Structural domination of graphs *

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ABSTRACT. In a graph $G = (V, E)$, a set S of vertices (as well as the subgraph induced by S) is said to be *dominating* if every vertex in $V \setminus S$ has at least one neighbor in S. For a given class $\mathcal{D}$ of *connected graphs*, it is an interesting problem to characterize the class $Dom(\mathcal{D})$ of graphs G such that each connected induced subgraph of G contains a dominating subgraph belonging to $\mathcal{D}$. Here we determine $Dom(\mathcal{D})$ for $\mathcal{D} = \{P_1, P_2, P_3\}$, $\mathcal{D} = \{K_t \mid t \geq 1\} \cup \{P_3\}$, and $\mathcal{D} = \{$connected graphs on at most four vertices$\}$ (where P_t and K_t denote the path and the complete graph on t vertices, respectively). The third theorem solves a problem raised by Cozzens and Kelleher [*Discr. Math.* 86 (1990), 101-116] It turns out that, in each case, a concise characterization in terms of forbidden induced subgraphs can be given.

1 Introduction

Though domination is a relatively young subfield of graph theory, it already has an extensive literature. It is also impressive how many other areas are related to it. For a detailed accout on the subject, we refer to the recent book [17] and the earlier edited volume [21].

In this paper we investigate three classes of graphs with respect to the following general problem, first studied in [4] and [5]. (Formal definitions and notation will be listed in the next section.)

*Research supported in part by the Hungarian Scientific Research Fund, OTKA grant no. T-032969

Basic Problem. *Given a (finite or infinite) class $\mathcal{D}$ of connected graphs, characterize the class $Dom(\mathcal{D})$ of those graphs in which every connected induced subgraph contains a dominating induced subgraph isomorphic to some $D \in \mathcal{D}$.*

More specifically, for a given $\mathcal{D}$, we wish to find a characterization of $Dom(\mathcal{D})$ in terms of *forbidden induced subgraphs*. In this setting, an already classic result of Seinsche [20] on "trivially perfect" graphs (i.e., those having a dominating vertex in each connected subgraph) can be formulated as the following equation between graph classes:

$$Dom(\{K_1\}) = Forb(P_4, C_4). \qquad (1)$$

Cozzens and Kelleher [12] and, independently and simultaneously, the present authors [4] characterized the existence of dominating cliques as follows:

$$Dom(\{K_t \mid t \geq 1\}) = Forb(P_5, C_5). \qquad (2)$$

Actually, one of the main motivations for the work [4] was the paper [9] on an extremal problem, where dominating cliques of *maximum size* played an important role. As shown in [9], such a clique can be found in every non-triangle-free connected graph without induced matchings of two edges, a subclass of graphs without induced P_5.

More generally, the class $Diam_d$ of graphs of diameter at most d admits a concise description as well, as proved in [6]:

$$Dom(Diam_2) = Forb(P_6, C_6),$$

and, for $t \geq 7$,

$$Dom(Diam_{t-4}) = Forb(P_t).$$

There is also an intermediate problem between (1) and (2): When does a graph contain some *small* dominating clique? This question was answered both in [12] and [4]:

$$\text{For } s \geq 2, \ Dom(\{K_t \mid 1 \leq t \leq s\}) = Forb(P_5, C_5, F(K_{s+1})),$$

where $F(K_{s+1})$ is the graph of order $2s+2$ obtained from K_{s+1} by attaching a pendant vertex to each vertex of the clique.

An interesting new point of view appears in a recent work of Penrice [19]: the *clique covering number* ($\theta(D) = \chi(\overline{D})$) of the dominating subgraph. With our notation, the result can be written in short as follows:

$$\text{For } t \geq 2, \ Dom(\{D \text{ is connected, } P_4\text{-free}, \theta(D) \leq t-1\}) = Forb(P_6, H_t),$$

where H_t is the graph consisting of t paths of length 2 starting from the same vertex (i.e., the subdivision of the star $K_{1,t}$).

Another approach, initiated by Liu and Zhou [18], is to find characterizations in restricted (e.g., triangle-free) graph classes.

Some further similar problems are considered in [1], [15] and [2]. Those results deal with subgraphs from which every vertex of the graph in question is at most a given distance apart. In this more general setting, however, most of the theorems provide sufficient conditions only. A positive exception is the paper [2] where a characterization is given for a subproblem of "distance-2 domination." One general method to attack structural domination problems has been given in [4, Lemma 7]. In the terminology of the present work, it claims:

Reduction Lemma. *Let $\mathcal{D}$ be a class of connected graphs, and let G be a minimal non-$\mathcal{D}$-dominated graph. Then*

1. *either G has a cut-vertex,*

2. *or G has no star-cutset.*

The Reduction Lemma already has several applications (see [19], [5]), and we shall use it here as well.

As regards new methods, one main contribution of the present work is formulated in the Cutpoint Lemma in Section 3 (proved in Section 4). In some sense, it completely settles Case 1 of the Reduction Lemma, by describing the non-2-connected minimal graphs G.

Applying this powerful tool, we shall solve three problems, giving a characterization for $Dom(\mathcal{D})$ where $\mathcal{D} = \{P_1, P_2, P_3\}$, $\mathcal{D} = \{K_t \mid t \geq 1\} \cup \{P_3\}$, and where $\mathcal{D}$ consists of the connected graphs on at most four vertices. The latter was raised a decade ago in [12] as an open problem.

In addition to the Cutpoint Lemma, in the solution of the first problem we introduce a method which one may call "building non-dominating subgraphs." This means the following. We prove for an increasing collection of particular graphs that they cannot be dominating in the original graph; and, along this way, we often use the fact that some other graphs have already been proved to be non-dominating. This approach is useful also in the second and third problems, but they need some more complex arguments.

2 Definitions and notation

Throughout the definitions below, $\mathcal{D}$ means a nonempty class of *connected* graphs. Moreover, as usual, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of graph G, respectively.

We say that a graph is

minimal non-$\mathcal{D}$-dominated if it is connected, and has no dominating induced subgraph belonging to $\mathcal{D}$, but each of its proper connected induced subgraphs does have one

minimal not in $\mathcal{D}$ if it is not in $\mathcal{D}$, but all of its proper induced connected subgraphs belong to $\mathcal{D}$

hereditarily dominated by $\mathcal{D}$ if each of its connected induced subgraphs is dominated by some graph in $\mathcal{D}$; the class of graphs hereditarily dominated by $\mathcal{D}$ is denoted by $Dom(\mathcal{D})$

F**-free** if it does not contain F as an *induced* subgraph; the class of F-free graphs will be denoted by $Forb(F)$. Similarly, for a given family $\mathcal{F}$ of graphs, $Forb(\mathcal{F})$ denotes the class of those graphs which are F-free for all $F \in \mathcal{F}$.

