Mod Sum Number of Wheels

Martin SUTTON
School of Management
University of Newcastle, NSW 2308, Australia
e-mail: mgms@cc.newcastle.edu.au

Anna DRAGANOVA
Department of Mathematics
Pomona College Claremont, California 91711, USA
e-mail: adraganova@pomona.edu

Mirka MILLER
Department of Computer Science and Software Engineering
University of Newcastle, NSW 2308, Australia
e-mail: mirka@cs.newcastle.edu.au

Abstract

A graph $G(V, E)$ is a *mod sum graph* if there is a labelling of the vertices with distinct positive integers so that an edge is present if and only if the sum of the labels of the vertices incident on the edge, modulo some positive integer, is the label of a vertex of the graph. It is known that wheels are not mod sum graphs. The *mod sum number* of a graph is the minimum number of isolates that, together with the given graph, form a mod sum graph. The mod sum number is known for just a few classes of graphs. In this paper we show that the mod sum number of the n spoked wheel, $\rho(W_n)$, $n \geq 5$, is n when n is odd and 2 when n is even.

1 Introduction

A graph $G = (V, E)$ is a *sum graph* if there exists a labelling, λ, of the vertices of G with distinct positive integers so that $uv \in E$ if and only if the sum of the labels assigned to u and v is the label of a vertex of G. A sum graph cannot be connected. There must always be at least one isolated vertex; the vertex with the largest label. The *sum number* $\sigma(H)$ of a connected graph H is the least number, r, of isolated vertices, $\overline{K_r}$, so that $G = H \cup \overline{K_r}$ is a sum graph. For more information about sum graphs see [1], [5], [8], [9], [10], [11] and [15].

Mod sum graph labelling was introduced by Bolland, Laskar, Turner and Domke [2] as a generalisation of sum graph labelling. A graph $G = (V, E)$ is a *mod sum graph* if there exists a positive integer z and a labelling, λ, of the vertices of G with distinct elements from $\{1, 2, 3, \ldots, z-1\}$ so that $uv \in E$ if and only if the sum, modulo z, of the labels assigned to u and v is the label of a vertex of G. Since all labels are distinct and zero is not allowed as a label we have $z \geq |V| + 1$. Any sum graph labelling can be considered as a mod sum graph labelling by choosing a sufficiently large modulus z. The converse is obviously not true.

A mod sum graph is related to a k-sum graph introduced by Chung [4], however k-sum labellings allow more than one vertex to have the same label. A mod sum graph is also related to sum graphs over $\mathbf{Z}_n$, introduced by Harary [7], however labellings of these graphs allow zero as a label.

Unlike in the case of sum graphs, there do exist mod sum graphs that are connected. For example, paths on $n \geq 3$ vertices, trees on $n \geq 3$ vertices, cycles on $n \geq 4$ vertices, cocktail party graphs, $H_{2,n}$, and some complete bipartite and multi-partite graphs have been shown to be mod sum graphs [2], [16].

On the other hand, there are connected graphs that are not mod sum graphs, for example, complete graphs K_n for $n \geq 2$ [2], wheels W_n for $n \geq 5$ [13] and $H_{m,n}$ for $n > m \geq 3$ [14].

The *mod sum number*, $\rho(H)$, of a connected graph, H, is the least number, r, of isolated vertices, $\overline{K_r}$, so that $G = H \cup \overline{K_r}$ is a mod sum graph.

The sum number of n spoked wheels was dealt with in [10], [11] and [15]. In this paper we determine the mod sum number of n spoked wheels: we show that $\rho(W_n) = 2$ for $n \geq 6$ and n even and $\rho(W_n) = n$ for $n \geq 5$ and n odd.

There are some interesting aspects to these results. Even wheels have a constant mod sum number whereas odd wheels have an $O(n)$ mod sum number. Furthermore, when comparing the more general mod sum graph labellings to sum graph labellings for wheels, we find that when the number of spokes is even $\rho(W_n)$ is a constant value whereas $\sigma(W_n)$ is $O(n)$ but when the number of spokes is odd there is no difference whatsoever as $\rho(W_n) = \sigma(W_n) = n$.

2 The Mod Sum Number of Wheels

We follow the graph theoretic notation and terminology of [7].

