Packings with block size five and index one: $v \equiv 2 \pmod 4$

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Abstract

We use the results on 5-GDDs to obtain optimal packings with block size five and index one. In particular, we prove that if $v \equiv 2, 6, 10 \pmod{20}$, there exists an optimal packing with block size five on v points with at most 32 possible exceptions. Furthermore, if $v \equiv 14, 18 \pmod{20}$, there exists an optimal packing with block size five on v points with a finite (large) number of possible exceptions.

1 Introduction

A (v, k, λ) packing design (henceforth, packing) is a pair $(\mathcal{X}, \mathcal{B})$ where $\mathcal{X}$ is a v-set, $\mathcal{B}$ is a collection of some k-subsets (called blocks) of $\mathcal{X}$ such that every pair $\{x, y\} \subset \mathcal{X}$ is contained in at most λ blocks of $\mathcal{B}$. The packing number $D(v, k, \lambda)$ is defined to be the maximum number of blocks in a (v, k, λ) packing. A (v, k, λ) packing with $D(v, k, \lambda)$ blocks is called a maximum packing.

The function $D(v, k, 1)$ is important in coding theory because the block incidence vectors of a $(v, k, 1)$ packing form the codewords of a binary code of length v with minimum distance $2(k-1)$ and constant weight k. Thus, $D(v, k, 1)$ is the maximum number of codewords in such a code.

Schoenheim [9] has shown that

$$D(v, k, \lambda) \leq \lfloor \frac{v}{k} \lfloor \frac{\lambda(v-1)}{k-1} \rfloor \rfloor = B(v, k, \lambda)$$

Other upper bounds on the function $D(v, k, 1)$ have been given by Johnson [7] and Best et al. [3]. Lower bounds on the function $D(v, k, \lambda)$ are generally given by the construction of (v, k, λ) packings. The values of $D(v, 3, \lambda)$ for all v and λ have been determined by Schoenheim [9], and Hanani [6]. The values of $D(v, 4, 1)$ have been determined for all v by Brouwer [5]. In this paper, we discuss the function for $D(v, 5, 1)$ when $v \equiv 2 \pmod 4$.

We proceed with some definitions.

A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

(1) $\mathcal{G}$ is a partition of a set X (of *points*) into subsets called *groups*,

(2) $\mathcal{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,

(3) every pair of points from distinct groups occurs in a unique block.

The *group-type (type)* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We usually use an "exponential" notation to describe group-type. For example, the group-type $1^i 2^r 3^k \ldots$ denotes i occurrences of 1, r occurrences of 2, etc.

If K is a set of positive integers, each of which is no less than 2, then we say that a GDD $(X, \mathcal{G}, \mathcal{B})$ is a K-GDD if $|B| \in K$ for every block B in $\mathcal{B}$.

We also require Wilson's "Fundamental Construction" [11]. A brief description is presented below.

Lemma 1.1 *Let $(X, \mathcal{G}, \mathcal{B})$ be a GDD, and let $w : X \to Z^+ \cup \{0\}$ be a weight function on X. Suppose that for every block $B \in \mathcal{B}$ there exists a k-GDD of type $\{w(x) : x \in B\}$. Then there exists a k-GDD of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

A *transversal design* (TD), denoted TD(k, m), is a k-GDD of group type m^k. A GDD is a transversal design if and only if each block meets every group in exactly one point. It is well-known that a TD(k, n) is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS). For the existence of TDs, see [1].

A *parallel class* in a design is a set of blocks that partition the point set. If the blocks of a design can be partitioned into parallel classes, then they are said to be *resolvable*. Henceforth, we shall write RTD and RGDD with the appropriate parameters to denote resolvable TD and GDD, respectively. The existence of a resolvable TD(k, n) is equivalent to the existence of a TD$(k + 1, n)$. The following is well-known.

Lemma 1.2 *For every prime power q, there exists a RTD(q, q).*

The following construction is simple but useful.