Furthermore, we shall use the following nonstandard terminology:

to put a leaf on a given vertex u of a graph means that we insert a new vertex u' and join it just to u (see Figure 1)

private dominated vertex: given a dominating induced subgraph D of G, and a vertex u of D, we say that u has a *private dominated vertex* – sometimes also called a *private neighbor* – if there exists some $u' \in V(G) \backslash V(D)$ such that u is the unique neighbor of u' in $V(D)$. In other words, a leaf has been put on u in D. If we know that $D - u$ is not dominating in G, then, obviously, u must have a private dominated vertex.

leaf-graph of a graph T, denoted $F(T)$, is the graph obtained from T by putting a leaf on each of its *non-cutting* vertices. Formally, this means that $F(T)$ has the vertex set $V(T) \cup \{u' \mid T - u \text{ is connected}\}$, while the edge set of $F(T)$ is $E(T) \cup \{uu' \mid T - u \text{ is connected}\}$ (see Figure 2). Note that for any leaf-graph $F = F(T)$, the graph T is uniquely determined.

For a family $\mathcal{T}$ of graphs, we use the shorthand $F(\mathcal{T})$ to denote $\{F(T) \mid T \in \mathcal{T}\}$.

partial leaf-graph of a graph T is obtained from T by putting leaves on some of its vertices. Contrary to the case of leaf-graphs, in the present situation one may put leaves on cutvertices, too; and some vertices may get no leaf. (Even not putting any leaf is allowed.)

compact class of graphs: closed under connected induced subgraphs.

star-cutset: a vertex subset $S \subset V(G)$ such that $G - S$ is disconnected and there is an $s \in S$ adjacent to all vertices of $S \backslash \{s\}$.

(The notion of star-cutset was introduced in connection with perfect graphs, in [8].)

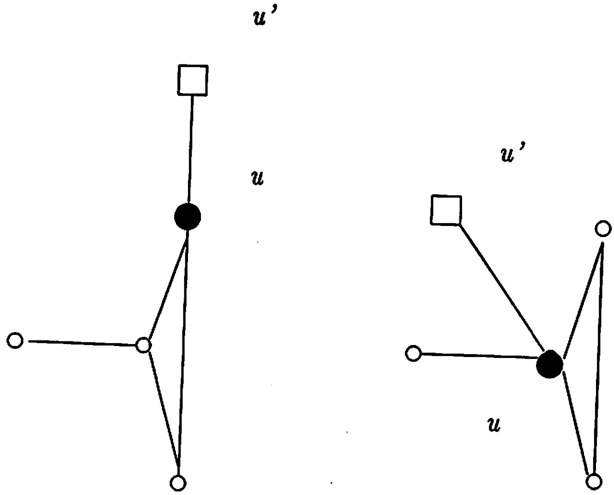
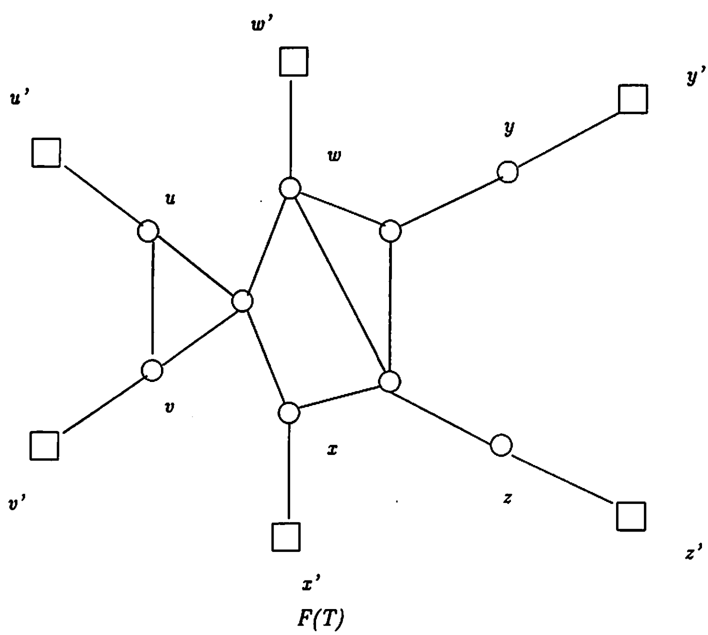


Figure 1



F(T)

Graph T: vertices denoted by circles.

Figure 2

Standard notation stands for particular types of graphs: K_n, P_n, and C_n denotes the complete graph, the path, and the cycle on n vertices, respectively. Moreover, some small graphs will be mentioned by name as follows. The *claw* is a star on four vertices (denoted by $K_{1,3}$). The *paw* is a graph on vertex set $\{a, b, c, d\}$ with edges ab, ac, ad, bc (hence, a partial leaf-graph of K_3). The *bull* is another partial leaf-graph of the triangle, with leaves on just two of its vertices. Finally, Y is a graph on the vertex set $\{a, b, c, d, e\}$, with edges ab, bc, cd, ce.

3 The results

We begin with an assertion that provides us with a general tool to handle dominating subgraphs of a given type.

Cutpoint Lemma *Let $\mathcal{D}$ be a compact class of connected graphs. A graph F with at least one cutpoint is minimal non-$\mathcal{D}$-dominated if and only if it is isomorphic to a leaf-graph $F(L)$ where L is a graph minimal not in $\mathcal{D}$.*

We shall apply this lemma for the following three collections of dominating subgraphs:

$$\mathcal{D}_1 = \{P_1, P_2, P_3\}$$
$$\mathcal{D}_2 = \{K_t \mid t \geq 1\} \cup \{P_3\}$$
$$\mathcal{D}_3 = \{\text{connected graphs on at most 4 vertices}\}$$

It is easily checked that the families $\mathcal{L}_i$ of minimal connected graphs not belonging to $\mathcal{D}_i$ $(i = 1, 2, 3)$ are

$$\mathcal{L}_1 = \{K_3, C_4, P_4, K_{1,3}\}$$
$$\mathcal{L}_2 = \{K_4 - e, PW, C_4, P_4, K_{1,3}\}$$
$$\mathcal{L}_3 = \{\text{connected graphs on 5 vertices}\}$$

According to the Cutpoint Lemma, a non-2-connected graph is a minimal forbidden induced subgraph for the class $Dom(\mathcal{D}_i)$ if and only if it has the form $F(L)$ for some $L \in \mathcal{L}_i$. Interestingly enough, it turns out that in either of the above three cases there exists precisely one further – 2-connected – minimal graph to exclude, namely a cycle of length 6 or 7. Note that cycles (of any length) are 2-connected and star-cutset-free, therefore they are natural candidates in characterizations for any $\mathcal{D}$.

