A note on the spatiality degree of graphs

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ABSTRACT. We introduce the notion of BP-spatial representation of a biconnected graph $G = (V, E)$. We show that the spatiality degree of a BP-spatial representable graph is $2(|E| - |V|)$. From this result, we derive the spatiality degree for planar and hamiltonian graphs.

1 Introduction

A *spatial representation* (in $\mathbf{R}^3$) [1] of a (undirected, simple, and connected) graph G is a collection of faces where: (1) each face is a finite simple surface whose boundary corresponds to a simple cycle of G, (2) no two faces can be bounded by the same cycle, (3) each edge belonging to a cycle is on the boundary of at least one face, (4) any two faces do not intersect but along the common edges, and (5) the union of all the faces is a set S on which any closed line can be continuously shrunk to a point always remaining on S (that is, such union is a simply connected topological space).

In Figure 1 a planar graph of eight vertices, ten edges, and four faces is shown while in Figure 2 a spatial representation of this graph with six faces is shown: three faces are bounded by cycles of four vertices and are plane, the fourth one is bounded by the cycle $ABCFGH$, the fifth one is bounded by the outer cycle, and the sixth one is bounded by the cycle $BCDEFG$ and is drawn below the plane of the graph.

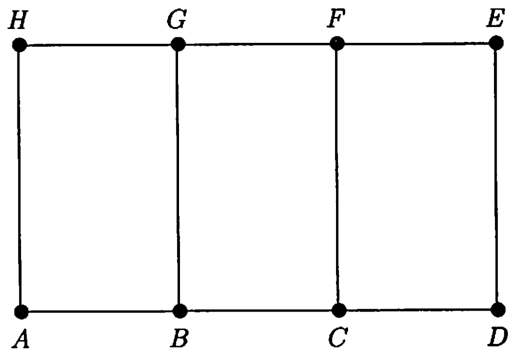

Figure 1: A planar graph

Observe that in the above example the faces identify four "cells", i.e., simple regions such that any two points inside the region can be connected with a line entirely contained into it. One of these cells is the infinite one.

In [1] it is shown that any graph admits a spatial representation. Such a representation has exactly one cell (i.e., the infinite one) and is also called a "tree-representation". For a biconnected and planar graph another spatial representation is the one obtained by drawing the graph onto the sphere. Such a representation has exactly two cells. However, graphs exist for which a tree-representation is the only one possible, that is, they do not admit a spatial representation with more than one cell (e.g. a chain of 3-vertex cycles joined by exactly one vertex). It thus seems reasonable to distinguish graphs according to the maximum number of cells that can be formed by spatial representations. In [1] this number is called the *spatiality degree* of a graph and it is denoted by $s(G)$.

The notion of spatial representation is strictly related to that of regular CW-complex (cell complex) [2]. In other words, it is essentially a regular 2-cell complex in a three-dimensional Euclidean space on a given graph

