Premature Partial Latin Squares

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ABSTRACT. We introduce the notion of premature partial latin squares; these cannot be completed, but if any of the entries is deleted, a completion is possible. We study their spectrum, i.e., the set of integers t such that there exists a premature partial latin square of order n with exactly t nonempty cells.

1 Introduction

A partial latin square of order n, PLS(n), is an $n \times n$ array whose cells are either empty, or contain an element of $N = \{1, 2, \ldots, n\}$ such that (1) each element occurs in at most one cell in each row and each column.

Recently, there has been some interest in maximal PLSs, and in critical PLSs. A PLS is *maximal* if no nonempty cell can be filled without violating (1) above.

If one can fill the empty cells of a PLS(n) with symbols of N so that a latin square of order n results, then the PLS is said to be *completable*, or to *have a completion*. Clearly, not every PLS is completable.

A PLS is *critical* if it has a unique completion but removing any of its entries destroys this property (i.e., there is then more than one completion).

In [HR], the second and the fourth author dealt with the problem of determining the spectrum of maximal PLSs, i.e., the set

$ML(n) = \{t\colon$ there exists a maximal PLS(n) with exactly t nonempty cells$\}$.

Similarly, there are several papers devoted to the determination of the spectrum for critical PLSs, i.e., the set

*Supported by Kuwait University Grant SM-168

$CL(n) = \{t\colon$ there exists a critical PLS(n) with exactly t nonempty cells$\}$

(see, e.g., [CR], [DC], [DH]).

In this paper we introduce and study so-called premature PLSs. A PLS(n) is *premature* if it cannot be completed but removing any of its entries destroys this property (i.e., there is then at least one completion).

The spectrum of premature PLSs is the set

$\mathcal{P}(n) = \{t\colon$ there exists a premature PLS(n) with exactly t nonempty cells$\}$.

The similarity in spirit of the critical and premature PLSs is readily apparent. In fact, if one defines a PLS(n) to be α-completable if it has exactly α distinct completions but after removing any of its entries it has (strictly) more than α distinct completions, then the premature PLSs and the critical PLSs are precisely the α-completable PLSs for $\alpha = 0$ and $\alpha = 1$, respectively. The similarity between maximal and premature PLSs is also apparent.

However, notwithstanding these similarities, it appears that the methods of attacking the respective spectra problems are quite different, and that the methods that have been employed so far in studying the spectrum problem of maximal and critical PLSs cannot be used for the present problem of studying $\mathcal{P}(n)$, the spectrum for premature PLSs. Thus premature PLSs are not just a "variation on a theme".

Let us observe that if one were to drop the minimality condition (i.e., that after removing an arbitrary entry of the PLS, there is a completion) and require only that the PLS be not completable, the spectrum problem for such PLSs becomes trivial: a noncompletable PLS(n) with t nonempty cells exists for all t in the interval $[n, n^2 - 2]$.

2 An upper bound

The truth of the Evans' conjecture (see, e.g., [S], [AH]) implies that every PLS(n) with at most $n-1$ cells filled can be completed. It is also well known that there exist PLS(n) with exactly n cells filled which cannot be completed (see, e.g., [AH]; for example, put symbol x in all but one cell of the main diagonal, and a different symbol y in its single remaining cell). Thus $\min \mathcal{P}(n) = n$.

On the other hand, how large can the maximum M_n of $\mathcal{P}(n)$ be? We show in the next section that M_n is asymptotic to n^2. We now proceed to obtain an upper bound on M_n.

First we observe that no premature PLS can have a line (= row or column) in which all cells are occupied; indeed, even if we deleted the entry from one cell of such a line, the same entry would still have to occupy the same cell in any completion which contradicts the prematurity. Equivalently (by employing conjugacy), no premature PLS(n) can contain an element occurring n times.

Similarly, we observe that one can fill the empty cells of any premature PLS until only two empty cells remain. Indeed, if k is the entry in the (i, j) cell of a premature PLS(n), delete this entry. Then, by definition, this new PLS can be completed to an $n \times n$ latin square. In the latter, the element k occurs in the cells (i, l), (m, j) where $l \neq j$, $m \neq i$; simply remove these two occurrences of k.