Theorem 1. *A graph is hereditarily dominated by $\mathcal{D}_1$ if and only if, among its induced subgraphs, there is no C_6 and no $F(L)$ where $L \in \mathcal{L}_1$.*

That is,

$$Dom(\mathcal{D}_1) = Forb(F(\mathcal{L}_1) \cup \{C_6\})$$

Theorem 2. *A graph is hereditarily dominated hy $\mathcal{D}_2$ if and only if, among its induced subgraphs, there is no C_6 and no graph $F(L)$ with $L \in \mathcal{L}_2$.*

That is,

$$Dom(\mathcal{D}_2) = Forb(F(\mathcal{L}_2) \cup \{C_6\})$$

Theorem 3. *In a graph G, each connected subgraph has some dominating subgraph from $\mathcal{D}_3$ if and only if, among the induced subgraphs of G, there is no C_7 and no $F(L)$ for any $L \in \mathcal{L}_3$.*

That is,

$$Dom(\mathcal{D}_3) = Forb(F(\mathcal{L}_3) \cup \{C_7\}).$$

The third theorem solves a problem raised by Cozzens and Kelleher [12], while the second one strengthens the following result of Frdös and El-Zahar [14]:

Theorem A. *In every $2K_2$-free connected graph, there exists a dominating P_3 or a dominating clique.*

Using our notation, this result can be written as

$$Forb(2K_2) \subseteq Dom(\mathcal{D}_2).$$

In [4], a generalization of Theorem A was proved, namely

Theorem B. $Forb(P_5) \subseteq Dom(\mathcal{D}_2)$

Along these lines, Theorem 2 above makes the description of $Dom(\mathcal{D}_2)$ complete.

4 Cutpoint Lemma

In this section we prove the Cutpoint Lemma:

> Let $\mathcal{D}$ be a compact class of connected graphs. A graph F with at least one cutpoint is minimal non-$\mathcal{D}$-dominated if and only if it is isomorphic to some $F(L)$ where L is a graph minimal not in $\mathcal{D}$.

First we prove the "only if" part.

Claim 1. *If F is a minimal non-$\mathcal{D}$-dominated graph, c is a cutpoint of F, d is a vertex of degree at least 2, and $cd \in E(F)$, then d is also a cutpoint.*

Proof: If the assertion does not hold true, then $H = F - d$ is connected and, by the minimality of F, it has some dominating induced subgraph $D \in \mathcal{D}$. Furthermore, $c \notin D$ because D cannot dominate the entire graph F. Since D is connected, it is the subgraph of some component K in the graph $F - c$. Let K' be the component containing d. If $K = K'$, then some other component remains undominated; and if $K \neq K'$, then the further points of K' remain undominated. (Such points exist, because of the degree condition on d.) This contradiction completes the proof of the claim. □

It is a well-known fact that any connected graph has some non-cutting point.

(One can choose any leaf of any spanning tree.) This implies:

> Both the set C of cutpoints and the set N of non-cutting points are nonempty in F.

Let us consider all those vertices which are not in C but have some neighbor in C. By Claim 1, they are leaves; we denote their (nonempty) set by S. Let $U = C \cup S$. There is no vertex outside U, for otherwise we would also have such a vertex adjacent to the set U, and then surely to some point $s \in S$, too. But s is a pendant vertex and we have obtained a contradiction.

So, giving the name L to the graph induced by the set C, it can be easily seen that F is isomorphic to the graph $F(L)$.

Of course, L is not in $\mathcal{D}$. To prove that it is minimal with respect to this property, we first state a simple but interesting lemma without proof:

Lemma 1. *If a connected graph Y is the subgraph of a connected graph X, and Y contains every non-cutting point of X, then $V(Y) = V(X)$.* □

Now, concerning the minimality of L, suppose for a contradiction that L has a proper induced connected subgraph P which is not in $\mathcal{D}$. By Lemma 1, some non-cutting vertex l of L does not belong to P. Consequently, $V(P) \subseteq V(L) - \{l\}$ and, by the compactness of $\mathcal{D}$, $H := L - l \notin \mathcal{D}$.

Furthermore, denoting by l' the leaf put on l, $F - l'$ has some dominating subgraph $D \in \mathcal{D}$, because of the minimality of F. Clearly, we may assume $V(D) \subseteq V(L)$.

From the definition of l, H is connected. We shall prove that D contains all the non-cutting points of H. By Lemma 1, this will imply $V(D) \supseteq V(H)$, and the proof will be done since, by compactness, the contradiction $H \in \mathcal{D}$ follows.

Let v be any non-cutting point in $H - D$. If it is also non-cutting in L, we are done since D, a dominating subgraph of $F - l'$ has to contain every vertex which is different from l and has some leaf. Otherwise, let v be a cutpoint in L. Since v is not in D and $V(D) \subseteq V(L)$, D is the part of some component of $L - v$. But this is impossible because it is dominating in $F - l'$ and consequently in L as well.

So we have deduced a contradiction from the assumption that some non-cutting point of H exists which is not in D. As we have seen above, this completes the proof for the minimality of L, and the whole "only if" part is done.

The "if" part can be deduced from Lemma 1 more easily.

Let us consider a connected graph L, minimal not in $\mathcal{D}$. We have to show that $F = F(L)$ is a minimal non-$\mathcal{D}$-dominated graph.

A graph D, dominating $F(L)$, contains all the non-cutting vertices of L. Thus, by Lemma 1, it contains $L \notin \mathcal{D}$. The compactness of $\mathcal{D}$ then implies $D \notin \mathcal{D}$.