Lemma 1.3 *Suppose that there exists a k-GDD of type* $\{s_i : 1 \le i \le r\}$. *Let* $a \ge 0$ *be an integer. If, for each* i *that satisfies* $1 \le i \le r-1$, *there exists a k-GDD of type* $\{s_{ij} : 1 \le j \le k(i)\} \cup \{a\}$ *where* $s_i = \sum_{1 \le j \le s(i)} s_{ij}$, *and there exist a k-GDD of type* $\{s_{rj} : 1 \le j \le k(r)\}$ *where* $s_r + a = \sum_{1 \le j \le s(i)} s_{rj}$, *then there is a k-GDD of type* $\{s_{rj} : 1 \le j \le k(i), 1 \le i \le r\}$.

2 $v \equiv 2, 6, 10 \pmod{20}$

We recall some known results on 5-GDDs. References can be found in [2].

Theorem 2.1 *1. There exists a 5-GDD of type* 4^n *for all* $n \equiv 0, 1 \pmod 5$.

2. There exists a 5-GDD of type $4^n 8^1$ *for all* $n \equiv 0, 2 \pmod 5$, $n \ge 7$ *except possibly when* $n = 10$.

3. There exists a 5-GDD of type $4^n 12^1$ *for all* $n \equiv 0 \pmod 5$ *and* $n \ge 10$.

4. There exists a 5-GDD of type $4^n 16^1$ *for all* $n \equiv 0, 3 \pmod 5$, $n \ge 13$ *except possibly when* $n \in \{15, 18, 30\}$.

5. There exists a 5-GDD of type $4^n 20^1$ *for all* $n \equiv 0, 1 \pmod 5$, $n \ge 16$.

6. There exists a 5-GDD of type $4^n 24^1$ *for all* $n \equiv 0, 4 \pmod 5$, $n \ge 19$.

We need the following results from [12].

Theorem 2.2 *There exists a 5-GDD of type* 2^n *for all* $n \equiv 1, 5 \pmod{10}$ *except when* $n = 11$ *and possibly when* $n \in \{15, 35, 71, 75, 85, 95, 111, 115, 135, 195, 215, 335\}$.

Lemma 2.3 *There exists a 5-GDD of type* 2^{85}.

Proof: Shen [10] has constructed a 4-RGDD of type 2^{16}. Extend the parallel classes to obtain a 5-GDD of type $2^{16}10^1$. Take a TD(5, 32) and apply Lemma 1.3, using a 5-GDD of type $2^{16}10^1$ for 4 groups and a 5-GDD of type 2^{21} for the last group. This yields a 5-GDD of type 2^{85} □

Theorem 2.4 *There exists a 5-GDD of type* $(20g)^n$ *for all* (g, n) *where* $n \ge 5$.

Theorem 2.5 *There exists a 5-GDD of type* $(4g)^n$ *for all* (g, n) *when* $n \equiv 0, 1 \pmod 5$.

The following lemma relates uniform GDDs to optimal packings.

Lemma 2.6 *If there exists a 5-GDD of type* 2^n, *then* $D(2n,5,1) = B(2n,5,1)$.

Proof: A simple computation reveals that the blocks of a 5-GDD of type 2^n form an optimal packing on $2n$ points. □

As a corollary, we obtain the following result.

Theorem 2.7 *If* $v \equiv 2, 10 \pmod{20}$, $v \neq 10, 22, 30, 70, 142, 150, 190, 222, 230, 270, 390, 430, 670$, *then* $D(v,5,1) = B(v,5,1)$.

In the remaining of the section, we focus on the case when $v \equiv 6 \pmod{20}$.

The following lemma can be obtained by a simple counting argument.

Lemma 2.8 *If there exists a 5-GDD of type* $2^n 6^1$, *then* $n \equiv 0 \pmod{10}$.

We need some direct constructions.

Lemma 2.9 *There exists a 5-GDD of type* $2^{40}6^1$.