Lemma 2.1. *Let L be a premature PLS(n) ($n \geq 4$) which contains a line having exactly $n-1$ cells occupied. Then L contains at least $3n-3$ empty cells.*

Proof: W.l.o.g., let the cells $(1, j)$, $j = 1, \ldots, n-1$ of the first row of L be occupied (and the cell $(1, n)$ be empty). Since L is premature, L^* obtained by deleting the entry in the cell $(1, j)$ can be completed, and thus no cell of the nth column of L can contain the entry in the cell $(1, j)$ of L, for $j = 1, \ldots, n-1$ (since in the completion of L^* the entry in the cell $(1, j)$ of L must be placed in the cell $(1, n)$). Let n be the element not occurring in the first row of L. Then either (i) the last column contains only the element n; but then L contains a total of n elements, or (ii) the last column is empty; but then in the completion L^*, the element n must occur in the cell $(1, j)$ (and the element from the cell $(1, j)$ of L must occur in the cell $(1, n)$ of L^*). Thus n does not occur at all in L. Thus each of the first $n-1$ columns of L has at least one empty cell. It is easy to see that there is a set S of n empty cells so that each row and each column contains one empty cell of S. If the jth column ($j = 1, \ldots, n-1$) contained only one empty cell, say, the cell (k, j), then arguing as before we get that the entire kth row is empty, and in total there are at least $3n-3$ empty cells. Otherwise, if for each $j = 1, \ldots, n-1$, the jth column contains at least two empty cells, then we have in total at least $3n-2$ empty cells. □

Theorem 2.2. *Any premature PLS(n) contains at least $3n-4$ empty cells.*

Proof: Let L be a premature PLS(n), and let t be the minimum number such that each line of L has at least t empty cells, and that there is a line of L with exactly t empty cells. If $t = 1$ then L has at least $3n-3$ empty cells by Lemma 2.1. If $t \geq 3$ then L has at least $3n$ empty cells. Thus it remains to consider the case $t = 2$. W.l.o.g. assume that the cells $(1, j)$, $j = 1, \ldots, n-2$ are occupied (and the cells $(1, n-1)$, $(1, n)$ are empty). Then by the same argument as above, each of the elements in the cells $(1, j)$, $j = 1, \ldots, n-2$ does not occur in at least one of the last two columns. Thus there are at least n empty cells in the last two columns. Since each of the first $n-2$ columns contains at least two empty cells, there are in total at least $3n-4$ empty cells. □

It follows that $M_n \leq n^2 - 3n + 4$. However, we feel that the following is true.

Conjecture. $M_n \leq n^2 - n^{\frac{3}{2}}$.

We can show:

Theorem 2.3. *Each premature PLS(n) contains a row and a column with at least $\sqrt{n}$ empty cells.*

Proof: We prove the statement only for columns.

By the same type of argument as in the proof of Lemma 2.1 we get: Let C_i be the set of columns of a premature PLS(n), L, which have an empty cell in the ith row of L. Then the total number of empty cells in columns of C_i is at least n.

For $j = 1, \ldots, n$, we denote by d_j the number of empty cells in the jth column of L. Using this notation, for $i = 1, \ldots, n$, the previous statement can be written in the form $\sum_j d_j \geq n$ where the sum runs over all columns from C_i. Summing over all rows yields

$$\sum_i \sum_j d_j \geq n^2. \qquad (*)$$

For a fixed index j the number d_j occurs exactly d_j times on the left side of (*) (each empty cell in the jth column contributes one to the number of occurrences of d_j in (*)). Thus (*) can be written as

$$d_1^2 + \cdots + d_n^2 \geq n^2$$

which in turn implies that there exists an index j (a column) so that $d_j \geq \sqrt{n}$. □

Let $E(n) = \{n, n+1, \ldots, n^2 - 3n + 4\}$. Then $\mathcal{P}(n) \subset E(n)$. In the next section we attempt to determine the values in $E(n)$ which belong to $\mathcal{P}(n)$.