Now, we prove the minimality of $F(L)$ among the non-$\mathcal{D}$-dominated graphs. Let R be any proper induced connected subgraph of $F(L)$. By Lemma 1, necessarily we have some vertex $l \in L$ with its leaf l' missing from R. The graph $R - l$ is connected. Omitting all the leaves from $R - l$, we obtain a connected proper subgraph R' of L which is in $\mathcal{D}$, because of the minimality of L. Furthermore, R' is dominating in R. Hence, we have shown that R is a $\mathcal{D}$-dominated graph, and so the minimality of $F(L)$ follows. This completes the proof of the Cutpoint Lemma. □

Remark. Without the assumption of compactness for $\mathcal{D}$, neither direction of the Cutpoint Lemma is valid.

In both of the next examples, we take $\mathcal{D} = \{P_1, P_3\}$.

Example 1. For $M = P_6$,.M is a minimal non-$\mathcal{D}$-dominated graph, but $M = F(P_4)$ and P_4 is not minimal in the sense that $P_2 \notin \mathcal{D}$ is its proper induced connected subgraph. Consequently, the "only if" part of the stronger version of the Cutpoint Lemma is not true.

Example 2. For $L = P_2$, $P_4 = F(L)$ is dominated by $\mathcal{D}$, although L is a graph minimal not in $\mathcal{D}$. So the "if" part of the stronger version is also false.

On the other hand, the following statement is valid:

> Let $\mathcal{D}$ be any class of graphs such that all $D \in \mathcal{D}$ are connected. If F is a minimal non-$\mathcal{D}$-dominated graph and has at least one cutpoint, then $F = F(L)$ for some $L \notin \mathcal{D}$.

The proof is the same as that for the first part of the Cutpoint Lemma.

5 Short dominating paths

In this section we prove Theorem 1, which states:

> A graph is hereditarily dominated by $\mathcal{D}_1 = \{P_1, P_2, P_3\}$ if and only if, among its induced subgraphs, there is no C_6 and no $F(L)$ where $L \in \mathcal{L}_1$.

(As described in the Introduction, $\mathcal{L}_1$ consists of the four graphs K_3, C_4, P_4, $K_{1,3}$.)

The "only if" part is immediately seen as neither C_6 nor any $F(L)$ ($L \in F(\mathcal{L}_1)$) contains a dominating induced path of length at most two.

For the "if" part, let us consider a minimal non-$\mathcal{D}_1$-dominated graph G.

If there is some cutpoint in G then, by the Cutpoint Lemma, $G = F(L)$ for some $L \in \mathcal{L}_1$, and the proof is done.

Otherwise, taking an arbitrary vertex x, the graph $H := G - x$ is connected. Obviously, by the minimality of G, it is enough to prove the following

Lemma 2. *Suppose that G is a non-$\mathcal{D}_1$-dominated star-cutset-free graph containing no induced $F(C_4)$, $F(P_4)$, $F(K_{1,3})$, and C_6. Omitting any vertex x from G, the remaining graph H is also non-$\mathcal{D}_1$-dominated.*

Remark. Let us emphasize that $F(K_3)$-freeness is not assumed in the lemma.

Proof of Lemma 2: Assume, on the contrary, that H has some dominating induced subgraph $D \in \mathcal{D}$.

First, suppose D is a cherry abc (a P_3, in other notation). Since G is connected, x has some neighbor y in H, and y has some neighbor in the dominating subgraph D of H. If y has only one neighbor in D and this is a or c, then G has a dominating P_4. Next, we prove a statement which exludes this possibility and will be applicable later, too.

Claim 2. *There is no dominating P_4 (in G).*

Proof: Omitting any one of the two endpoints u, v of a dominating P_4 if it exists, the remaining graph – namely a cherry – cannot be dominating. Consequently, the endpoint considered has a private dominated vertex. So we put leaves u' and v' on the endpoints and we get a C_6 or a P_6, according as $u'v'$ is or is not an edge. Since C_6 and P_6 $(= F(P_4))$ are forbidden, we have got a contradiction proving the claim. □

Now, suppose y has two neighbors in D, and these are a and c. This means that the C_4, induced by $\{a, b, c, y\}$, is dominating. We need some observations, in order to exclude this case.

Claim 3. *There is no dominating P_5.*

Proof: By Claim 2, omitting one of the endpoints of such a P_5, say u, the remaining P_4 is not dominating. Thus, u has a private dominated vertex, and we get a P_6 which is forbidden. □

Claim 4. *The graph in Figure 7/L5 cannot foe dominating.*

Proof: By Claim 3, omitting u, the remaining P_5 iS not dominating and u has a private dominated vertex. We have obtained an $F(K_{1,3})$ which is forbidden. □

Claim 5. *There is no dominating Y.*

Proof: (See Figure 7/L4) The graph remaining after the deletion of u is a P_4. Consequently, by Claim 2, it is not dominating and u has a private dominated vertex. But in this way we obtain a dominating subgraph shown in Figure 7/L5, contradicting Claim 4. □

Now we are in a position to prove

Claim 6. *There is no dominating C_4.*

Proof: Let us suppose for a contradiction that the $C_4 = abcy$, as constructed above, is dominating. Due to the preceding observations, several vertices will have private neighbors, as follows. There is some leaf on vertex y, that we may denote by x without ambiguity. Then, there is a leaf on c because of Claim 2, one on b because of Claim 5, and finally one on a because of Claim 4. Hence, we have found an $F(C_4)$ which is forbidden. This contradiction proves Claim 6. □

Thus, we have proved that whenever there exists some vertex y adjacent to x but not adjacent to the midpoint b of the cherry, then the proof is done. So, it can be easily seen that we shall be able to achieve the case $D = P_3$ if we prove

Lemma 3. *Let x, b be two vertices at distance 2 in the star-cutset-free graph G. If every neighbor of x is the neighbor of b, then all vertices different from x and b are adjacent to both x and b.*

Proof: If x has a non-neighbor different from b, then we get the contradiction that b and its neighbors adjacent to x form a star-cutset. □

One may note that the situation described in Lemma 3 can occur, e.g. in a C_4.

We are now very near to proving Lemma 2. By assumption, G is star-cutset-free. Furthermore, there are at least three vertices nonadjacent to x in G. Consequently, by Lemma 3, the case $D = P_3$ is settled.

Next, let D be a P_2 with vertices a, b. We may assume that there is no dominating induced P_3 in H, i.e., every vertex in H is adjacent to both a and b. Then the two-element set $\{a, y\}$ – inducing a P_2 – is dominating in the whole graph G, which contradicts the definition of this graph. Thus, the case of P_2 is done.