An n spoked *wheel* W_n is a graph $G = (V, E)$ with a vertex set $V = \{v_c, v_1, v_2, \ldots, v_n\}$ so that $v_c v_i \in E$ for $i = 1, 2, \ldots, n$, $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, n-1$ and $v_n v_1 \in E$.

The vertex v_c of degree n is called *centre*, and the other n vertices $v_1, v_2, \ldots, v_n$, each of degree 3, are called the ***rim vertices***. An edge incident on the centre and a rim vertex is called a ***spoke***, and an edge incident on two rim vertices is called a *rim edge*.

We shall use the label of each vertex to denote the vertex itself so that rather than refer to a vertex v and its corresponding label $a = \lambda(v)$ we shall simply use the label a to denote the vertex. This is possible since the definition of a mod sum graph guarantees that the label assigned to each vertex is distinct. The label c is reserved for the centre so that $c = \lambda(v_c)$. We shall use the term *edge sum*, written as $\{a, b\}$ to mean the sum of the labels of the two vertices incident on the edge so that $\{a, b\} \in V$ is the same as $a + b \in V$ (since all arithmetic is performed modulo z, strictly speaking we should say $a + b \bmod z \in V$). Traditionally, the same notation $\{a, b\}$ is used to denote an edge joining vertices a and b. Consequently, for mod sum graphs, the statements $\{a, b\} \in E$ and $\{a, b\} = a + b \in V$ both indicate the same fact: there is an edge joining the vertices a and b.
For convenience, we will assume from now on that the equation $a + b = d$ means $a + b \equiv d \bmod z$, where z is the modulus.

We say that a vertex d of the connected component H is a *working* vertex if, for some distinct $a, b, d \in V(H), \{a, b\} \in E(H)$ and $d = a + b$.
We say that two edges are *adjacent* if there is a vertex common to both edges.

There are no wheels with less than 3 spokes. The mod sum number of the 3 spoked wheel $\rho(W_3) = \rho(K_4) = 4$ [14]. The mod sum number of the 4 spoked wheel $\rho(W_4) = 0$ [6]. From now on we consider n spoked wheels where $n \geq 5$. For $n \geq 5$ it is known that $\rho(W_n) \geq 1$ [13].

We first establish a few basic lemmas that will be used to determine the mod sum number.

Lemma 1 *In a mod sum graph labelling of* $W_n \cup \overline{K_s}$, $n \geq 5$, $s \geq r$, *the centre is not a working vertex.*

Proof. The centre, c, cannot be the edge sum of a spoke, $\{c, a\}$ say, as this would imply that $c = a + c$ but, by definition, no vertex of the graph has a label of zero. To show that the centre cannot be the sum of two vertices incident on a rim edge, we assume the contrary so that a and b are two adjacent rim vertices and $c = a + b$ is the centre. Let d be the second rim vertex adjacent to b and f be the second rim vertex adjacent to a so that f, a, b, d are four consecutive rim vertices. The edge sum of the spoke

$\{c, d\} = \{a + b, d\} = a + b + d$, shows that $a + b + d$ must be a vertex of the graph. Since the vertex a and the rim edge $\{b, d\} = b + d$ are both vertices of the graph, we deduce that one of the following two conditions must hold.

i) Vertices a and $b + d$ are distinct ($a \neq b + d$) so that the edge $\{a, b + d\}$ exists.
ii) Vertex a is the same vertex as $b + d$ ($a = b + d$) and no additional edge is implied.

Similarly the spoke $\{c, f\} = \{a+b, f\} = a+b+f$ implies that either $\{b, a+f\}$ exists or $b = a + f$.

Allowing for symmetry, there are three cases to consider:-

Case 1. Both $\{a, b + d\}$ and $\{b, a + f\}$ exist.
Case 2. $a = b + d$ and $b = a + f$.
Case 3. $\{a, b + d\}$ exists and $b = a + f$.

Case 1. When $\{a, b + d\}$ exists, the vertex $b + d$ must be identical to one of the three vertices adjacent to a. Clearly $b + d \neq b$ and $b + d \neq a + b$ so that $b + d = f$. Similarly $a + f = d$. But now $f = b + d = b + (a + f) = a + b + f \Rightarrow a + b = 0$, a contradiction since $a + b$ is a label of the graph.