3 Constructions

In this section, we present some fairly general constructions of premature PLS(n), both for "small" and for "large" values within the set $E(n)$. We rely heavily on Ryser's theorem which deals with completing partial latin squares which are latin rectangles.

Theorem 3.1. (Ryser [R]). *An $r \times s$ latin rectangle L can be completed to a latin square of order n if and only if each element occurs in L at least $r + s - n$ times.*

We also need the following simple observation:

(A) If an element of a PLS(n), L, which occurs in r rows (columns) of L can be placed in (unoccupied cells) of at most $n - r - 1$ further rows (columns) then L cannot be completed.

Our first construction is especially simple.

Construction A. *Let n, r, s be positive integers such that $2 \le r \le s \le n-r$. Then $rs+n-r-s+1 \in \mathcal{P}(n)$.*

Proof: Let R be an $r \times s$ latin rectangle based on $\{1,\dots,s\}$ occupying the first r rows and the first s columns of L, a PLS(n). Furthermore, let the cells $(r+\delta, s+\delta)$, $\delta = 1,\dots,n-r-s+1$, of L be occupied by the symbol $s+1$, and let all other cells of L be empty. Then L is a premature PLS(n) with exactly $rs+n-r-s+1$ nonempty cells.

Indeed, the symbol $s+1$ occurs in exactly $n-r-s+1$ cells of L. If L is to be completed, $s+1$ must be placed in each of the first r rows but there are only $r-1$ columns (the last $r-1$ columns) in which it can be placed. Thus, by (A), L is not completable. Clearly, deleting any entry of L (whether in R, or from among the cells $(r+\delta, s+\delta)$) renders L completable.

Construction A'. *Let n, r, s be positive integers such that $2 \le r \le s < \frac{n}{2}$. Then $n(r-1)-r^2+r+s+1 \in \mathcal{P}(n)$.*

Proof: Let C_1 be a circulant latin square of order s on $\{1,\dots,s\}$, with elements of the first column in natural order, and let C_2 be a circulant $(n-s) \times s$ latin rectangle on $\{s+1,\dots,n\}$, again with elements of the first column in natural order. Let R_1 be obtained from C_1 by deleting all its entries in the first $r-2$ columns and all its entries in the last $s-r$ rows, and R_2 from C_2 by deleting all its entries in the last $s-r+2$ columns. Let now L be a PLS(n) with R_1 in its upper left corner and R_2 in its lower left corner. The only other nonempty cells of L occur in the $(s+1)$st column where the cells $(j, s+1)$, $j = r+1, r+2, \dots, n-s+1$ contain $n-r-s+1$ elements of $\{s+1,\dots,n\}$, and any such element differs from all elements occurring in its row in C_2. Since any $(n-s) \times s$ latin rectangle with $s < \frac{n}{2}$ can be extended by one column, the cells in the $(s+1)$st column can always be filled in this manner.

Clearly, L cannot be completed, as the first r cells of the $(s+1)$st column can contain only elements of $\{s+1,\dots,n\}$ but there are only $r-1$ such elements available. On the other hand, if any entry of L is deleted, completion is possible. To see this, consider three cases.

Case 1. Suppose we delete an element t from the cell $(j, s+1)$. Then $j \in \{r+1,\dots,n-s+1\}$ and $t \in \{s+1,\dots,n\}$. The first r cells of the $(s+1)$st column can now be filled with the $r-1$ elements of $\{s+1,\dots,n\}$ that do not occur in the cells of the $(s+1)$st column of L, together with the "freed" element t. Complete now R_1 back to C_1 and R_2 back to C_2 to obtain L'. There are still s unfilled cells of the $(s+1)$st column to be filled, namely the cell $(j, s+1)$ and the last $s-1$ cells. If $j \in \{s+1,\dots,n-s+1\}$, we can fill these cells arbitrarily with the s elements of $\{1,\dots,s\}$ to obtain an $n \times (s+1)$ latin rectangle L'' on $\{1,\dots,n\}$. Otherwise, i.e. when $j \in \{r+1,\dots,s\}$, let d ($d \in \{1,\dots,s\}$) be the element in the cell (j, s) of