Finally, the case of P_1 can be completed in the same way, finishing the proof of Lemma 2. □

As we have already seen, Lemma 2 makes the proof of Theorem 1 complete. □

6 Dominating cliques and cherries

In this section we prove Theorem 2:

> A graph is hereditarily dominated by $\mathcal{D}_2 = \{K_t \mid t \geq 1\} \cup \{P_3\}$ if and only if among its induced subgraphs there is no C_6

and no graph $F(L)$ with $L \in \mathcal{L}_2$

(All members of $\mathcal{L}_2$ have four vertices; it consists of $K_4 - e$, C_4, P_4, $K_{1,3}$, and the paw.)

The "only if" part is immediately seen as neither C_6 nor any $F(L)$ ($L \in F(\mathcal{L}_2)$) contains a dominating subgraph from the given class.

Now we prove the "if" part.

We shall apply some parts from the proof of Theorem 1, too.

Let G be a minimal non-$\mathcal{D}_2$-dominated graph which does not contain the given subgraphs, and x some vertex of G. We know that G has no cutpoint; what is more, it is star-cutset-free. Then the graph $H := G - x$ is connected. Also, x has some neighbor y in H, and y has some neighbor in a dominating induced subgraph $D \in \mathcal{D}$ of H.

Case I D is a P_1 or P_2 or P_3.

In this case, the proof is done by Lemma 2.

So, we may assume

Case II D is a clique of size at least 3.

We are going to formulate two assertions in incerasing strength, both implying the impossibility of Case II. For this purpose, we need some preliminaries:

Notation. For $0 < i < n$, let $G_{n,i}$ denote the graph obtained from the complete graph K_{n+1} by deleting i edges incident to the same vertex y. The vertices nonadjacent to y will usually be called "non-neighbors." We also put $G_{n,0} := K_n$.

Proposition 1. *There is no dominating $G_{n,i}$ in G.*

It will be easier to prove

Proposition 2. *Let $n \geq 2$, D be a partial leaf-graph of $G_{n,i}$ such that $D = G_{n,i}$ or y has a leaf, and if there is any leaf on any neighbor of y then all the non-neighbors have leaves. Such a D cannot be dominating in G.*

Definition. In the situation described in Proposition 2, $G_{n,i}$ is called the *base* of D.

Proof of Proposition 2: We apply induction on n. For $n = 2$, D can be one of the graphs exhibited in Figure 3/a-d (or $D = K_2$, settled in Case I). For these graphs we have already proved that they cannot be dominating in G (see the proof of Lemma 2). Now we prove that the validity of the proposition for $2, \ldots, n-1$ implies its validity for n.

Let us pick some dominating graph D satisfying the conditions. Suppose D has base $G_{n,i}$, $n \geq 3$.

Claim 7. *We have $i = 1$.*

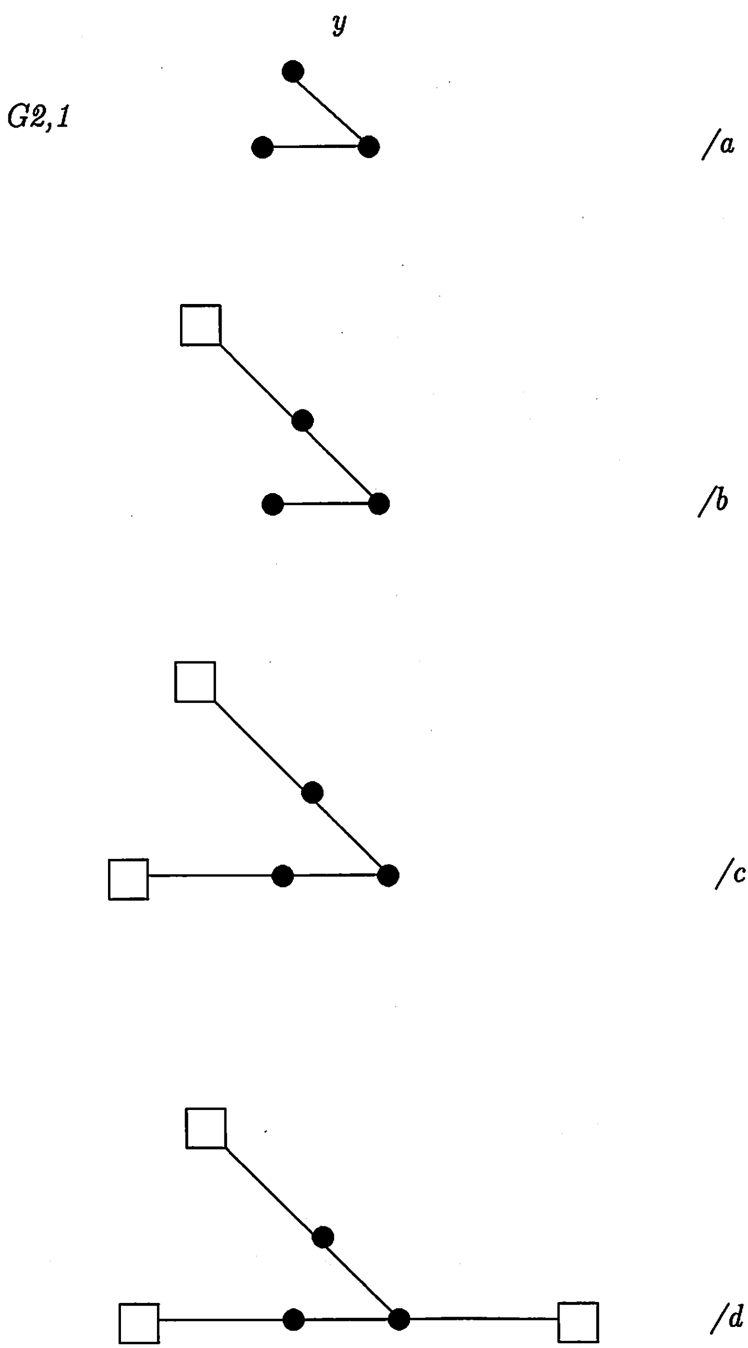


Figure 3

Proof: By the definition of G, Proposition 2 is valid for $i = 0$. Let us suppose, for a contradiction, that $i \geq 2$. We are able to put leaves on two of the non-neighbors s, t because, omitting one of them, say s, by the induction hypothesis of Proposition 2 the remaining graph is not dominating, therefore we can put a leaf on s; and the second leaf can be put similarly. But then we obtain an $F(PW)$ as an induced subgraph since y has at least one neighbor in $G_{n,i}$. This is a contradiction proving Claim 7.