Case 2. When $a = b+d$ and $b = a+f$, we note that $a = b+d = (a+f)+d \Rightarrow d + f = 0$. The four consecutive rim vertices may be written as $f, a, a + f, d$ and the centre as $2a + f$. The spoke $\{c, a\} = \{2a + f, a\} = 3a + f$ and the spoke $\{c, d\} = \{2a + f, d\} = 2a + f + d = 2a$ imply that either $\{2a, a + f\}$ exists or $2a = a + f$. Now, $2a = a + f \Rightarrow a = f$, which is not true, and so $2a$ must be one of the three vertices adjacent to $a + f$. Clearly $2a \neq a$ and $2a \neq 2a + f$ so that $2a = d$. But now the centre, $2a + f$ implies $\{2a, f\} = \{d, f\}$, a contradiction when $n \geq 5$.

Case 3. As in Case 1, $b + d = f$ and we note that $b = a + f = a + (b + d) \Rightarrow a + d = 0$. Since $f = b + d = (a + f) + d = f$, the four consecutive rim vertices may be written as $f, a, a + f, d$ and the centre as $2a + f$. The spoke $\{c, f\} = \{2a+f, f\} = 2a+2f$ and the spoke $\{c, a+f\} = \{2a+f, a+f\} = 3a+2f$ imply that either $\{a, 2a + 2f\}$ exists or $a = 2a + 2f$. Now $2a + 2f \neq f$ otherwise $2a + f = 0$ which is not true since $2a + f$ is a label of the graph. Similarly $2a + 2f \neq 2a + f$ and $2a + 2f \neq a + f$ so that $a = 2a + 2f \Rightarrow a + 2f = 0$. The spoke $\{c, a\} = \{2a+f, a\} = 3a+f$ and the spoke $\{c, a+f\} = \{2a+f, a+f\} = 3a+2f = 2a$ now imply that either $\{2a, a+f\}$ exists or $2a = a+f$. Now $2a \neq a+f$ otherwise $a = f$ which is not true. Clearly $2a \neq a$ and $2a \neq 2a+f$ so that $2a = d$. But now the centre, $2a + f$ implies $\{2a, f\} = \{d, f\}$, a

contradiction when $n \geq 5$.
□

Remark 1 *From Lemma 1 it is obvious that, if v is the edge sum of a spoke or v is the edge sum of a rim edge, $v \neq c$. Consequently, from now on, we need only consider the possibility that any working vertex is a rim vertex.*

Lemma 2 *If, in a mod sum labelling of $W_n \cup \overline{K_s}$, $s \geq r$, $n \geq 4$, the edge sum of a spoke is an isolate then the rim edges adjacent to the spoke are also isolates.*

Proof. Assume the contrary so that a and b are adjacent rim vertices, the spoke $\{c, a\} = a + c$ is an isolate and rim edge $\{a, b\} = a + b$ is a working vertex.
The spoke $\{c, a + b\} = a + b + c$ implies that either $\{a + c, b\}$ exists or $b = a + c$. Neither case can happen since b is a vertex of the connected component whereas $a + c$ is an isolate.
□

Corollary 1 *If, in a mod sum labelling of $W_n \cup \overline{K_s}$, $n \geq 4$, $s \geq r$, the edge sum of any spoke is an isolate then there are at least three isolates.*

Lemma 3 *$\rho(W_n) \geq 2$ for $n \geq 5$.*

Proof. For $n \geq 5$, it was proved in [13] that W_n is not a mod sum graph, that is, at least one isolate is required, so that $\rho(W_n) \geq 1$. We assume that $\rho(W_n) = 1$ and show that this assumption leads to a contradiction. Let x be the only isolate. It is clear that x is not the edge sum of any spoke since Corollary 1 of Lemma 2 tells us that this can only be true if $\rho(W_n) \geq 3$ so that x is the edge sum of one or more rim edges.
Let a, b and d be consecutive rim vertices and let $x = \{b, d\} = b + d$. Clearly $\{a, b\} = a + b \neq x$ and so $a + b$ must be a rim vertex. Both the spokes $\{c, a\} = a + c$ and $\{c, b\} = b + c$ exist so that the spoke $\{c, a + b\} = a + b + c$ implies that both of the following conditions hold.

i) Either $\{a, b + c\}$ exists or $a = b + c$,
and
ii) either $\{b, a + c\}$ exists or $b = a + c$.