L' and let (k, s) be the cell containing the element t. Create now from L' an $n \times (s+1)$ latin rectangle L'' by replacing d in the cell (j, s) with t, placing d in the cells $(j, s+1)$ and (k, s), and filling the remaining $s-1$ cells of the $(s+1)$st column arbitrarily with the $s-1$ elements of $\{1, \dots, s\}$ other than d. To show that L'' is indeed a latin rectangle, that is, that no row or column of L'' contains more that one occurence of the same element (or that all rows and columns in L'' are 'latin'), we only need to examine those rows and columns in L'' which are different from the corresponding rows and columns in L'. In particular, we need to examine rows j and k and columns s and $s+1$. Row j and column s are obtained from the corresponding row and column in L' by swapping two entries, and thus are latin. Row k is latin if the cell $(k, s+1)$ does not contain the element d. If the cell $(k, s+1)$ is occupied in L', then it contains an element $d' \in \{s+1, \dots, n\}$; if the cell $(k, s+1)$ is empty in L', then in L'' it contains the element $d' \in \{1, \dots, s\}$, $d' \neq d$. Finally, column $s+1$ is obviously latin.

Case 2. Suppose we delete now an element t from the cell (j, k) where $k \in \{r-1, r, \dots, s\}$. Then $j \in \{1, \dots, r\}$ and $t \in \{1, \dots, s\}$. Complete R_1 back to C_1 and R_2 back to C_2 to obtain L'. Suppose q ($q \in \{s+1, \dots, n\}$) is the element in the cell $(s+1, k)$ of L' (say). Create now L'' by replacing t in the cell (j, k) with q, replacing q in the cell $(s+1, k)$ with t, and also placing t in the cell $(j, s+1)$; fill the cells $(i, s+1)$, $i = 1, \dots, r$; $i \neq j$, with the $r-1$ elements of $\{s+1, \dots, n\}$ that do not occur in the cells of the $(s+1)$st column of L. Fill the last $s-1$ cells of the $(s+1)$st column arbitrarily with the $s-1$ elements of $\{1, \dots, s\}$ other than t. Arguing similarly as in Case 1, L'' is an $n \times (s+1)$ latin rectangle on $\{1, \dots, n\}$.

Case 3. Finally, suppose we delete an element t from the cell (j, k) where $k \in \{1, \dots, r-2\}$. Then $j \in \{s+1, \dots, n\}$ and $t \in \{s+1, \dots, n\}$. Complete R_1 back to C_1 and R_2 to C_2 to obtain L'. Suppose q is the element in the cell $(1, k)$ of L' (say). Create L'' from L' by replacing t in the cell (j, k) with q, ($q \in \{1, \dots, s\}$), and also placing q in the cell $(1, s+1)$, and placing t in the cell $(1, k)$. Proceed now as in Case 2.

In all cases, we obtain an $n \times (s+1)$ latin rectangle on $\{1, \dots, n\}$ which can be completed to a latin square of order n by Theorem 3.1. This completes the proof that L is a premature PLS(n). □

Construction A''. *Let n, r, s be positive integers such that $1 \leq r < s \leq \frac{n}{2}$. Then $n(r+1) - r^2 - r - s + 1 \in \mathcal{P}(n)$.*

Proof: Let D_1 be a circulant $(s-1) \times s$ latin rectangle on $\{1, \dots, s\}$, and let D_2 be a circulant $(n-s+1) \times s$ latin rectangle on $\{s+1, \dots, n, n+1\}$, with both D_1, D_2 having elements of the first column in natural order. Let S_1 be obtained from D_1 by deleting its entries in the first r columns and the last $s-r-1$ rows, and let S_2 be obtained from D_2 by deleting all entries in the last $s-r$ columns, and all entries equal $n+1$.

