On Clique Graph Recognition

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Abstract

The clique operator K maps a graph G into its *clique graph*, which is the intersection graph of the (maximal) cliques of G. Recognizing clique graphs is a problem known to be in NP, but no polynomial time algorithm or proof of NP-completeness is known. In this note we prove that this recognition problem can be reduced to the case of graphs of diameter at most two.

Keywords: combinatorics, computational complexity, diameter.

1 Introduction

The *clique operator* K transforms a graph G into a graph $K(G)$ having as vertices all the cliques of G, with two cliques being adjacent when they intersect. The graph $K(G)$ is called the *clique graph* of G. (This and other definitions can be found in Section 2.) A characterization of clique graphs was given by Roberts and Spencer in 1971, based on certain families of complete sets, called RS families here [1]. This characterization can be used to show that recognizing clique graphs is in NP. The problem is not known to be NP-complete, and may be polynomial for all we know.

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In this note we prove that it suffices to have a recognition procedure for graphs of diameter at most 2. If such a procedure is available, the results below show how to recognize whether an arbitrary graph is in $K(\mathcal{G})$, with only a polynomial extra amount of effort.

2 Definitions

In this note all graphs are simple, i.e., without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. An edge with x and y as extremes is denoted by xy. We say that x and y are *incident* to edge xy.

A set C of vertices of G is *complete* when any two vertices of C are adjacent. A maximal complete subset of $V(G)$ is called a *clique*.

The *neighborhood* $N_G(x)$ of a vertex x of G is the set of all vertices adjacent to x. For a non empty set A of vertices of G we define the *graph induced by* A, denoted by $G[A]$, as the graph with vertex set A and whose edges are those of G that link two vertices of A. For a non empty set E of edges of G we define the *graph induced by* E, denoted by $G[E]$, as the graph whose vertices are those of G incident to at least one edge of E, and with edge set E. The notation $G - E$ will be used to denote $G[E(G) \setminus E]$

A *path* in a graph G is a sequence $x_0, x_1, ..., x_k$ of different vertices of G such that for every i, $i = 0, ..., k-1$, $x_i x_{i+1} \in E(G)$. A graph is *connected* when for every pair of vertices x, y there is a path containing x and y in G. The *distance* $d_G(x,y)$ between the vertices x and y in a graph G is the minimum number of edges on a path from x to y in G, or $+\infty$ if no path such exists. The *diameter* of a connected graph G is the maximum distance between vertices in G.

Let $\mathcal{F} = (F_i)_{i \in I}$ be a finite family of finite, nonempty sets. The sets F_i are the *members* of $\mathcal{F}$.

A family $\mathcal{F}$ of arbitrary sets satisfies the *Helly property*, or is *Helly*, when for every subfamily $J \subseteq \mathcal{F}$ such that any two members A, B of J intersect, we have $\bigcap_{A \in J} A \neq \emptyset$.

The *clique operator* K transforms a graph G into a graph $K(G)$ having as vertices all the cliques of G, with two cliques being adjacent when they intersect. The graph $K(G)$ is called the *clique graph* of G. Let $\mathcal{G}$ be the class of all graphs. Then $K(\mathcal{G})$ is the class of all clique graphs.

Roberts and Spencer [1] proved that a graph G is in $K(\mathcal{G})$ if and only if there is a family $\mathcal{F}$ of complete sets in G such that:

1. $\mathcal{F}$ covers all the edges of G (i.e., if $xy \in E(G)$, then $\{x, y\}$ is contained in some element of $\mathcal{F}$).

2. $\mathcal{F}$ satisfies the Helly property.

We call an *RS family* of G a family of complete sets in G that fulfills the hypothesis of the Roberts and Spencer characterization.

3 Metric results

Our main result in this note follows.

Theorem 1 *Let G be a graph and E a set of edges of G such that $\emptyset \neq E \neq E(G)$ and, for all $x, y \in V(G[E])$, we have*

$$d_{G-E}(x, y) > 2.$$

Then G is a clique graph if and only if both $G[E]$ and $G - E$ are clique graphs.

For the proof, we need the following lemma.

Lemma 1 *Let G and E be as in Theorem 1. Then for every complete set C of G, the induced subgraph $G[C]$ has all its edges in E or none of its edges in E.*

Proof: Suppose that $G[C]$ has one edge in E, say, edge xy. We will first show that $C \subseteq V(G[E])$. Notice that x and y are in $V(G[E])$. Any other vertex $z \in C$ must also be in $V(G[E])$. Indeed, one of the edges xz, yz must be in E, otherwise $d_{G-E}(x, y) = 2$, a contradiction.

