The Splitting Number and Skewness of $C_n \times C_m$

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Abstract

The skewness of a graph G is the minimum number of edges that need to be deleted from G to produce a planar graph. The splitting number of a graph G is the minimum number of splitting steps needed to turn G into a planar graph; where each step replaces some of the edges $\{u, v\}$ incident to a selected vertex u by edges $\{u', v\}$, where u' is a new vertex. We show that the splitting number of the toroidal grid graph $C_n \times C_m$ is $\min\{n, m\} - 2\delta_{n,3}\delta_{m,3} - \delta_{n,4}\delta_{m,3} - \delta_{n,3}\delta_{m,4}$ and its skewness is $\min\{n, m\} - \delta_{n,3}\delta_{m,3} - \delta_{n,4}\delta_{m,3} - \delta_{n,3}\delta_{m,4}$. Here, δ is the Kronecker symbol, i.e., $\delta_{i,j}$ is 1 if $i = j$, and 0 if $i \neq j$.

Keywords: topological graph theory, graph drawing, toroidal mesh, planarity.
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1 Introduction

The skewness $sk(G)$ and splitting number $sp(G)$, defined below, are two natural measures of the non-planarity of a graph G. These topological invariants play important roles in automatic graph drawing and circuit design [8, 9, 15, 18, 22, 23].

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sp					
	3	4	5	6	7
3	1	2	3	3	3
4	2	4	4	4	4
5	3	4	5	5	5
6	3	4	5	6	6
7	3	4	5	6	7

sk					
	3	4	5	6	7
3	2	2	3	3	3
4	2	4	4	4	4
5	3	4	5	5	5
6	3	4	5	6	6
7	3	4	5	6	7

Table 1: Values of *sp* and *sk* for small values of n and m

In this paper, we determine exact values for the skewness and splitting number of the graphs $C_n \times C_m$, where C_n is the chordless cycle on n vertices. These graphs can be drawn as regular latitude-longitude grids on the torus, and thus are also known as 'toroidal rectangular grids' or similar names. They occur often as interconnection diagrams of multiprocessor computers and cellular automata [15, 18], and so our results are relevant to the physical design of such machines.

It turns out that the obvious upper bound $\min\{n, m\}$ is always tight except for $C_3 \times C_3$ and $C_3 \times C_4$. Specifically, we show that

$$sp(C_n \times C_m) = \min\{n,m\} - 2\delta_{3,n}\delta_{3,m} - \delta_{3,n}\delta_{4,m} - \delta_{4,n}\delta_{3,m} \quad (1)$$

$$sk(C_n \times C_m) = \min\{n,m\} - \delta_{3,n}\delta_{3,m} - \delta_{3,n}\delta_{4,m} - \delta_{4,n}\delta_{3,m}, \quad (2)$$

where δ is the Kronecker identity symbol, i.e., $\delta_{i,j}$ is 1 if $i = j$, and 0 if $i \neq j$.

Table 1 shows these bounds explicitly for small values of n and m.

Our strategy to prove these results is as follows. In section 4, we prove that $sp(G) \leq sk(G)$, for any graph G. In section 5, we show that formula (1) is a lower bound for the splitting number *sp*, and in section 6, we prove that formula (2) is an upper bound for the skewness *sk*. It follows that the two invariants coincide except for $C_3 \times C_3$. To complete the proof we show in sections 5 and 6 that $sp(C_3 \times C_3) = 1$ and $sk(C_3 \times C_3) = 2$, respectively.

2 Notation and definitions

For basic concepts—*graph, path, cycle, complete graph,* etc.—we borrow the definitions and nomenclature from Bondy and Murty [4].

Two graphs G and H are said to be *isomorphic* if there is a bijection $\alpha: V(G) \to V(H)$, such that $\{u, v\} \in E(G)$ if and only if $\{\alpha(u), \alpha(v)\} \in E(H)$. The bijection α is called an *isomorphism* from G to H. An *automorphism* of a graph is an isomorphism from the graph to itself.

Additionally, we define an *open arc* as a bounded subset of the plane $\mathbb{R}^2$ homeomorphic to the real line $\mathbb{R}$ in the standard topology. A *drawing* of a graph G is a mapping ϕ of the vertices of G to points of the plane, and of the edges of G to open arcs—the *vertices* and *edges* of the drawing, respectively—such that (1) the vertices of the drawing are pairwise distinct, and disjoint from all its edges; (2) any two edges of the drawing are either disjoint, or cross at a single point; (3) for every edge $e = \{u, v\}$ of G, the external frontier of $\phi(e)$ is $\{\phi(u), \phi(v)\}$; and (4) no three edges of the drawing go through the same point.