Let L be a PLS(n) with S_1 in its upper left corner and S_2 in its lower left corner. The only other nonempty cells of L occur in the $(s+1)$st column. Let D_2' be a circulant $(n-s+1)\times(n-s+1)$ latin square on $\{s+1,\ldots,n+1\}$ having the first column in natural order, where all entries equal to $n+1$ are deleted. Then entries $(r+1,s+1),(r+2,s+1),\ldots,(n-s+2,s+1)$ in L are equal to the entries $(1,r+2),(2,r+2),\ldots,(n-s-r+2,r+2)$ in D_2', respectively. Note that the cell $(n-s,s+1)$ in L is empty, and thus there are $n-r-s+1$ occupied cells in the column $s+1$. Again, L cannot be completed but if any entry of L is deleted, completion is possible.

Case 1. Suppose we delete an element t from the cell $(j,s+1)$. Then $t\in\{s+1,\ldots,n\}$, and the first r cells of the $(s+1)$st column can now be filled with the $r-1$ elements of $\{s+1,\ldots,n\}$ that do not occur in the cells of the $(s+1)$st column of L, together with the "freed" element t. Complete now S_1 back to D_1 and S_2 back to D_2; replace now element $n+1$ in column 1 with s, in column 2 with $1,\ldots$, in column s with $s-1$, to obtain L'. Note that in L' a row $n-j$, $j\in[0,s-1]$, contains only one element from the set $\{1,\ldots,s\}$, namely the element j if $j>0$, and the element s if $j=0$. Also note that rows $n-j$, $j\in[s,n-s]$ do not contain any of the first s elements; whatever the value of s, row s in L' will never contain any of the first s elements.

There are still s unfilled cells of the $(s+1)$st column to be filled, namely the cells $(j,s+1)$ and $(n-s,s+1)$, and the last $s-2$ cells.

If $j\in\{s,\ldots,n-s+2\}$, we can always fill these cells with the s elements of $\{1,\ldots,s\}$ to obtain an $n\times(s+1)$ latin rectangle L'' on $\{1,\ldots,n\}$; to see this, recall that each of the last $n-s+1$ rows in L' contains at most one element $d'\in\{1,\ldots,s\}$, and that $s\geq 2$.

Otherwise, i.e. when $j\in\{r+1,\ldots,s-1\}$, create from L' an $n\times(s+1)$ latin rectangle L'' by replacing the element j in the cell $(j,1)$ with $s+1$, placing j in the cells $(j,s+1)$ and $(s,1)$ and filling the remaining empty cells of the $(s+1)$st column with the $s-1$ elements of $\{1,\ldots,s\}$ other than j. Note that we have in fact swapped the entries in the cells $(j,1)$ and $(s,1)$, and thus column 1 is latin; also, row s is latin, as it originally did not contain any of the first s elements.

Case 2. Suppose we delete now an element t from the cell (j,k) where $k\in\{r+1,r+2,\ldots,s\}$. Then $j\in\{1,\ldots,r\}$ and $t\in\{1,\ldots,s\}$. Complete now S_1 back to D_1 and S_2 back to D_2; replace all entries equal to $n+1$, as in Case 1, to obtain L'. Create now L'' by placing t in the cell $(j,s+1)$, and swapping elements in the cells (j,k) and (s,k). Fill the cells $(i,s+1)$, $i=1,\ldots,r$; $i\neq j$, with the $r-1$ elements of $\{s+1,\ldots,n\}$ that do not occur in the cells of the $(s+1)$st column of L. Fill the remaining empty cells in $(s+1)$st column with the $s-1$ elements of $\{1,\ldots,s\}$ other than t. Then L'' is an $n\times(s+1)$ latin rectangle on $\{1,\ldots,n\}$.

Case 3. Finally, suppose we delete an element t from the cell (j,k) where $k \in \{1,\dots,r\}$. Then $j \in \{s,\dots,n\}$ and $t \in \{s+1,\dots,n\}$. Complete now S_1 back to D_1 and S_2 back to D_2; replace now all entries equal to $n+1$, as in Case 1, to obtain L'.