Now we will show that all edges of $G[C]$ are in E. If ab is any edge of $G[C]$, we know that a and b are in $V(G[E])$. If $ab \notin E$, we have $d_{G-E}(a, b) = 1$, a contradiction. Therefore $ab \in E$ and the proof is completed. □

Proof: (of Theorem 1) Suppose that $G \in K(\mathcal{G})$. We will prove that both $G[E]$ and $G - E$ are in $K(\mathcal{G})$. Let $\mathcal{F}$ be an RS family of G. By Lemma 1, for every member A of $\mathcal{F}$, either (1) $G[A]$ has all its edges in E or (2) none of them in E. Construct $\mathcal{F}'$ with the members A of $\mathcal{F}$ that satisfy (1), and $\mathcal{F}''$ with the members A of $\mathcal{F}$ that satisfy (2). Since $\mathcal{F}$ covers all the edges of G, $\mathcal{F}'$ covers the edges of $G[E]$ and $\mathcal{F}''$ covers the edges of $G - E$.

On the other hand, families $\mathcal{F}'$ and $\mathcal{F}''$ are obviously Helly, because they are subfamilies of a Helly family. This shows that $\mathcal{F}'$ and $\mathcal{F}''$ are RS families for $G[E]$ and $G - E$, respectively, implying that these graphs are in $K(\mathcal{G})$.

Conversely, suppose $G[E]$ and $G - E$ are in $K(\mathcal{G})$. By this hypothesis, there are RS families $\mathcal{F}'$ and $\mathcal{F}''$ of $G[E]$ and $G - E$, respectively . Let $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$. We claim that $\mathcal{F}$ is an RS family of G. Obviously, the members of $\mathcal{F}$ are complete sets of G, and that $\mathcal{F}$ covers all edges of G. We must show it is Helly. Let $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_l$ be an intersecting subfamily of $\mathcal{F}$, where the A_i's come from $\mathcal{F}'$ and the B_j's come from $\mathcal{F}''$. If $k = 0$ or $l = 0$ the subfamily has a common intersection since $\mathcal{F}'$ and $\mathcal{F}''$ are Helly. If there are members from both families, let x be any vertex in $A_1 \cap B_1$. We claim that x belongs to all members of the subfamily.

In fact, let i be any index between 1 and k. Let $y \in A_i \cap B_1$. If $x \neq y$, then $d_{G-E}(x, y) = 1$, because $xy \in E(G[B_1])$ contradicting the hypothesis, since x and y are both in $V(G[E])$. Therefore $x = y$ and hence $x \in A_i$.

Now consider and index j between 1 and l. Let $y \in A_1 \cap B_j$. If $x \neq y$, take $z \in B_1 \cap B_j$. This vertex z cannot be equal to x or y, otherwise the edge xy would be in E and out of E simultaneously. Then z can be used to show that $d_{G-E}(x, y) = 2$, a contradiction, because x and y are in $V(G[E])$. Therefore $x = y$ and $x \in B_j$. We conclude that the subfamily has a common element x and therefore $\mathcal{F}$ is Helly, and is an RS family of G. □

As a special case, when E has just one edge, we have the following result.

Corollary 1 *Let G be a connected graph and x, y two vertices of G with $d_G(x, y) > 2$. Then $G \in K(\mathcal{G})$ if and only if $G + xy \in K(\mathcal{G})$.*

We write $G \lhd H$ when there is a pair of vertices x, y in G with $d_G(x, y) > 2$ and $H \cong G + xy$. Extend this relation to a symmetric relation by defining $G \sim H$ if and only if $G \lhd H$ or $H \lhd G$. Now extend this relation to an equivalence relation by defining $G \overset{*}{\sim} H$ if and only if there is a series $G_0, G_1, \ldots, G_k$ of graphs such that

$$G = G_0 \sim G_1 \sim \cdots \sim G_k = H.$$

The following result shows that the problem of recognizing clique graphs can be reduced to graphs of diameter at most 2.

Corollary 2 *Let G a connected graph. There is a graph H with diameter at most 2 such that $G \overset{*}{\sim} H$. Then $G \in K(\mathcal{G})$ if and only if $H \in K(\mathcal{G})$.*

Proof: If G has diameter at most 2 we take $H = G$. Otherwise, there are two vertices x and y with $d_G(x, y) > 2$. Let $G_1 = G + xy$. If G_1 still has vertices at distance greater than two, we continue with this process. Since G has a finite number of vertices, this process will stop with a graph G_k with diameter at most 2. By Corollary 1, $G \in K(\mathcal{G})$ if and only if $G_k \in K(\mathcal{G})$. So, G_k is the H sought. □

References

[1] F. S. Roberts and J. H. Spencer. A characterization of clique graphs. *J. Combin. Theory. Series B*. 10:102 108. 1971.