We say that a graph is *planar* if it has a drawing without crossing edges.

We denote by K_n the complete graph on n vertices, and by $K_{m,n}$ the complete bipartite graph between m vertices and n vertices. In our proofs, we rely heavily on Kuratowski's theorem [21], which says that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph. In fact, we always prove that a graph is not planar by showing that it contains a subdivision of $K_{3,3}$, shown in figure 1.

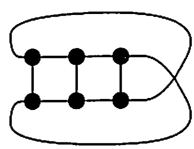


Figure 1

We also make use of the fact that a planar graph remains planar if an edge is deleted or contracted.

The *skewness* $sk(G)$ is the minimum number of edges that must be removed from G to produce a planar graph.

A *vertex splitting operation*, or *splitting* for short, consists in replacing some of the edges $\{u, v\}$ incident to a selected vertex u by edges $\{u', v\}$, where u' is a single new vertex. The (*vertex*) *splitting number* $sp(G)$ is the minimum number of splittings needed to turn G into a planar graph.

Note that, for any sequence of splittings, there is a sequence of the same length that produces the same graph, and is such that the vertex u affected by each splitting is always an original vertex of G, not one of the vertices introduced by previous steps.

For $n \geq 3$, we denote by C_n the chordless cycle with n vertices and n edges. The $n \times m$ *toroidal grid* $C_n \times C_m$ is the graph-theoretic product of C_n and C_m; that is, the graph with nm vertices $\{v_{ij} : 0 \leq i < n,\ 0 \leq j < m\}$, and $2nm$ edges $\{\{v_{ij}, v_{(i+1) \bmod n, j}\}, \{v_{ij}, v_{i,(j+1) \bmod m}\} : 0 \leq i < n,\ 0 \leq j < m\}$.

In our drawings of $C_n \times C_m$, vertex v_{ij} is represented by a point on the plane with coordinates (i, j). Based on this convention, we call the two families of edges above *horizontal* and *vertical*, respectively.

A cycle of $C_n \times C_m$ is called a *meridian* if it uses only vertical edges, and a *parallel* if it uses only horizontal ones. Thus the $n \times m$ toroidal grid has n meridians isomorphic to C_m, and m parallels isomorphic to C_n.

Let $\mathcal{F}$ be a family of isomorphic subgraphs of a graph G. We say that G is *$\mathcal{F}$-transitive* if for any two elements F and H of $\mathcal{F}$ there is an automorphism of G that takes F to H. Note that $C_n \times C_m$ is meridian-transitive, and parallel-transitive.

3 Previous results

The problems of verifying and computing the invariants *sk* and *sp* for general graphs have been shown to be respectively NP-complete [11, 14] and MAX SNP-hard [6, 10], even for cubic graphs. However, it can be checked in polynomial time whether the skewness *sk* is equal to a fixed k [13]. We have shown [6] that the same holds for the splitting number *sp*, by the results of Robertson and Seymour [26].

The difficulty in computing the invariants *sk* and *sp* for general graphs justifies their analysis for special families of graphs. Exact explicit formulas have been found for the splitting number of complete graphs and complete bipartite graphs [17, 19], and for the skewness of the n-cube Q_n [5].

For the toroidal grid $C_n \times C_m$, in particular, there are only a few partial results concerning these invariants. The upper bounds $sk(C_n \times C_m) \leq \min\{n, m\}$ and $sp(C_n \times C_m) \leq \min\{n, m\}$ are fairly obvious, too (see lemma 19).

The splitting number $sp(C_n \times C_m)$ was determined exactly by Schaffer in his 1981 thesis [28], but not published elsewhere. The special case of $C_4 \times C_4$, which is isomorphic to the 4-cube Q_4, was proved by Faria et al. [12]. In this article we give a new proof of Schaffer's result, and also an exact formula for the skewness $sk(C_n \times C_m)$.