Case 1. Either $a \neq b + c$ or $b \neq a + c$.
The rim vertices $a + c$ and $b + c$ can only be adjacent when $n \geq 5$ if both $a = b + c$ and $b = a + c$ so that if $a \neq b + c$ and/or $b \neq a + c$ then $a + c$ and

$b+c$ are not adjacent. The spoke $\{c, a+b\} = a+b+c$ must be an isolate otherwise the spoke $\{c, a+b+c\} = a+b+2c$ implies $\{a+c, b+c\}$. By Corollary 1 of Lemma 2, $\rho(W_n) \geq 3$ when any spoke is an isolate.

Case 2. $a \equiv b+c \bmod z$ and $b \equiv a+c \bmod z$.
The equations $a \equiv b+c \bmod z$ and $b \equiv a+c \bmod z$ are both true only when $c = \frac{z}{2}$, $b = a+\frac{z}{2}$ and $a = b+\frac{z}{2}$. This results in a configuration with a centre $c = \frac{z}{2}$, two adjacent rim vertices $a, b = a+\frac{z}{2}$ and a third, distinct, rim vertex $a+b = 2a+\frac{z}{2}$. We retain the label d for the second rim vertex adjacent to $b = a+\frac{z}{2}$ even though d and $a+b = 2a+\frac{z}{2}$ may not be distinct. The isolate $x = b+d = a+d+\frac{z}{2}$.
The spoke $\{c, d\} = \{\frac{z}{2}, d\} = d+\frac{z}{2}$ must be a rim vertex since d implies $\{c, d+\frac{z}{2}\} = \{\frac{z}{2}, d+\frac{z}{2}\}$. The isolate $a+d+\frac{z}{2}$ implies that either $\{a, d+\frac{z}{2}\}$ exists or $a = d+\frac{z}{2}$. Now $a \neq d+\frac{z}{2}$, otherwise $a+\frac{z}{2} = (d+\frac{z}{2})+\frac{z}{2} = d$, which is not true since $b = a+\frac{z}{2}$ and d are distinct. Clearly, $d+\frac{z}{2} \neq a+\frac{z}{2}$ so that $d+\frac{z}{2}$ must be the second rim vertex adjacent to a. When $n \geq 5$, there is a second rim vertex $e \neq a$ adjacent to $d+\frac{z}{2}$. Using similar arguments to those just given, it is easy to show that the spoke $\{c, e\} = \{\frac{z}{2}, e\} = e+\frac{z}{2}$ is a sixth, distinct, rim vertex adjacent to d. The rim edge $\{e, d+\frac{z}{2}\} = d+e+\frac{z}{2}$ must be an isolate, otherwise the spoke $\{c, d+e+\frac{z}{2}\} = \{\frac{z}{2}, d+e+\frac{z}{2}\} = d+e$ implies $\{d, e\}$ which is not possible when $n \geq 6$. Clearly, the isolates $a+d+\frac{z}{2}$ and $d+e+\frac{z}{2}$ are distinct.
□

2.1 Mod Sum Number of Even Wheels

Labelling 1 *The following is a labelling of* $W_n \cup \overline{K_2}$ *for* n *even,* $n \geq 6$.

Let the centre c be $\frac{z}{2}$ where z is the modulus.
Denote n by $2t$ where $t \geq 3$ when $n \geq 6$. Now, let $z = 10n = 20t$, leading to $c = \frac{z}{2} = 5n = 10t$. Furthermore, let the following be the values of $b_1, \ldots, b_t$.

$$\begin{array}{lcl}
b_1 = 1 & & 10+b_1 = 10+1 \\
b_2 = 1+20 & & 10+b_2 = 10+1+20 \\
b_3 = 1+2\times 20 & \text{so that} & 10+b_3 = 10+1+2\times 20 \\
\vdots & & \vdots \\
b_{t-1} = 1+(t-2)\times 20 & & 10+b_{t-1} = 10+1+(t-2)\times 20 \\
b_t = 1+(t-1)\times 20 & & 10+b_t = 10+1+(t-1)\times 20
\end{array}$$

The b_j 's form an arithmetic progression with initial term $b_1 = 1$ and common difference $d = 20$, satisfying $b_j = b_{j-1}+20$ for $j = 2, 3, \ldots, t$. Notice also that $b_t+20 = b_1$.