Proof: Let $V = Z_{40} \times \{0,1\}$. The groups are $\{(i,j),(20+i,j)\}$ for $i = 0,1,2,\ldots,19$ and $j = 0,1$. The starter blocks are

$$\{(0,0),(2,0),(3,0),(2,1),(7,1)\},\ \{(0,1),(2,1),(3,1),(6,0),(27,0)\},$$
$$\{(0,0),(4,0),(12,0),(30,0),(23,1)\},\ \{(0,0),(16,0),(23,0),(29,0),(14,1)\},$$
$$\{(0,1),(4,1),(10,1),(28,1),(18,0)\},\ \{(0,1),(8,1),(15,1),(29,1),(20,0)\},$$
$$\{(0,0),(5,0),(6,1),(29,1)\},\ \{(0,0),(9,0),(30,1),(17,1)\},$$
$$\{(0,0),(15,0),(18,1),(27,1)\}.$$

The last three blocks of size four generate six parallel classes on V. Attach an infinite point to each of the six parallel classes, and the group of six infinite points gives a 5-GDD of type $2^{40}6^1$. □

Lemma 2.10 *There exists a 5-GDD of type* $2^{60}6^1$.

Proof: Let $V = Z_{60} \times \{0,1\}$. The groups are $\{(i,j),(30+i,j)\}$ for $i = 0,1,2,\ldots,29$ and $j = 0,1$. The starter blocks are

$$\{(0,0),(1,0),(3,0),(7,0),(12,0)\},\{(0,0),(8,0),(18,0),(31,0),(0,1)\},$$
$$\{(0,0),(14,0),(33,0),(1,1),(3,1)\},\{(0,0),(16,0),(36,0),(2,1),(7,1)\},$$
$$\{(0,0),(17,0),(5,1),(11,1),(14,1)\},\{(0,0),(22,0),(20,1),(35,1),(45,1)\},$$
$$\{(0,0),(26,0),(25,1),(41,1),(53,1)\},\{(0,0),(28,0),(4,1),(8,1),(44,1)\},$$
$$\{(0,0),(6,1),(24,1),(32,1),(43,1)\},\{(0,0),(12,1),(19,1),(33,1),(50,1)\},$$
$$\{(0,0),(15,0),(10,1),(37,1)\},\{(0,0),(21,0),(38,1),(39,1)\},$$
$$\{(0,0),(25,0),(21,1),(34,1)\}.$$

The last three blocks of size four generate six parallel classes on V. Attach an infinite point to each of the six parallel classes, and the group of six infinite points gives a 5-GDD of type $2^{60}6^1$. □

Lemma 2.11 *If there exists a 5-GDD of type 2^n and a TD$(5, 2n-1)$, then there exists a 5-GDD of type $2^{5(n-1)}6^1$.*

Proof: Take a TD$(5, 2n-1)$ which contains a block of size five, say $\{a, b, c, d, e\}$, and a new point x. For each group, we put a 5-GDD of type 2^n on the $2n-1$ points together with the new point x. We identify the point x and y, which are in the same group in the 5-GDD, and where y is one of the $\{a, b, c, d, e\}$. This gives a 5-GDD of type $2^{5(n-1)}6^1$. □

Lemma 2.12 *There exists a 5-GDD of type $2^{10n}6^1$ for $n = 4, 6, 10, 12, 15$.*

Proof: When $n = 4, 6$, a 5-GDD of type $2^{10n}6^1$ is constructed in Lemmas 2.9 and 2.10. When $n = 10, 12, 15$, a 5-GDD of type $2^{10n}6^1$ is constructed by Lemma 2.11, using a 5-GDD of type 2^q where $q = 21, 25, 31$, respectively. □

Lemma 2.13 *There exists a 5-GDD of type $2^{10n}6^1$ for $n = 26, 29, 31, 34, 38, 47, 49, 58, 69, 71, 79, 81, 83, 85, 87, 89, 91, 93, 95, 97, 99, 101, 105, 107$.*

Proof: Take a TD$(6, k)$ and truncate a group to m points to obtain a $\{5, 6\}$-GDD of type k^5m^1. Give weight four to the GDD to obtain a 5-GDD of type $(4k)^5(4m)^1$. If there exist a 5-GDD of type 2^{2k+1} and a 5-GDD of type $2^{2m-2}6^1$, then there exists a 5-GDD of type $2^{10k+2m-2}6^1$ by Lemma 1.3. The following table gives the applications of the above construction.