There are many partial results about the *crossing number* $cr(G)$ (the minimum number of edge crossings in any drawing of G) for $G = C_n \times C_m$. Harary et al. [16] conjectured that $cr(C_n \times C_m) = (n-2)m$, for all n, m satisfying $3 \leq n \leq m$. This has been proved only for n, m satisfying $m \geq n$, and $n \leq 5$ [3, 7, 20, 24, 25], and for the special cases $n = m = 6$ [1], and $n = m = 7$ [2]. A recent result [27] based on the asymptotic behaviour of the minimum crossing numbers of wide classes of drawings for $C_n \times C_m$ also supports the conjecture. The general conjecture $cr(C_n \times C_m) = (n-2)m$ remains open for all but a finite number of values of n. It can be shown that $cr(G)$ is always an upper bound for $sk(G)$ and $sp(G)$ [12]. However, for $C_n \times C_m$ this bound is not tight, and so the results above cited are not directly useful for our problem.

4 Skewness versus splitting

The following general properties of skewness and splitting numbers are easily proved:

Lemma 1 *If H is a subgraph of G, then* $\mathrm{sp}(H) \leq \mathrm{sp}(G)$ *and* $\mathrm{sk}(H) \leq \mathrm{sk}(G)$.

Lemma 2 *If H is a subdivision of G, then* $\mathrm{sp}(H) = \mathrm{sp}(G)$ *and* $\mathrm{sk}(H) = \mathrm{sk}(G)$.

Lemma 3 *If a vertex v of a graph G has at most one neighbor, then* $\mathrm{sp}(G) = \mathrm{sp}(G - v)$.

Proof. Consider a minimum sequence of splittings that turns $G' = G - v$ into a planar graph H'. Since these splittings do not affect the edge $\{u, v\}$, if we apply the same splittings to G, then we will get a graph H equal to H' with the extra vertex v and extra edge $\{u, v\}$; which is obviously planar like H'. Thus $sp(G) \leq sp(G - v)$. The claim then follows by lemma 1. □

We also need the following inequality between the invariants:

Lemma 4 *For every graph G, we have* $\mathrm{sp}(G) \leq \mathrm{sk}(G)$.

Proof. We prove the lemma by induction on $sk(G)$. If $sk(G) = 0$, then G is planar and therefore $sp(G) = 0$. Otherwise, there is some edge $e = \{u, v\}$ such that $sk(G - e) = sk(G) - 1$. Now let H be the result of adding a vertex u' to G and replacing the edge e by $e' = \{u', v\}$, as shown in figure 2. This is a splitting step, so $sp(G) \leq sp(H) + 1$. By lemma 3, $sp(H) = sp(H - u') = sp(G - e)$. Since $sp(G - e) \leq sk(G - e)$ by the induction hypothesis, we conclude that $sp(G) \leq (sk(G) - 1) + 1 = sk(G)$. □

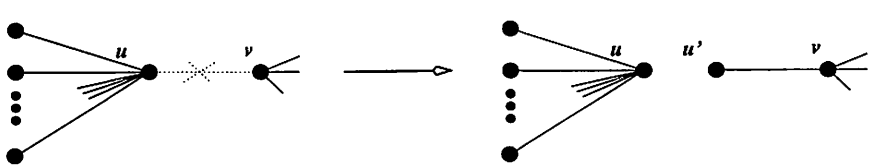


Figure 2

5 A lower bound for the splitting number

Lemma 5 *The splitting number of $C_3 \times C_3$ is* 1.

Proof. The graph $C_3 \times C_3$ has a subdivision of the $K_{3,3}$ as shown in figure 3(a), where the edges belonging to the subdivision of the $K_{3,3}$ are thicker and vertices are emphasized. It follows that $sp(C_3 \times C_3) \geq 1$. On the other hand, we can obtain a planar graph from $C_3 \times C_3$ with a single splitting as shown in figure 3(b) which implies that $sp(C_3 \times C_3) \leq 1$. Therefore, $sp(C_3 \times C_3) = 1$. □

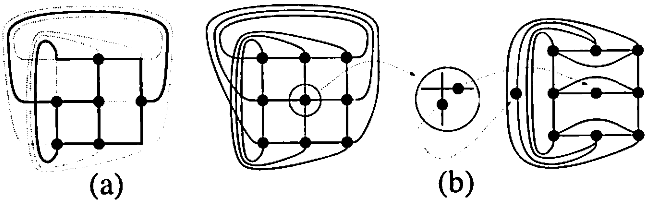


Figure 3

Lemma 6 *The splitting number of $C_3 \times C_4$ is at least* 2.