Label the rim as the cycle, C_1, given by

$$C_1 = \{b_1, 10 + b_t, b_2, 10 + b_{t-1}, b_3, 10 + b_{t-2}, \ldots, 10 + b_2, b_t, 10 + b_1\}$$

Note that $(b_1, 10 + b_1)$ is the only rim edge of the form $(b_k, 10 + b_k)$.
Label the two isolates 12 and -8.

Lemma 4 *Labelling 1 is a mod sum graph labelling of $W_n \cup \overline{K_2}$ for n even, $n \geq 6$.*

Proof. We must ensure that all sums of the form $u+v$, where $\{u, v\} \in E$, are labels of the graph and that when $\{u, v\} \notin E$, $u+v$ does not appear as a label. Since $c = \frac{z}{2} = 10t$, $c + b_j = b_{j+t/2}$ for $n \equiv 0 \bmod 4$ and $c + b_j = 10 + b_{j+(t-1)/2}$ for $n \equiv 2 \bmod 4$. Both of these are labels in V. Similarly $c + (10 + b_j) \in V$ for all $j = 1, 2, \ldots, t$ so that all sums arising from the spokes are present. The rim, on the other hand, consists of the cycle, C_1, given by

$$C_1 = \{b_1, 10 + b_t, b_2, 10 + b_{t-1}, b_3, 10 + b_{t-2}, \ldots, 10 + b_2, b_t, 10 + b_1\}$$

Apart from $\{b_1, 10 + b_1\}$, each edge sum is of the form $b_i, 10 + b_j$ where, alternately, $i + j = t + 1$ and $i + j = t + 2$. This leads to alternate edge sums of $12 + (t-1) \times 20$ and $12 + t \times 20$ which equal -8 and 12 respectively since $z = 20t$. These are the labels of the two isolates. The edge sum $\{b_1, 10 + b_1\}$ is also equal to 12 so that all rim edges are accounted for.
We note that all rim vertices are equivalent to 1 mod 10, the two isolates are both equivalent to 2 mod 10 and the centre and modulus are both equivalent to 0 mod 10. No edges are implied between the isolates and the rim vertices since there is no label equivalent to 3 mod 10. Similarly, $\{+12, -8\}$ is not implied since there is no label equivalent to 4 mod 10. To see that $\{c, 12\}$ and $\{c, -8\}$ are not implied we note that $\{c, 12\} = \{\frac{z}{2}, 12\} = \frac{z}{2} + 2$ and $\{c, -8\} = \{\frac{z}{2}, -8\} = \frac{z}{2} - 8$, both of which are equivalent to 2 mod 10. The only labels equivalent to 2 mod 10 are the two isolates and when $n \geq 6$ we have $c = \frac{z}{2} \geq 30$ so that it is easy to see that neither $\frac{z}{2} + 2$ nor $\frac{z}{2} - 8$ are equal to either 12 or -8.
The only other possibility is an implied edge of the form $\{b_i, b_j\}$ where $i + j \neq t+1, t+2$. But $b_i + b_j \equiv 2 \bmod z \Rightarrow b_i + b_j = 12$ or $b_i + b_j = -8$ which is true only if $i + j = t + 1$ or $i + j = t + 2$, a contradiction. Thus, no "unwanted" edges are present and the claim holds.
□

Theorem 1 $\rho(W_n) = 2$ *for n even, $n \geq 6$.*

Proof. Lemma 3 shows that at least 2 isolates are necessary and, by Lemma 4, for all $n \geq 6$ when n is even, Labelling 1 is a mod sum graph labelling of W_n using only 2 isolates.
□

2.2 Mod Sum Number of Odd Wheels

Lemma 5 *If, in a mod sum graph labelling of* $W_n \cup \overline{K_s}$, $n \geq 5$, $s \geq r$, *the edge sum of a spoke is an isolate then the rim vertices adjacent to this spoke are not the edge sums of spokes.*

