x and y are adjacent. In this case, $G = K_r - e \cup I_1$ for $r > 2$ or $G = I_q$.

To prove the converse, note that if $G = K_r \cup I_q$ for $r > 1, q > 0$, then the construction of D in the proof of Proposition 3.7 gives the needed acyclic digraph since $C^*(2)$ implies $C^*(4)$. Suppose that $G = K_r - e \cup I_1$ for $r > 2$, where the missing edge is $\{x, y\}$. Define D on the vertices of G by including all arcs (v, a) for $v \in V(K_r) - \{y\}$ and a the vertex of I_1 and also all arcs (v, x) for $v \in V(K_r) - \{x\}$. To verify $C^*(4)$, note that every 4-element subset S of vertices contains at least three vertices of $K_r - e$ and so at least one of these is not x or y. This can serve as $H(S)$. Q.E.D.

Theorem 3.10 *Let G be a graph. Then G is the competition graph of an acyclic digraph satisfying condition $C^*(5)$ if and only if $G = I_q$ or $G = K_r \cup I_q$ for $r > 1, q > 0$ or $G = K_r - e \cup I_2$ for $r > 2$, or $G = K_r - P_3 \cup I_1$ for $r > 3$ or $G = K_r - K_3 \cup I_1$ for $r > 3$.*

Proof. Suppose that $G = C(D)$ for $D = (V, A)$ acyclic and satisfying $C^*(5)$. By acyclicity, G has at least one isolated vertex. Hence, by Lemma 3.6, part 1), with $p = 5, q \geq 1$, a maximum-sized independent set T of non-isolated vertices of G has at most three vertices. If T has at most one vertex, then the argument is as in the proof of Proposition 3.7. If T has three vertices, then $q = 1$ by Lemma 3.6, part 2). Thus, $G = K_r - K_3 \cup I_1$ for $r \geq 3$. The case $r = 3$ reduces to $G = I_4$.

Next, suppose T has two vertices x and y. Since T has maximum size, Lemma 3.6, part 1), implies that $q = 1$ or $q = 2$. Also, since T has maximum size, every vertex not isolated in G is adjacent to x or y. Now suppose that there are two vertices x_1 and x_2 that are adjacent to x but not to y. Then there are arcs $(x, a), (x_1, a), (x, b), (x_2, b), (y, c)$ in D. Since D is acyclic, one can show that at least one of a, b, c is not in $U = \{x, x_1, x_2, y\}$. To see why, suppose the contrary and note that $b \neq x, x_2$. If $b = x_1$, then $a \neq x_2$ because otherwise there is a cycle x_1, x_2, x_1. Hence, $a = y$. Then $c \in U$ gives us a cycle. A similar conclusion holds if $a = x_2$. Thus, we conclude that if a and b are both in U, each is y. But then $c \in U$ again gives us a cycle. We consider two cases, $a \notin U$ and $c \notin U$. (The case $b \notin U$ is analogous to the case $a \notin U$.) Let S be obtained from U by adding either a or c. By Proposition 3.2, $H(S) = a$ or c. However, in the former case, $(x, a) \in A$ implies $(a, a) \in A$ and in the latter case, $(y, c) \in A$ implies $(c, c) \in A$, both contradictions. We conclude that at most one vertex is adjacent to x and not y. Similarly, at most one vertex is adjacent to y and not x.

Now suppose that x' is adjacent to x and not y and y' is adjacent to y and not x. Let a, b be chosen so that $(x, a), (x', a), (y, b), (y', b) \in A$. Acyclicity implies that either a or b is not in $U = \{x, x', y, y'\}$, say a without loss of generality. Then $S = U \cup \{a\}$ cannot have a head. We conclude that there is at most one vertex adjacent to exactly one of x and y.

Now take a vertex v adjacent to both x and y and a vertex z adjacent to exactly one of x and y. Arguing as above, we find a vertex c so that $(x, c), (v, c) \in A$ or $(y, c), (v, c) \in A$ or $(x, c), (z, c) \in A$ or $(y, c), (z, c) \in A$ and $c \notin U = \{x, y, v, z\}$. Then $U \cup \{c\}$ must have a head and v is the only candidate. In particular, this implies that v and z are adjacent. Next, take vertices v_1 and v_2 adjacent to both x and y. We find a vertex d so that $(x, d), (v_1, d) \in A$ or $(x, d), (v_2, d) \in A$ or $(y, d), (v_1, d) \in A$ or $(y, d), (v_2, d) \in A$ and $d \notin U = \{x, y, v_1, v_2\}$. Then $U \cup \{d\}$ must have a head and v_1 and v_2 are the only candidates. It follows that v_1 and v_2 are adjacent in G. We have shown that any two vertices adjacent to both x and y are adjacent and any such vertices are adjacent to any vertex adjacent to exactly one of x and y.

We know that every vertex not isolated in G is adjacent to either x or y. If there is no z adjacent to exactly one of x and y, then G must be the graph $K_r - e \cup I_q$ where the missing edge is $\{x, y\}$. The case $r = 2$ is impossible since this implies that x is isolated in G. Suppose there is such a z adjacent to exactly one of x and y. We have already shown that there cannot be another $w \neq z$ adjacent to exactly one of x and y. We conclude that G is $K_r - P_3 \cup I_q$, where the missing edges are $\{x, y\}$ and $\{z, x\}$ or $\{z, y\}$. We have already observed that $q = 1$ or $q = 2$. The case $r = 3$ is impossible since this implies that x or y is isolated in G. If $r > 3$, then $q = 1$. For, if there are two vertices a and b in I_q, then $\{x, y, z, a, b\}$ can have no head, which is a contradiction.

To prove the converse, we use the constructions in the proofs of Propositions 3.7 and 3.9 to take care of the cases $K_r \cup I_q$ and $K_r - e \cup I_q$. Suppose next that $G = K_r - P_3 \cup I_1$, where $r > 3$. Suppose that the two missing edges in K_r are $\{x, y\}, \{y, z\}$. Define digraph D on the vertex set of G by including all arcs (v, a) for $v \in V(K_r) - \{x, z\}$, where a is the vertex of I_1, and all arcs (v, y) for $v \in V(K_r) - \{y\}$. Since every 5-element set of vertices has at least 4 elements from $K_r - P_3$, it has a vertex other than x, y, z and this is its head. Finally, suppose that $G = K_r - K_3 \cup I_1$. Suppose the three vertices whose pairwise edges are omitted are x, y, z. Define D on the vertex set of G by taking arcs (v, a) for $v \in V(K_r) - \{y, z\}$, where a is the element of I_q, (v, x) for $v \in V(K_r) - \{x, z\}$ and (v, y) for $v \in V(K_r) - \{x, y\}$. Since every 5-element set of vertices has at least 4 elements from $K_r - K_3$, it has a vertex other than x, y, z and this is its head. Q.E.D.

4 Concluding Remarks

This paper has left often the problem of characterizing those graphs G for which $G = C(D)$ for some digraph D satisfying condition $C(p)$ without requiring that D be acyclic. In particular, this is open in the case that the union of nontrivial components of G has less than p vertices. The paper has also left open in general the characterization of graphs G for which $G = C(D)$ for some acyclic digraph satisfying condition $C^*(p)$ for general p.

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