Proof. Let H be the graph obtained from $C_3 \times C_4$ by a single vertex splitting. Without loss of generality, we may assume that the split vertex is $v_{2,0}$ (indicated by $\times$ in figure 4). That splitting leaves untouched the subdivision of $K_{3,3}$ shown in figure 4. It follows that $sp(C_3 \times C_4) \geq 2$. □

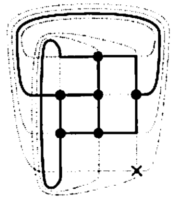


Figure 4

Lemma 7 *If G can be obtained from $C_3 \times C_5$ by two splittings on the same vertex u, then* $\mathrm{sp}(G) \geq 1$.

Proof. As in the proof of lemma 6, the two splittings in the vertex $v_{2,0}$ will not destroy the copy of $K_{3,3}$ shown in figure 5. Therefore G is not planar, and $sp(G) \geq 1$. □

Lemma 8 *If G can be obtained from $C_3 \times C_5$ by two splittings on distinct vertices of $C_3 \times C_5$, which belong to the same parallel or to adjacent parallels, then* $\mathrm{sp}(G) \geq 1$.

Proof. If the two vertices are on the same parallel (C_3), then without loss of generality we may assume that they are $v_{1,0}$ and $v_{2,0}$. In that case the copy of $K_{3,3}$ shown in figure 5 is not affected by the splittings. The same is true if u and v belong to consecutive parallels: we can always map them by an automorphism to two of the vertices marked $\times$ figure 5, which can be split without destroying the $K_{3,3}$. Therefore G is not planar, and $sp(G) \geq 1$. □

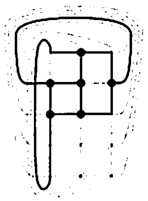


Figure 5

As shown in figure 6, there are at most seven different ways to split a vertex of $C_n \times C_m$ (assuming we do not care which of the two resulting vertices is the new one). We need this fact to prove the next two lemmas.

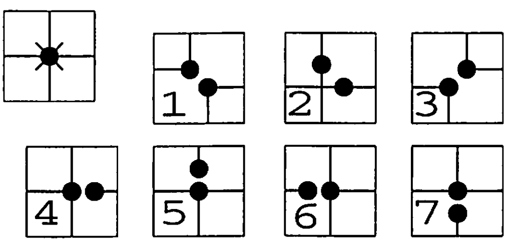


Figure 6

Lemma 9 *If G is obtained from $C_3 \times C_5$ by splitting two non-adjacent vertices on the same meridian of $C_3 \times C_5$, then* $\mathrm{sp}(G) \geq 1$.

Proof. Without loss of generality, we may assume that the two vertices are $v_{2,0}$ and $v_{2,2}$. Figure 7 shows all $7 \times 7 = 49$ possible ways to split these two vertices, grouped into five cases. In each case there is a subdivision of $K_{3,3}$, shown in figure 7, that is contained in $C_3 \times C_5$ and is not destroyed by the splits. Therefore G is not planar, and $sp(G) \geq 1$. □

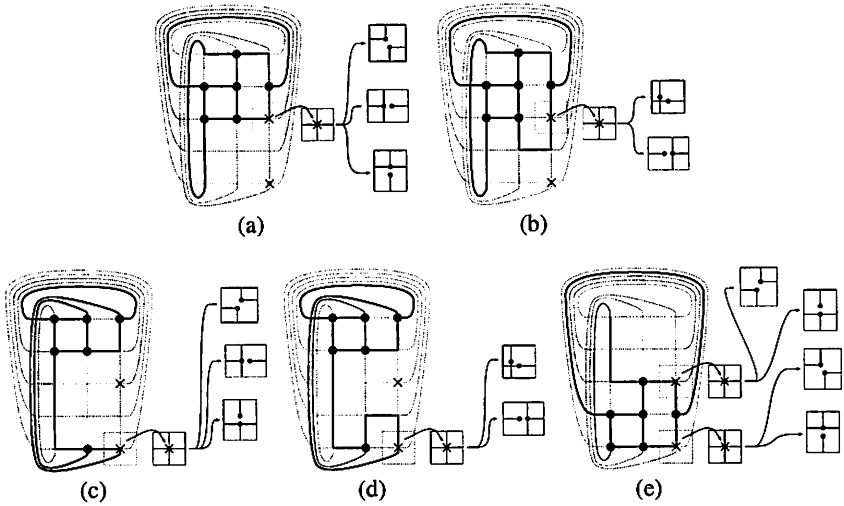


Figure 7

Lemma 10 *If G is the result of splitting two vertices of $C_3 \times C_5$ that lie at distance 3 from each other, then* $\mathrm{sp}(G) \geq 1$.