Proof. Assume that a and b are adjacent rim vertices and the spoke $\{c, a\} = a+c$ is an isolate. Let d be a third rim vertex so that a, b and d are all distinct. Clearly, $b \neq \{c, a\} = a+c$ as $a+c$ is an isolate and $b \neq \{c, b\} = b+c$ as $c \neq 0$. Now, $b \neq \{c, d\} = d+c$ otherwise the rim edge $\{a, b\} = \{a, d+c\} = a+d+c$ would imply that either $\{d, a+c\}$ exists or $d = a+c$. Neither case can be true since d is a vertex of the connected component whereas $a+c$ is an isolate.
□

Lemma 6 *In a mod sum graph labelling of* $W_n \cup \overline{K_s}$, n *odd*, $n \geq 5$, $s \geq r$, *the spokes are either all working or all non-working.*

Proof. Recall that a spoke $\{c, a\} = a+c$ is *working* if $a+c$ is a rim vertex and *non-working* if $a+c$ is an isolate since the centre, c, is not working (Lemma 1).
All rim vertices are distinct and, hence, the edge sums of all spokes are distinct modulo z. It follows that the size of the set of all working spokes is the same as the size of the set of all rim vertices that are the edge sums of spokes. (We note here that there may be additional rim vertices that are working but represent rim edges and not spokes). Conversely, the size of the set of all spokes that are not working is the same as the size of the set of all rim vertices that are not the edge sums of spokes.
Assume that we have a valid mod sum graph labelling of $W_n \cup \overline{K_s}$ where $s \geq r$ and $n \geq 5$.
If all the spokes are working then we are finished. If any of the spokes are not working (isolates) then we distinguish between two different situations.

Case 1. At least one non-working spoke is incident on a rim vertex that is not the edge sum of a spoke.
Case 2. Every non-working spoke is incident on a rim vertex that is the edge sum of a spoke.

Case 1.
Consider a non-working spoke incident on a rim vertex that is not the edge sum of a spoke. The two rim vertices adjacent to this non-working spoke are not themselves the edge sum of a spoke (Lemma 5) so that three consecutive

rim vertices are not the edge sum of a spoke. If there are three rim vertices that are not the edge sum of a spoke then there must also be three non-working spokes. Each additional non-working spoke contributes at least one extra rim vertex that is not the edge sum of a spoke (Lemma 5) so that, unless all spokes are not working, there must be fewer spokes that are not working than rim vertices that are not the edge sum of some spoke.

Case 2.
When a non-working spoke is incident on a rim vertex that is the edge sum of a spoke, there are two vertices adjacent to the non-working spoke that are not the edge sum of a spoke (Lemma 5). This implies that two non-working spokes cannot be adjacent to each other since, by hypothesis, all non-working spokes are incident on a rim vertex that is the edge sum of a spoke. The simple counting argument used in Case 1 now eliminates all configurations except one. This configuration is where non-working spokes alternate with working spokes and there are exactly half of each type. Clearly, this is not possible when n is odd.
□

Theorem 2 *$\rho(W_n) = n$ for $n \geq 5$ and n odd.*

Proof. All the spokes are working or all the spokes are isolates (Lemma 6) and we assume for the moment that all the spokes are isolates.
The mod sum number must be at least n since the edge sums of the n spokes are all distinct modulo z. The mod sum number cannot be greater than n since the sum number of the n spoked wheel is known to be n [11] and we can always convert a sum graph labelling to a mod sum graph labelling by selecting a large enough modulus so that $\rho(W_n) \leq \sigma(W_n) = n$. The sum graph labelling for the odd wheels given in [11] may also be used as a mod sum graph labelling given a suitably high modulus.

It remains to show that the spokes cannot all be working.
Assume the contrary so that all spokes are working. Recall that all rim vertices are distinct and, hence, the edge sums of all spokes are distinct so that the set of all rim vertices is the same as the set of labels generated by the edge sums of all spokes. It is now clear that the rim vertices must be partitioned into sets of equal size, ℓ, so that the elements of each set are closed under addition of c modulo z. There are only two possible configurations to consider.