k	m	$10k + 2m - 2$
22	21	260
25	21	290
27	21	310
30	21	340
32	31	380
45	21	470
45	31	490
52	31	580
65	21	690
65	31	710
75	21	790
75	31	810
77	31	830
75	51	850
75	61	870
77	61	890
87	21	910
87	31	930
80	76	950
82	76	950
87	51	970
87	61	990
95	31	1010
95	51	1050
95	61	1070

□

Lemma 2.14 *There exists a 5-GDD of type* $2^{10n}6^1$ *for* $n = 55, 57, 61, 63, 67, 73, 75, 103$.

Proof: If d is a prime power of at least 15, then a 5-GDD of type $60^d(w + 4a+12b)^1$ exists, whenever there exists a 5-GDD of type $4^d w^1$ and $a+b \leq d-1$ (Construction 3.11 in [12]). If there exists a 5-GDD of type $2^{\frac{w}{2}+2a+6b-2}6^1$, Lemma 1.3 gives a 5-GDD of type $2^{30d+\frac{w}{2}+2a+6b-2}6^1$. Applications are as follows:

d	w	a	b	$w+4a+12b$	$30d+\frac{w}{2}+2a+6b-2$
17	8	1	6	84	550
17	8	2	9	124	570
19	4	2	6	84	610
19	4	0	10	124	630
19	24	0	15	204	670
23	16	2	5	84	730
23	16	0	9	124	750
31	0	0	17	204	1030

Lemma 2.15 *Let $Q = \{4, 6, 10, 12, 15\} \cup \{4r : r \geq 5\}$. If there exists a GDD on n points with group sizes in Q and block sizes at least five, then there exists a 5-GDD of type $2^{10n}6^1$.*

Proof: We first show that if $q \in Q$, then there exists a 5-GDD of type $2^{10q}6^1$. If $q = 4, 6, 10, 12, 15$, a 5-GDD of type 2^q is obtained in Lemma 2.12. Suppose $q = 4k$ and $k \geq 5$, a 5-GDD of type 80^k is constructed in [12]. A 5-GDD of type $2^{40k}6^1$ can be constructed by applying Lemma 1.3 to a 5-GDD of type 80^k using a 5-GDD of type $2^{40}6^1$.

Finally, if there exists a 5-GDD on n points with group sizes in Q, then we give weight 20 and apply Lemma 1.3 to obtain the resulting GDD. □

Lemma 2.16 *If n is even and $n \notin \{2, 8, 10, 14, 16, 18, 22\}$, then there exists a 5-GDD of type $2^{10n}6^1$.*

Proof: We take a TD$(q+1, q)$ and remove one block in the TD. This gives a GDD of type $(q-1)^{q+1}$ with block sizes q. We truncate the points in all but six groups to sizes either $0, 4, 6, 10, 12$ or a multiple of 4 at least 20. By Lemma 2.15, we can obtain a 5-GDD of type $2^{10n}6^1$ where n is the number of points in the GDD. The following table gives the application of the above construction. Note that n is always even in this construction.

q	n even
11	64-116
13	76-168
25	170-500

Finally, take a TD$(6, 4n)$ for $n \geq 21$ and truncate in one group to size g where $64 \leq g \leq 84$ and g is even. Give weight 20 and apply Lemma 1.3 to obtain a 5-GDD of type $2^{10(20n+g)}6^1$ for $n \geq 21$. This proves that there exists a 5-GDD of type $2^{10n}6^1$ for all $n \geq 64$ and $n \equiv 0 \pmod{2}$. When $n = 30, 42, 54$, a 5-GDD of type $2^{10n}6^1$ can be constructed by using a 5-GDD of type $120^{\frac{n}{6}}$ and Lemma 1.3. When $n = 46$, take a 7-GDD of type 6^8 which can be obtained from a TD$(7, 7)$. Delete two points in a group to obtain a $\{6,7\}$-GDD of type $6^7 4^1$ and apply Lemma 2.15. If $n = 50, 62$, a 5-GDD of type $2^{10n} 6^1$ can be constructed by Lemma 2.11. If $n = 26, 34, 38, 58$, a 5-GDD of type $2^{10n} 6^1$ is constructed in Lemma 2.13. □

Next, we deal with the case when n is odd.