Proof. Without loss of generality, we may assume that one of the vertices is $v_{1,2}$. There are four vertices at distance 3 from $v_{1,2}$, namely $v_{0,0}$, $v_{2,0}$, $v_{0,4}$, and $v_{2,4}$. Without loss of generality, we may assume the other split vertex is $v_{2,0}$. Figure 8 shows all $7 \times 7 = 49$ possible ways to split these two vertices, grouped into five cases. In each case there is a subdivision of $K_{3,3}$ contained in $C_3 \times C_5$ that is not destroyed by the splittings. Therefore G is not planar, and $sp(G) \geq 1$. □

Lemma 11 *The splitting number of $C_3 \times C_5$ is at least* 3.

Proof. Consider a sequence of splittings that turns $C_3 \times C_5$ into a planar graph. We may assume that all splittings are applied to vertices of $C_3 \times C_5$. By lemma 6, the sequence has at least two steps; let u and v be the affected vertices, and d their distance in $C_3 \times C_5$. If $d = 0$, then $u = v$, and lemma 7 applies. If $d = 1$, then u and v lie on the same parallel or on adjacent parallels, and lemma 8 applies. If $d = 2$, then they either lie on adjacent parallels, or are non-adjacent vertices of the same meridian, and either lemma 8 or lemma 9 applies. Finally, if $d = 3$, then lemma 10 applies. Since there are no pairs of vertices with $d > 3$, we conclude that two splittings are not enough to turn $C_3 \times C_5$ into a planar graph. □

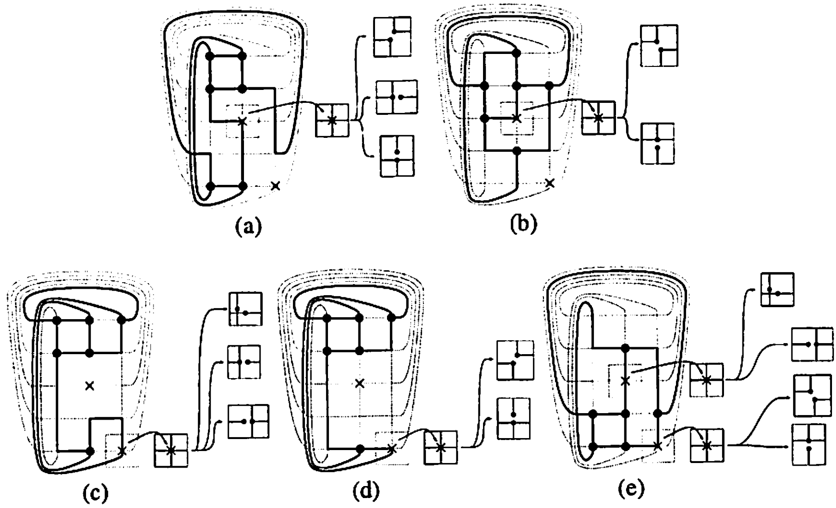


Figure 8

Lemma 12 *The splitting number of $C_3 \times C_m$, for $m \geq 5$, is at least* 3.

Proof. This result follows from lemmas 1, 2 and 11, since $C_3 \times C_m$ contains a subgraph that is isomorphic to a subdivision of $C_3 \times C_5$. □

Lemma 13 *The splitting number of $C_4 \times C_4$ is* 4.

Proof. The graph $C_4 \times C_4$ is isomorphic to the 4-cube Q_4; the result $sp(Q_4) = 4$ was proved by Faria, Figueiredo, and Mendonça [12]. □

Lemma 14 *The splitting number of $C_k \times C_k$, for $k \geq 4$, is at least k.*

Proof. We prove this assertion by induction on k. The induction basis is the case $k = 4$, proved by lemma 13.

Now let k be greater than 4, and let Z be any sequence of splittings that turns $G = C_k \times C_k$ into a planar graph H. We may assume that all splittings in Z are applied to vertices of G. Let v be one of the vertices split by Z, and let G' be the graph $G - v$. It is easy to see that the graph G' contains a subgraph that is isomorphic to a subdivision of $C_{k-1} \times C_{k-1}$; hence, by induction, $sp(G') \geq k - 1$. It follows that the sequence Z has at least $k - 1 + 1 = k$ steps. □

Lemma 15 *The splitting number of $C_n \times C_m$, for $n, m \geq 4$, is at least* $\min\{n, m\}$.