Suppose q is the element in the cell (p,k) of L', where $p < s$ and q does not occur in the row j of L'. That is, $p < s$ and $q \neq n-j$ if $n-s < j < n$, and $q \neq s$ if $j = n$. If $s > 2$ or $n - j \neq 1$, such p always exists. Create L'' from L' by replacing t in the cell (j,k) with q, $(q \in \{1,\dots,s\})$, and placing q in the cell $(p, s+1)$, and t in the cell (p,k). Proceed now as in Case 2.

However, if $s = 2$ and $n - j = 1$, then we replace q in the cell $(1,1)$ by t, we place 2 in the cell $(n-1,1)$, and 1 in the cells $(n,1)$ and $(1,s+1)$. (Note that $r = k = 1$.)

In either case, we see that L is a PLS(n). □

Corollary 3.2. *$[n+1, \frac{n^2}{4}+\delta] \subset \mathcal{P}(n)$ where $\delta = 1$ if n is even, and $\delta = \frac{3}{4}$ if n is odd.*

Proof: Due to Construction A' and A'' it suffices to show that the sets $\{n(r-1)-r^2+r+s+1 \colon 2 \le r \le s < \frac{n}{2}\}$ and $\{n(r+1)-r^2-r-s+1 \colon 1 \le r < s \le \frac{n}{2}\}$ cover the interval $[n+1, \frac{n^2}{4}+1]$ for n even, and the interval $[n+1, \frac{n^2+3}{4}]$ for n odd. For any fixed r, let $f(r) = \{n(r-1)-r^2+r+s+1 \colon 2 \le r \le s < \frac{n}{2}\}$, and let $g(r) = \{n(r+1)-r^2-r-s+1 \colon 1 \le r < s < \frac{n}{2}\}$. Clearly, both $f(r)$ and $g(r)$ are intervals, so $f(r) = [f(r)_m, f(r)_M]$, $g(r) = [g(r)_m, g(r)_M]$. A routine calculation yields $f(r) = [n(r-1)-r^2+2r+1, n(r-\frac{1}{2})-r^2+r]$ and $g(r-1) = [n(r-\frac{1}{2})-r^2+r+1, nr-r^2+1]$ for n even (that is, $f(r)_M+1 = g(r-1)_m$), and $f(r) = [n(r-1)-r^2+2r+1, n(r-\frac{1}{2})-r^2+r+\frac{1}{2}]$ and $g(r-1) = [n(r-\frac{1}{2})-r^2+r+\frac{3}{2}, nr-r^2+1]$ for n odd (that is, $f(r)_M+1 = g(r-1)_m$). In order to complete the proof, now one only needs to observe that $f(r+1)_m = g(r-1)_M+1$, $g(\frac{n}{2}-2)_M = g(\frac{n}{2}-1)_m - 1$ for n even, $f(2)_m = n+1$, and $g(\frac{n}{2}-1)_M = \frac{n^2}{4}+1$ for n even, $g(\frac{n-3}{2})_M = \frac{n^2+3}{4}$ for n odd. □

Construction B. *Let n, r, s, x, α be positive integers such that $r+s-n = \alpha > 0$, $2 \le r \le s \le n-x$, $x \le r$, and, moreover,*

(i) $x\alpha - 1 \le (x-1)r$ [equivalently, $r \le x(n-s)+1$]

(ii) $rs - (x\alpha - 1) \ge (\alpha+1)(n-x)$.

Then $rs - x\alpha + 1 \in \mathcal{P}(n)$.

It follows that either $x > 1$, or $x = 1$ and $\alpha = 1$.

Proof: We construct a PLS(n), L, all of whose nonempty cells are contained within an $r \times s$ rectangle R in the upper left corner of L. However, not all cells of R are filled; in fact, exactly $\alpha - 1$ of these are empty (we refer to these empty cells as "holes").