Lemma 2.17 *If n is odd, then there exists a 5-GDD of type $2^{10n} 6^1$ for all $n \geq 109$.*

Proof: We take a TD$(q+1, q)$ and remove one block to obtain a q-GDD of type $(q-1)^{q+1}$. We truncate in all but seven groups to either $0, 4, 6, 10, 12, 15$ or a multiple of 4 at least 20. By Lemma 2.15, we obtain a 5-GDD of type $2^{10n} 6^1$ where n is the number of points in the GDD.

q	i	odd n
16	1	109-239
25	1	159-591
29	1	183-799
37	1	231-1311
41	1	255-1615
49	1	303-2319
53	1	327- 2719

Take a TD$(6, 4n+1)$ for $n \geq 40$. Truncate a group to at least 109 points and give weight 20. Applying Lemma 1.3 inductively yields the following result: If $k \geq 109$ and $k \equiv 1 \pmod 2$, then there exists a 5-GDD of type $2^{10k} 6^1$. □

Lemma 2.18 *If n is odd, then there exists a 5-GDD of type $2^{10n} 6^1$ for all n except possibly when $n \in \{1, 3, 5, 7, 9, 11, 13, 17, 19, 23, 33, 35, 39, 41, 43, 51, 53, 59\}$.*

Proof: If $n \geq 109$, then there exists a 5-GDD of type $2^{10n} 6^1$ by Lemma 2.17. If $n = 21$, a 5-GDD of type $60^7 4^1$ is constructed in [12] (Lemma 4.10). We add two infinite points, fill in the each group of size 60 with a 5-GDD of type 2^{31} to obtain a 5-GDD of type $2^{210} 6^1$. When $n = 25, 27, 45, 65, 77$, we obtain a 5-GDD of type $2^{10n} 6^1$ by Lemma 2.11 with a 5-GDD of type 2^q for $q = 51, 55, 91, 131, 155$, respectively. When $n = 29, 31, 47, 49, 69, 71, 79, 81, 83, 85, 87, 89, 91, 93, 95, 99, 101, 105, 107$, a 5-GDD of type $2^{10n} 6^1$ is constructed in Lemma 2.13. When $n = 37$, take a TD$(6, 11)$ and truncate four points in a group to obtain a $\{5,6\}$-GDD of type $11^5 7^1$. Use one of the truncated point to define a $\{5, 6, 11\}$-GDD of type $5^{11} 7^1$. Give weight 12 to obtain a 5-GDD of type $60^{11} 84^1$. Apply Lemma 1.3 with two infinite points to obtain a 5-GDD of type $2^{370} 6^1$. When $n = 55, 57, 61, 63, 67, 73, 75, 103$, a 5-GDD of type $2^{10n} 6^1$ is constructed in Lemma 2.14. □

Lemma 2.19 *If there exists a* $5-GDD$ *of type* $2^{10n}6^1$*, then* $D(20n+6,5,1) = B(20n+6,5,1)$.

Proof: In a group of size six, we add an additional block of size five. The result follows by a simple counting argument. □

As a corollary, we have

Theorem 2.20 $D(20n+6,5,1) = B(20n+6,5,1)$ *for all* n *but possibly* $n \in \{2,8,10,14,16,18,22\} \cup \{1,3,5,7,9,11,13,17,19,23,33,35,39,41,43,51,53,59\}$.

3 $v \equiv 14, 18 \pmod{20}$

In this section, we discuss the asymptotic behavior of $D(v,5,1)$ when $v \equiv 14, 18 \pmod{20}$.

Before we proceed, we need a result on $5-GDD$ of type $2^{10n}14^1$ and $2^{10n}18^1$.

Lemma 3.1 *If there exists a* $5-GDD$ *of type* 2^n *and a* $TD(6, \frac{n-1}{2})$*, then there exists a 5-GDD of type* $2^{5(n-1)}a^1$ *for* $a = 14, 18$.