Claim 8. *We have* $n - i = 1$.

Proof: Since $i \leq n-1$, it is enough to prove $n - i \leq 1$. Let us suppose, on the contrary, the existence of at least two neighbors. In this case, we are able to put a leaf on y (if it does not have one yet) because $D - y$ is a clique and cannot be dominating. By Claim 7, there is exactly one non-neighbor. Due to the induction hypothesis of Proposition 2, we may put a leaf on it. Let us pick now some neighbor and delete it. Using the induction hypothesis again, the remaining graph cannot be dominating. (Observe that this is a graph investigated in Proposition 2 because we have supposed $n - i \geq 2$).

So we may put a leaf also on this neighbor. Similarly, we may put a leaf on some other neighbor. But in this way we have constructed an $F(K_4 - e)$ which is forbidden. This contradiction proves Claim 8. □

We have obtained $i = 1$ and $n - i = 1$, thus $n = 2$, despite we have assumed $n \geq 3$. Hence, Proposition 2 is proved, and so is Theorem 2 as well. □

7 Connected dominating subgraphs of order 4

In this section we prove Theorem 3:

> In a connected graph G, each connected subgraph has some connected dominating subgraph with at most 4 vertices if and only if, among the indiced subgraphs of G, there is no C_7 and no $F(L)$ for any connected 5-vertex graph L.

First we introduce a notion:

Definition. A *truncated leaf-graph* of a graph T is a graph obtained from $F(T)$ by deleting some of the leaves. Deleting no leaf or delete all of them is also allowed.

The "only if" part of Theorem 3 can be obtained essentially from Lemma 1. The "if" part will be proved by first reducing it to stronger and stronger assertions in several steps and then proving the last version applying the method of "building non-dominating subgraphs."

Suppose, for a contradiction, that G is a minimal non-$\mathcal{D}_3$-dominated graph. The notation for the beginning of the proof will be the same as that in the proofs of Theorems 1 and 2, except that the dominating subgraph found in H will be denoted by T. This T is some connected four-vertex graph. Obviously, the connected five-vertex graph $G|_{V(T)\cup\{y\}} = D$, induced by the vertex subset $V(T) \cup \{y\}$, is dominating in G. Consequently, it is enough to prove that no connected five-vertex dominating graph exists in G. We shall prove more: no truncated leaf-graph of any connected five-vertex graph can be dominating in G.

First, let us consider the following statement:

Lemma A. *No partial leaf-graph of any connected four-vertex graph can be dominating in G.*

Now we prove that Theorem 3 can be reduced to Lemma-A. We show this by induction on the number of leaves in decreasing order; namely, if there does not exist any dominating truncated leaf-graph of a connected five-vertex graph, with a given number of leaves, then there does not exist any with one fewer leaf. The maximal truncated leaf-graph $F(D)$ of D, of course, cannot be dominating in G because it is excluded as an induced subgraph. Consequently, we can begin the induction.

Let us consider some truncated leaf-graph and some missing leaf in it, namely some non-cutting vertex u where we have not put any leaf. By assumption, $T = D - u$ is connected and, by Lemma A, the given partial leaf-graph of T cannot be dominating. Thus, u has a private dominated vertex u'; i.e., the leaf u' can be put on u. Thus, using Lemma A, we have obtained that a further leaf can be attached to the original (dominating) truncated leaf-graph. This proves that Theorem 3 can be reduced to Lemma A.

Next, we need a definition.

Definition. A *partial 2-leaf-graph* of a graph T means a graph L which is constructed from T by putting one "leaf of length 2" – that is, a pendant $P_3 = uu'u''$ where u' is a leaf on u and u'' is one on u' – and some leaves on some other vertices; the latter ones are not obligatory. We may put leaves on cutpoints, too! (See Figure 4.)

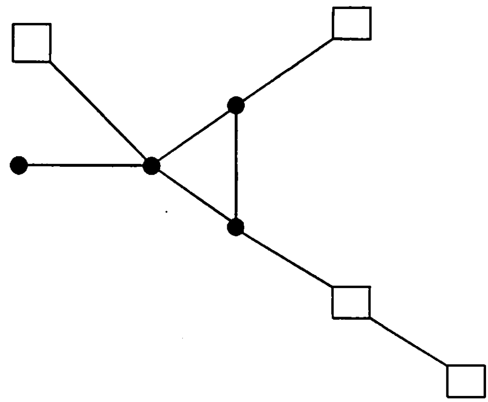


Figure 4

Let us consider next the following statement:

Lemma B. *No dominating induced subgraph can be a partial 2-leaf-graph of a connected four-vertex graph.*

We prove that Lemma B implies Lemma A. We apply induction on the number of leaves, in increasing order. We can begin the induction because the graph of Lemma A without leaves is a four-vertex graph that cannot be dominating in G.

Let us take a partial leaf-graph of a connected four-vertex graph T, and some leaf u' of it hanging on the vertex u. Omitting u', we obtain a partial leaf-graph with a smaller number of leaves; it is not dominating in G, by the induction hypothesis. Thus, u' has a private dominated vertex and we obtain a dominating partial 2-leaf-graph, contradicting Lemma B. The reduction of Lemma A to Lemma B is done.

Finally, we consider the following assertion:

Lemma C. *Let G be a 2-connected, minimal non-$\mathcal{D}_3$-dominated graph, and L a partial 2-leaf-graph of a connected three-vertex graph. Then L cannot be dominating in G.*

We prove that Lemma C implies Lemma B. Suppose that the latter is false. If the partial 2-leaf-graph L of the connected four-vertex graph T has a leaf on every noncutting vertex, then the proof is done because L contains an $F(D)$ where D is the five-vertex graph obtained from T by adding the middle vertex of the 2-leaf. (See Figure 5.) Otherwise we have some noncutting vertex u of T, such that no leaf hangs on it. Omitting u, we get a partial 2-leaf-graph of some connected three-vertex graph, contradicting Lemma C. Hence, Lemma B is reduced to Lemma C.