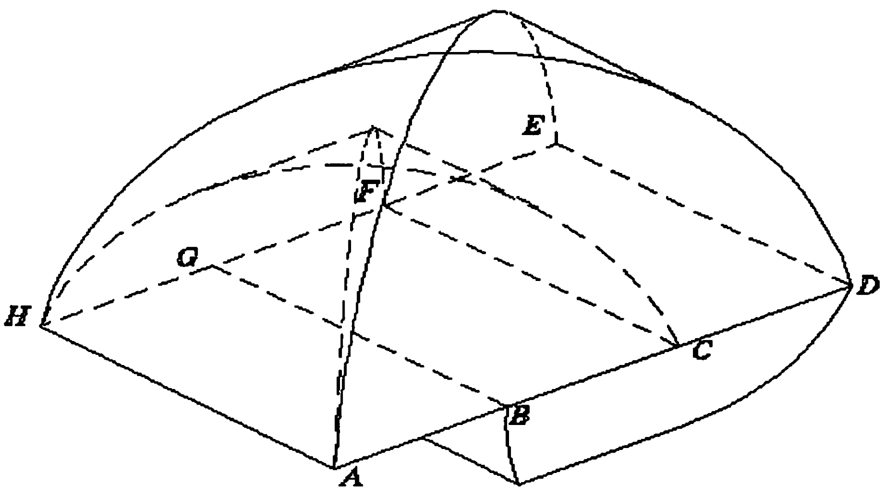

Figure 2: An example of a spatial representation of the graph of Figure 1

$G = (V, E)$. For some set of simple cycles, an open disk can be sewed on each so that every non-cut-edge is on a face and so that the resulting topological space is simply connected. The goal is to maximize the number of 3-cells. We conjecture that this number is $2(|E| - |V|)$ (each 3-cell is delimited by exactly three faces) if G is biconnected and is not an edge or a simple cycle. We prove it for planar and hamiltonian graphs. In section 2 we introduce the notion of BP-spatial representation of a biconnected graph $G = (V, E)$. We show that the spatiality degree of a BP-spatial representable graph is $2(|E| - |V|)$. We also prove that the spatiality degree of any G is equal to the sum of the spatiality degrees of the biconnected components minus the number of these components plus one. In section 3, from these results the spatiality degree for planar and hamiltonian graphs is derived.

2 General Results

A *BP-spatial representation* of a graph G is a spatial representation such that (1) any face belongs to exactly two cells and (2) the subgraph of G corresponding to the boundary of any cell is a biconnected planar subgraph of G. A graph is BP-spatial representable if it admits a BP-spatial representation. In this section we show how a set of faces can be added to a BP-spatial representation so that any cell of the new representation is delimited by exactly three faces. The following two lemmas are needed to define such a suitable set of faces.

Let F be a face of a planar graph G drawn onto a sphere. An *ear* of F is another face F' such that the intersection between the boundaries of F and F' is a simple path of at least one edge.

Lemma 1.1. For any biconnected planar graph G with at least 3 faces, any face has two ears.

Proof: Let us assign to any face with non-empty edge-intersection with F a distinct color. Let us then color the edges of the boundary of F according to the color of the face they belong to. A subpath of the boundary of F is then said to be a "segment" if all its edges have the same color and no adjacent edge has the same color. A segment is said to be "single" if it is the only one with that color. Since G is biconnected, the boundary of any face is a simple cycle. Given that there are at least 3 faces, it follows that F has at least two segments. In fact, if there were only one segment the boundary of F would be part of the boundary of another face F'. Then, the boundary of F' would not be a simple cycle. If all segments have different colors, then there are at least two single segments. Otherwise, let s_1 and s_2 be two segments of the same color. Consider the two paths obtained by removing the edges of s_1 and s_2 from the boundary of F. Since pairs of equally-colored segments with distinct colors cannot interleave, both of those paths must contain single segments. □

Lemma 1.2. Let G be a biconnected planar graph. Let G' be the graph obtained by deleting in G the internal vertices of the intersection of the boundary of a face with the the boundary of one of its ears. Then, G' is a biconnected planar graph.

Proof: Clearly, G' is planar since it has been obtained from G by deleting a path. Let v be any vertex of G'. We show that $G' - v$ is connected. Let u_1 and u_2 be any two vertices of $G' - v$. In $G - v$ there is a path p between u_1 and u_2. Let t_1 and t_2 be the endpoints of the path deleted from G. If one of the deleted edges belongs to p, then t_1 and t_2 must belong to p. This follows from the fact that the internal vertices of the deleted path have degree two. W.l.o.g., u_1 and u_2 are connected by subpaths of p to t_1 and t_2, respectively in $G' - v$. Moreover, since t_1 and t_2 belong to the boundary of a face of G' they are connected in $G' - v$. □

For any spatial representation of a graph $G = (V, E)$, the following equality holds:

$$|V| - |E| + f = c.$$

where f and c are the number of faces and the number of cells of the spatial representation, respectively [1]. Observing that any face may delimit at most two cells and that, if $c \geq 2$, each cell must be delimited by at least three faces, it follows that, for any graph G,

$$s(G) = 1 \quad \text{or} \quad 1 < s(G) \leq 2(|E| - |V|).$$

Now, we can state the following theorem.

Theorem 2.1. If $G = (V, E)$ is BP-spatial representable then $s(G) = 2(|E| - |V|)$.