Proof: Take a TD$(6, \frac{n-1}{2})$ and truncate the points in a group to three or four points. Give weight four and apply Wilson's Fundamental Construction to obtain a 5-GDD of type $(2(n-1))^5 12^1$ or $(2(n-1))^5 16^1$. Add two infinite points and fill in each of the group by a 5-GDD of type 2^n. □

The following lemma is a simple application of Lemma 3.1.

Lemma 3.2 *There exists a 5-GDD of type* $2^{10n}a^1$ *for* $n = 12, 15, 20, 22, 25$ *and* $a = 14, 18$.

Lemma 3.3 *There exists a 5-GDD of type* $2^{10n}a^1$ *for* $a = 14, 18$ *and* $n \geq 181$ *or* $n = 137$.

Proof: Take a TD$(26, 25)$ and truncate 21 groups to sizes $\{0, 12, 15, 20, 22, 25\}$. Give weight 20 and fill in each group with a 5-GDD of type $2^n a^1$ for some $n \in \{120, 150, 200, 220,$
$250\}$ and $a \in \{14, 18\}$ to obtain a 5-GDD of type $2^{10n}a^1$ for $a = 14, 18$ and $181 \leq n \leq 500$. It is not too hard to show that there exists a 5-GDD of type $2^{10n}a^1$ for all $a = 14, 18$ and $n \geq 181$, using a similar argument but with a larger TD and the method of induction. For $n = 137$, we use a TD$(6, 25)$, truncate the group to 12, and give weight 20. □

Lemma 3.4 $D(2574, 5, 1) = B(2574, 5, 1)$ *and* $D(2078, 5, 1) = B(2078, 5, 1)$.

Proof: In [10], a 4-RGDD of type 3^8 is known to exist. By completing all parallel classes, we obtain a 5-GDD of type $3^8 7^1$. Give weight 67 to obtain a 5-GDD of type $201^8 469^1$. Add a point at infinity and fill in each group by a 5-GDD of type 2^{101} or a 5-GDD of type 2^{235}. It is easy to verify that it is an optimal packing on 2078 points. Next, we give weight 83 to the 5-GDD of type $3^8 7^1$. It becomes a 5-GDD of type $249^8 581^1$. Add one point and fill in each of the group by a 5-GDD of type 2^{125} or a 5-GDD of type 5-GDD of type 2^{291}. This gives an optimal packing on 2574 points. □

Theorem 3.5 *$D(20n + 2574, 5, 1) = B(20n + 2574, 5, 1)$ and $D(20n + 2078, 5, 1) = B(20n + 2078, 5, 1)$ for all $n \geq 751$.*

Proof: Take a TD(138, 137), truncate one group to size 128 or 103, and 132 groups to sizes $\{0, 12, 15, 20, 22, 25, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130\}$. Give weight 20 and fill in each of the groups with 14 or 18 infinite points corresponding to the case when the one group has size 128 or 103, respectively. Fill in all other groups by a 5-GDD of type $2^n a^1$ for some n and $a = 14, 18$. (Such a 5-GDD is known to exist by Lemma 3.1). This gives a 5-GDD of type $2^{10n} b^1$ for $751 \leq n \leq 10000$ and $b \in \{2078, 2574\}$. Simple induction proves that there exists a 5-GDD of type $2^{10n} b^1$ for all $n \geq 751$. Fill in the group of size 2078 or 2574 with an optimal packing on the same number of points. The result then follows easily by simple computing. □

In this paper, we show that if $v \equiv 2, 6, 10 \pmod{20}$ then there exists an optimal packing on v points with block size five with few possible exceptions. Also, we have established an asymptotic existence of optimal packings in the case when $v \equiv 14, 18 \pmod{20}$. In this particular instance, however, much work remains to be done in order to reduce the number of possible exceptions. A main challenge is to construct an optimal packing with a small number of points. Our result for $v \equiv 14, 18 \pmod{20}$ can likely be improved easily by a more careful analysis of the existence of certain 5-GDDs. Yet, any substantial improvement is unlikely unless a smaller optimal packing can be obtained.

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