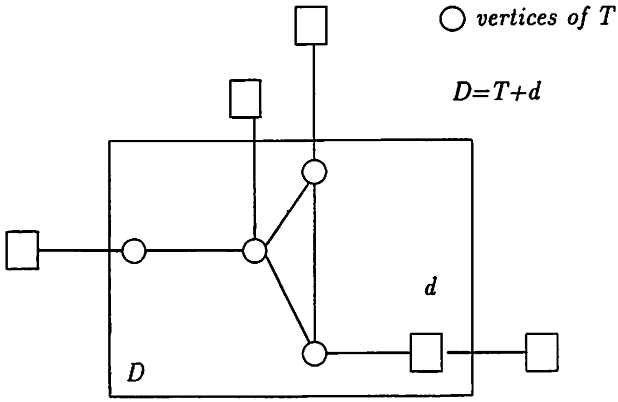


Figure 5

The rest of the paper is devoted to the

Proof of Lemma C: There are two possibilities for a connected three-vertex graph, namely the triangle and the cherry. All their partial 2-leaf-graphs are exhibited in Figures 6, 7, 8/L1-L10. We denote those graphs by $L_1, \ldots, L_{10}$. We will prove, for all $1 \leq i \leq 10$ (though not in increasing order of subscripts), that L_i cannot be dominating in G.

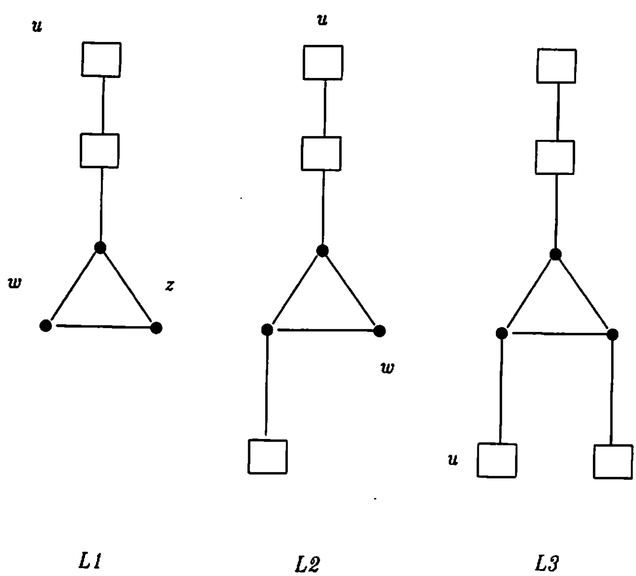


Figure 6

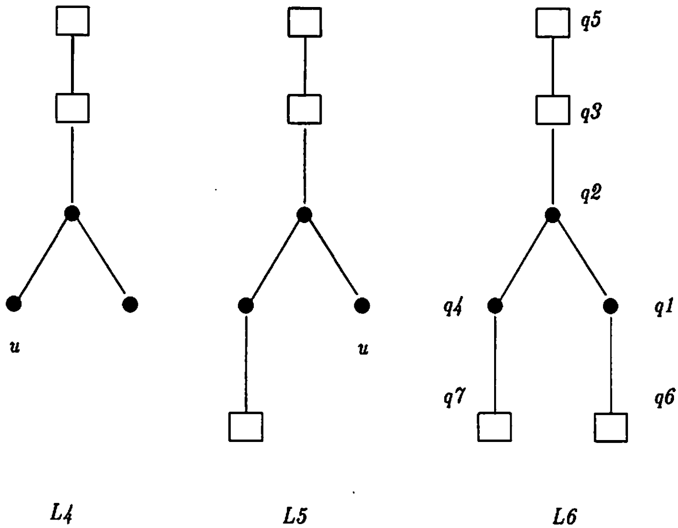


Figure 7

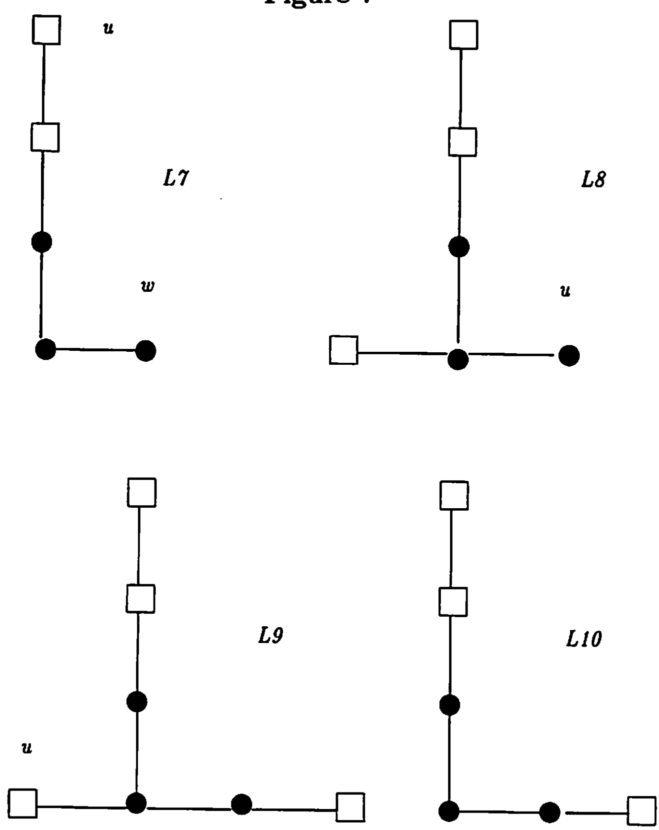


Figure 8

Claim 9. *No induced path can be dominating.*

Proof: It is enough to prove that neither P_5 nor P_6 can be dominating, since P_7 is forbidden not only as a dominating but also as an induced subgraph. In our list, P_5 and P_6 are the graphs L_7 and L_{10}.

Suppose that P_5 is dominating, and let us pick one of its endpoints, say u. Omitting u, we get a four-vertex graph. It is not dominating, thus u has a private dominated vertex u'; and so does the other endpoint w, too, with some vertex w'. We have obtained a C_7 or a P_7, depending on $u'w'$ being an edge or non-edge. This is a contradiction, and the case of P_5 is done.

Now the assertion follows for P_6 as well, by considering just one endpoint, because its removal cannot yield a dominating P_5. □

Claim 10. *The graphs L_8 and L_9 are not dominating.*

Proof: Let us consider an L_9 first. Deleting the vertex u, we obtain a P_6 which cannot be dominating, by Claim 9, and thus u has a private dominated vertex. But so we have found an $F(Y)$ which is forbidden.