Case 1. $\ell = n$, all rim vertices are of the form $a + ic$, $i = 0 \ldots n-1$, $a < c$ and the modulus is nc.

Case 2. $\ell < n$, ℓ divides n and the modulus is ℓc. The rim vertices belong to one of $\frac{n}{\ell}$ sets each of size ℓ. Each set is of the form $a+ic$, $i = 0\ldots\ell-1$ where $a < c$ and each set has a different value for a.

Case 1. All rim vertices are of the form $a+ic$, $i = 0\ldots n-1$, $a < c$
Assume that vertex a is adjacent to vertex $a+ic$ and vertex $a+jc$. This results in two labels, $\{a, a+ic\} = 2a+ic$ and $\{a, a+jc\} = 2a+jc$ that must be isolates since $2a+ic \neq a+tc$, $t \in \{0\ldots n-1\}$. There cannot be a third, distinct isolate due to a rim edge, $2a+kc$ say, otherwise the edge $\{a, a+kc\}$ is implied and a can only be adjacent to two rim vertices. Now, two consecutive rim edges cannot have the same edge sum since if, for example, p, q, r are consecutive rim vertices and the edge sum $\{p,q\} = p+q$ is the same as the edge sum $\{q,r\} = q+r$ then $p = r$ which is contrary to the definition of a mod sum graph. Clearly, when n is odd, it is not possible to avoid having two consecutive edge sums that are the same when only two different edge sums are present.

Case 2. All rim vertices belong to one of $\frac{n}{\ell}$ sets each of size ℓ and of the form $a+ic$, $i = 0\ldots\ell-1$. Each set has a different value for a, the smallest element, and we can label the members of each set so that each $a < c$. For convenience, we use the smallest label of each set to characterise the entire set and simply call $\{a, a+c, a+2c, \ldots, a+(\ell-1)c\}$ the set a.

We now make two observations.
Our first observation is:- *An edge between any vertex belonging to one set and any vertex belonging to another set implies a matching between the vertices of the first set and the vertices of the second set.*
Assume there is an edge between vertex $a+ic$ from set a and vertex $b+jc$ from set b where $i, j \in 0\ldots\ell-1$. The edge sum $\{a+ic, b+jc\} = a+b+(i+j)c$ also implies the edge $\{a+(i-1)c, b+(j+1)c\}$ since the vertices are from two different sets and so $a+(i-1)c \neq b+(j+1)c$. In fact, the edge sum $\{a+ic, b+jc\} = a+b+(i+j)c$ joins every vertex of the a set with a unique vertex of the b set because the edge sum $a+b+(i+j)c$ not only implies $\{a+ic, b+jc\}$ and $\{a+(i-1)c, b+(j+1)c\}$ but also $\{a+(i-2)c, b+(j+2)c\}, \{a+(i-3)c, b+(j+3)c\}, \ldots, \{a+(i+1)c, b+(j-1)c\}$.

Our second observation is:- *It is not possible for a rim vertex to be adjacent to any vertex that belongs to the same set.*
Assume the contrary so that vertices $a+ic$ and $a+jc$ are adjacent rim vertices belonging to set a where $i, j \in 0\ldots\ell-1$ and $i \neq j$. Using a similar argument to that used in the first observation, we see that $\{a+ic, a+jc\} \in V$ also implies edges $\{a+(i-1)c, a+(j+1)c\}, \{a+(i-2)c, a+(j+2)c\}$ and so

on. A total of $\frac{1}{2}(\ell - 1)$ edges is implied between different pairs of vertices belonging to the set a but one vertex is left unpaired since ℓ, the size of the set, is odd. This unpaired vertex is $a + \frac{1}{2}(i + j)c$ when $i + j$ is even and $a + \frac{1}{2}(\ell + i + j)c$ when $i + j$ is odd.
Consider the two rim vertices adjacent to this unpaired vertex.
If $\ell = 3$ and both vertices are also from the set a then a cycle of size 3 is formed, a contradiction when $n \geq 5$.
For any other situation, the extra edges implied ensure that at least one rim vertex from the set a is adjacent to three other rim vertices, a contradiction.