Let R be an $r \times s$ latin rectangle based on elements $\{1, \ldots, n-x\}$ with exactly $x\alpha - 1$ entries deleted in such a way that no line contains more than $x-1$ holes (thus condition (i) must be satisfied). There are x elements $n-x+1, \ldots, n$ that do not occur in R at all. Now if R contains each of $1, \ldots, n-x$ at least $\alpha+1$ times, (thus condition (ii) must be satisfied), then if we delete another entry to obtain R' with $x\alpha$ holes, each of the elements $1, \ldots, n-x$ will still occur in R' at least α times. If we now fill the $x\alpha$ holes of R' (so that each of the x missing elements is placed in exactly α holes, and no element is placed in two holes in the same row or column) to obtain R^* then R^* satisfies the conditions of Ryser's theorem, and thus can be completed. Since R is clearly not completable, R is a premature PLS(n) with exactly $rs - x\alpha + 1$ nonempty cells.

Thus all that is needed is an $r \times s$ latin rectangle R as above. We may start, for example, with an equitable $r \times s$ latin rectangle based on $1, \ldots, n-x$ (which always exists; see, e.g., [M], [W]) and delete from it the entries of no more than $x-1$ disjoint (partial) transversals. □

Corollary 3.3. *The maximum M_n of $\mathcal{P}(n)$ is asymptotic to n^2.*

Proof: Assume for simplicity of calculations that n is a perfect square (when it is not, the argument is similar). In Construction B, choose $r = s = n - \sqrt{n}$ and $x = \sqrt{n}$. Then $\alpha = n - 2\sqrt{n}$; $x\alpha - 1 = n^{\frac{3}{2}} - 2n - 1$, and conditions (i) and (ii) are satisfied. Thus $t_n = (n - \sqrt{n})^2 - n^{\frac{3}{2}} + 2n + 1 = n^2 - 3n^{\frac{3}{2}} + 3n + 1 \in \mathcal{P}(n)$. But $t_n \sim n^2 - o(n^2)$, and since $M_n \geq t_n$, the corollary follows. □

Example. Let $n = 16$, $r = s = 12$, $x = 4$. Then $\alpha = 8$, $x\alpha - 1 = 31$. Let L be any latin square of order 12 with three disjoint transversals, and let T_1, T_2, T_3 be three disjoint transversals of L. Delete all entries in the cells of T_1 and T_2 from L as well as any 7 entries from the cells of T_3. The resulting PLS(12) L' has 113 nonempty cells, and considered as a PLS(16), is clearly premature. Thus $113 \in \mathcal{P}(16)$.

Construction B is quite general, and covers many values in the "upper part" of $E(n)$. For example, we get:

Corollary 3.4. *For n even, $[\frac{n^2}{4} + 1, \frac{n^2}{4} + n - 2] \subset \mathcal{P}(n)$.*

Proof: In Construction B, choose $\alpha \in \{1, 2\}$, $r \leq s$, $s - r \leq 3$, $x \in \{\alpha, \alpha+1, \ldots, n-s\}$. □

Combining Constructions A and B, we get for small values of n:
$\mathcal{P}(2) = \{2\}$, $\{3, 4\} = \mathcal{P}(3)$,
$\{4, 5, 6\} \subset \mathcal{P}(4)$, $\{5, 6, 7, 8\} \subset \mathcal{P}(5)$,
$[6, 13] \subset \mathcal{P}(6)$, $[7, 17] \subset \mathcal{P}(7)$;
$[8, 25] \backslash \{23, 24\} \subset \mathcal{P}(8)$,
$[9, 31] \backslash \{26, 29, 30\} \subset \mathcal{P}(9)$,

$[10, 42] \backslash \{36, 39, 40\} \subset \mathcal{P}(10)$,
$[11, 49] \backslash \{42, 46, 47, 48\} \subset \mathcal{P}(11)$,
$[12, 63] \backslash \{47, 50, 54, 55, 59, 60, 62\} \subset \mathcal{P}(12)$.

4 Conclusion

While several further corollaries similar to Corollary 3.4 can be obtained from Construction B, it is also clear that Constructions A', A'' and B alone leave many values of $E(n)$ uncovered.

Acknowledgements. Thanks to Chris Rodger for providing needed references. Part of this work was done while the second and fourth author were visiting the Department of Computer Science at the University of Newcastle, NSW. They would like to thank the department for its hospitality.

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