Proof. Without loss of generality suppose that $n \leq m$. The assertion follows from the fact that $C_n \times C_m$ contains a subgraph that is isomorphic to a subdivision of $C_n \times C_n$, which has splitting number at least n. □

Lemma 16 *The splitting number of* $C_n \times C_m$ *is at least* $\min\{n, m\} - 2\delta_{n,3}\delta_{m,3} - \delta_{n,4}\delta_{m,3} - \delta_{n,3}\delta_{m,4}$.

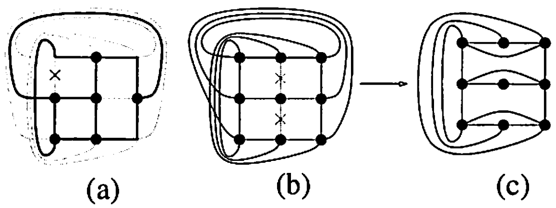
Proof. The assertion follows from lemmas 5–15. □

6 An upper bound for the skewness

Lemma 17 *The skewness of* $C_3 \times C_3$ *is* 2.

Proof. Let e be any edge of $C_3 \times C_3$; without loss of generality, we may assume that e is the vertical edge $\{v_{0,1}, v_{0,2}\}$, marked with $\times$ in figure 9(a). Deleting e from $C_3 \times C_3$ does not affect the subdivision of $K_{3,3}$ indicated in the figure; therefore $C_3 \times C_3 - e$ is not planar, and $sk(C_3 \times C_3) > 1$.

On the other hand, the removal of the two edges marked $\times$ in figure 9(b) results in a planar graph, as shown in figure 9(c). Therefore $sk(C_3 \times C_3) = 2$. □



(a) (b) (c)

Figure 9

Lemma 18 *The skewness of* $C_3 \times C_4$ *is at most* 2.

Proof. Figure 10 exhibits two edges of $C_3 \times C_4$ whose removal results in a planar graph. □

Lemma 19 *The skewness of* $C_n \times C_m$ *is at most* $\min\{n, m\}$.

Proof. Suppose without loss of generality that $n \leq m$. Figure 11 exhibits a set of n edges of $C_n \times C_m$ whose removal obtains a planar graph. □

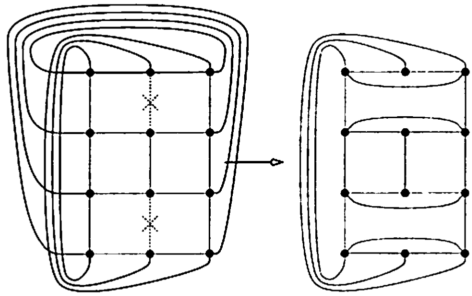


Figure 10

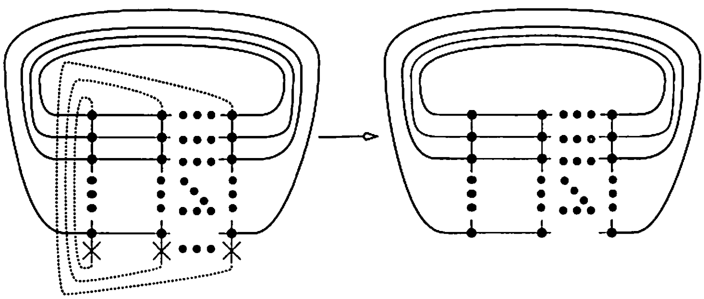


Figure 11

Theorem 1 *The splitting number and the skewness of* $C_n \times C_m$ *are:*

$$\mathrm{sp}(C_n \times C_m) = \min\{n,m\} - 2\delta_{3,n}\delta_{3,m} - \delta_{3,n}\delta_{4,m} - \delta_{4,n}\delta_{3,m} \quad (3)$$
$$\mathrm{sk}(C_n \times C_m) = \min\{n,m\} - \delta_{3,n}\delta_{3,m} - \delta_{3,n}\delta_{4,m} - \delta_{4,n}\delta_{3,m} \quad (4)$$

Proof. For all cases except $n = m = 3$, formulas (3) and (4) follow from the inequality $sp(G) \leq sk(G)$ (lemma 4) and from the fact that the lower bound for sp (lemma 16) equals the upper bound for sk (lemma 19).

For the case $n = m = 3$, the formulas are shown valid by lemmas 5 and 17. □

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