Let us consider next an L_8. The graph $L_8 - u$ is isomorphic to P_5, not dominating, thus u has a private dominating vertex. We have got an L_9, contradicting the first part of the proof. □

Claim 11. *No L_6 is dominating.*

Proof: The set $H = \{q_1, q_2, q_3, q_4\}$ induces a connected four-vertex graph which is not dominating. Thus, there exists some vertex r not dominated by H, and so it is dominated by $\{q_5, q_6, q_7\}$. If it has only one neghbor in this set, then we obtain an $F(Y)$, thus we may suppose it has at least two neighbors. By symmetry reasons, it is enough to look at two cases. In the first case, the neighbors of r are q_6 and q_7. In the second case, all the three vertices are adjacent to r.

In the first case, $\{q_5, q_3, q_2, q_1, q_6, r, q_7\}$ induces a P_7. In the second case (Figure 10), $\{q_5, r, q_6, q_1, q_2, q_4\}$ induces a P_6 which is not dominating, thus there exists some vertex s which is adjacent only to q_3 and/or q_7. When it is adjacent to exactly one of them, we get a P_7. When it is adjacent to both, we get a C_7, by omitting q_4 and q_5. □

Claim 12. *The graphs L_4 and L_5 cannot be dominating.*

Proof: Let us pick an L_5 first. Then $L_5 - u$ is isomorphic to P_5, not dominating, so u has a private dominated vertex. We have obtained a dominating L_6, contradicting Claim 11.

Let us pick now an L_4 (which is the graph Y). Then $L_4 - u$ is connected and has four vertices, not dominating, thus u has a private dominated vertex. We obtain a dominating L_5, contradicting the previous case. □

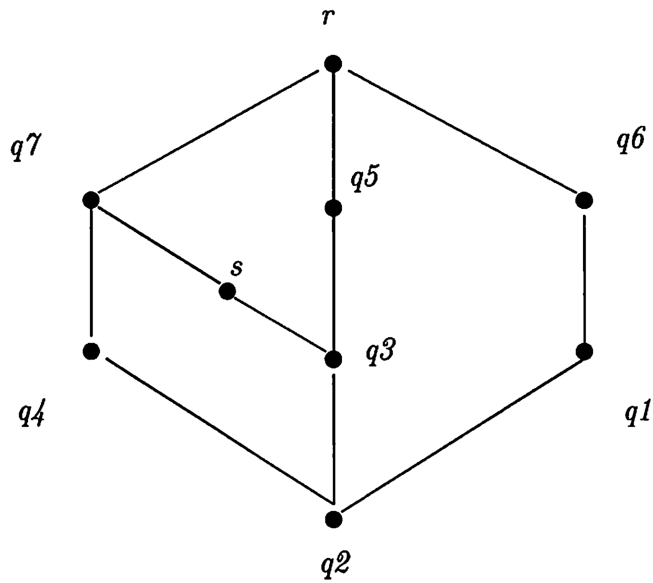


Figure 10

Claim 13. *No L_1 is dominating.*

Proof: Since $L_1 - u$ is not dominating, u has a private dominated vertex u'. Denote $M = V(L_1) \cup \{u'\}$. The subgraph induced by $M - \{w\}$ is isomorphic to P_5, not dominating, thus w has a private dominated vertex w'. Then $M \cup \{w'\} - \{z\}$ induces P_6, not dominating, so z has a private dominated vertex. The resulting graph is an $F(L_1)$, a contradiction. □

Lemma 4. *There is no dominating bull in G.*

Proof: Let B be the bull (see Figure 9, on vertices s, t, u, w, z). Since $B - u$ is not dominating, u has a private dominated vertex u'. Let $V(B) \cup \{u'\}$ be denoted by M. Then $M - \{w\}$ induces L_1, not dominating (by Claim 13), so w has a private dominated vertex w'. Furthermore, $M \cup \{w'\} - \{z\}$ induces P_6, not dominating, consequently z has a private dominated vertex. The new graph is $F(B)$ where B is a bull, a contradiction. □

Claim 14. *No L_2 is dominating.*

Proof: The subgraph $L_2 - u$ is isomorphic to the bull, which is not dominating by Lemma 4. So u has a private dominated vertex u'. Denote $M = V(L_2) \cup \{u'\}$. Then $M - \{w\}$ induces P_6, not dominating, consequently w has a private dominated vertex. So we have got an $F(L_1)$. □

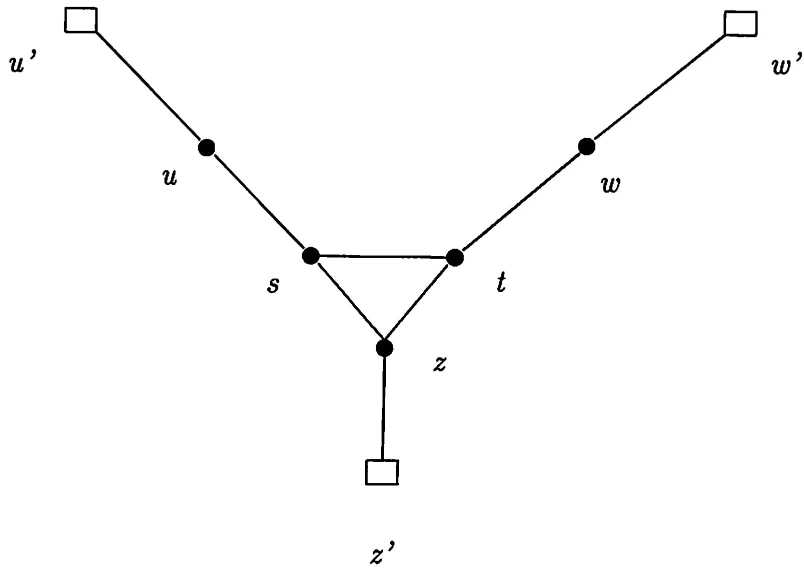


Figure 9

Claim 15. *No L_3 is dominating.*

Proof: The subgraph $L_3 - u$ is isomorphic to L_2, not dominating, so u has a private dominated vertex. An $F(B)$ has been constructed where B is a bull. □

Thus, we have proved Lemma C that completes the proof of Theorem 3, too. □

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