Proof: Let F be a face of a planar graph and let F' be an ear of F. Let B and B' be the boundaries of F and F' respectively. The *sum* of B and B', denoted as $B + B'$, is a cycle determined by the symmetric edge-difference between B and B'. It suffices to show that, for any cell delimited by more than three faces, it is possible to "split" the cell adding a new face. Indeed, select any face F of the graph corresponding to the boundary of the cell (which is biconnected and planar). From Lemma 2.1 it follows that F has two ears F_1 and F_2. Let B, B_1, and B_2 be the boundary of F, F_1, and F_2 respectively. Either $B + B_1$ or $B + B_2$ is not the boundary of a face of the spatial representation. In fact, since the cell is delimited by more than three faces neither $B + B_1$ nor $B + B_2$ may be a boundary of a face of the cell. It follows that they cannot be both boundaries of faces, otherwise they would intersect. Thus we can add a new face with this boundary and then split the cell. From Lemma 2.2 the representation obtained is still a BP-spatial representation. Then, by iterating this operation the statement of the theorem follows. □

The next result relates the spatiality degree of a graph to the spatiality degrees of its biconnected components.

Theorem 2.2. For any graph G,

$$s(G) = \sum_{i=1}^{k} s(B_i) - k + 1$$

where $B_1, \ldots, B_k$ are the biconnected components of G.

Proof: We prove that $s(G) = \sum_{i=1}^{k} s(B_i) - k + 1$ by induction on the number k of the biconnected components of G. If $k = 1$, it is trivially true. Suppose that G has $k > 1$ biconnected components $B_1, \ldots, B_k$. B_k is connected to the other part of the graph by a single vertex v. Let G' be the graph obtained by deleting from G all the vertices of B_k but v. By the inductive hypothesis, G' admits a spatial representation with a maximum number of cells equal to $\sum_{i=1}^{k-1} s(B_i) - (k-1) + 1$. Let C be a cell of this representation such that v belongs to it. Consider a spatial representation of B_k with $s(B_k)$ cells and such that v belongs to the infinite cell. Then there exists a representation of G where the spatial representation of B_k is inside C. Such a representation has $\sum_{i=1}^{k} s(B_i) - k + 1$ cells. This is the maximum number of cells since any cycle of G is included in some biconnected component. □

3 The Spatiality Degree of Planar and Hamiltonian Graphs

From the general results shown in the previous section, we derive the following theorems.

Theorem 3.1. Let $G_1 = (V_1, E_1), \ldots, G_h = (V_h, E_h)$ be the biconnected components of a planar graph G, which are not edges or simple cycles. Then,

$$s(G) = 2\sum_{i=1}^{h}(|E_i| - |V_i|) - h + 1.$$

Proof: Clearly, G_i is BP-spatial representable. It follows from Theorem 2.1 that $s(G_i) = 2(|E_i| - |V_i|)$. Then the statement follows from Theorem 2.2. □

Theorem 3.2. For any hamiltonian graph $G = (V, E)$ which is not a simple cycle, $s(G) = 2(|E| - |V|)$.

Proof: Let $v_1, v_2, \ldots v_n, v_1$ be the sequence of nodes visited on a hamiltonian cycle, where $|V| = n$ and let such cycle be a face of the spatial representation we are constructing. Consider any edge, say (v_i, v_j), which does not belong to the hamiltonian cycle, with $i < j$. Then, add as faces of the spatial representation the cycles $v_1, \ldots, v_i, v_j, v_{j-1}, \ldots, v_1$ and $v_i, \ldots, v_n, v_{n-1}, \ldots, v_j, v_i$. Obviously, these three faces can be seen as the boundary of a cell. Observe that another edge which is not in the hamiltonian cycle can provide another pair of faces, as in the previous case. This new pair of faces can form a new cell with the two previously added faces. Since all the vertices are still on the boundary of the infinite cell, we can iterate this operation for each edge which is not in the hamiltonian cycle.

Hence, we obtain a BP-spatial representation and the statement follows from Theorem 2.1. □

References

[1] F. Luccio and L. Pagli, Introducing Spatial Graphs, *Congressus Numerantium* **110** (1995), 33–41.

[2] A.T. Lundell and S. Weigram (1969), *The Topology of CW Complexes*, Van Nostrand, 1967.