We are now ready to show that all the spokes cannot be working.
Find a pair of adjacent vertices from different sets (such a pair must always exist). As we have seen in our first observation, once we have found such a pair we have also found a total of ℓ pairs each with the same edge sum. Without loss of generality, assume the first pair of vertices to be $a + ic$ and $b + jc$, where $i, j \in 0 \dots \ell - 1$. Consider the second rim vertex adjacent to $b + jc$. It cannot be another vertex of the set a since this would join every vertex of the set b to a second vertex of the set a, forming a cycle of maximum length 2ℓ, however $2\ell < n$ since n is odd and ℓ divides n. The second rim vertex adjacent to $b + jc$ cannot be another vertex from the set b and we conclude that it must belong to a third set, and assume, without loss of generality, that it is $d + kc$ where $k \in 0 \dots \ell - 1$. This also adds a third vertex from the set d to each of our $\frac{n}{\ell}$ chains.
It is clear that this argument may be repeated until one member of each set is added to each chain of vertices. At each stage we cannot join the terminal vertex to another member of the same set, we cannot join the terminal vertex to the starting vertex of another chain (from the set a) without forming a cycle of length less than n, and we cannot join the terminal vertex to any non-terminal chain vertex, one from the set b, say, as each is already adjacent to two rim vertices.
When all the $\frac{n}{\ell}$ chains are fully formed, we need to join them end to end to form the C_n rim vertices. Our second observation tells us that we cannot join vertices from the same set and, without loss of generality, we assume that $a + ic$, the first vertex of the chain $a + ic, b + pc, \dots, g + rc$, is adjacent to $g + sc$, the terminal vertex of the chain $a + jc, b + qc, \dots, g + sc$. Also assume, again without loss of generality, that $i = j + k$.
Because the vertices of each chain have the same pairwise sums, the vertices of the first chain are alternately kc larger and kc smaller than the corresponding vertex of the second chain. As an example, consider the first two vertices of each chain. Now $(a + ic) - (a + jc) = kc$, by hypothesis, and since $(a + ic) + (b + pc) = (a + jc) + (b + qc)$ we have immediately that $(b + pc) - (b + qc) = -kc$.

Since each chain is of odd length we deduce that $(g + rc) - (g + st) = kc$. But now $(a + ic) - (a + jc) = kc = (g + rc) - (g + st)$ which rearranges to $(a+ic)+(g+st) = (a+jc)+(g+rc)$. This immediately implies that $a+jc$ is adjacent to $g + rc$, thus forming a cycle of length 2ℓ, which is less than n. □

Remark 2 *Mod sum graphs are a generalisation of sum graphs and, intuitively, it seems "obvious" that there should always be at least one more isolate required in a minimal mod sum graph labelling compared to the number of isolates required in a minimal sum graph labelling of the same graph. Looked at another way, since every sum graph labelling has at least one isolate, that is, the largest label, it is "obvious" that we can convert a given sum graph labelling into a (not necessarily minimal) mod sum graph labelling with one less isolate by choosing the modulus so that this largest label maps onto another label of the graph. This approach does work for some classes of graphs, for example, for trees, so that $\rho(T_n) = 0$ while $\sigma(T_n) = 1$.*

However, in this paper we have proved that for odd wheels with at least 5 spokes, the mod sum number and the sum number are exactly the same with $\rho(W_n) = n$ and $\sigma(W_n) = n$. This is the first known instance of a class of graphs where the mod sum number is the same as the sum number.

Interestingly, the situation is very different for even wheels. As we proved in this paper, for even wheels with at least six spokes the difference between mod sum number and sum number is even more striking, with $\rho(W_n) = 2$ and $\sigma(W_n) = \frac{n}{2} + 2$, a difference in the order of magnitude.

Open Problem 1 *Are there any classes of graph, apart from the odd wheels, for which $\rho(G) = \sigma(G)$?*

The following table is a summary of the current state of knowledge of mod sum numbers.

Graph Class, H	Restrictions	$\rho(H)$	Citation
Trees, T_n	$n \geq 3$	0	[2]
Cycles, C_n	$n \geq 4$	0	[2]
$K_{p+1,q}$	$q \geq 1$ and $p \geq r_q + r_{q-1} - 1$ *	0	[2]
$K_{2,p}$	$p \geq 1$	0	[2]
$K_{n_1,n_2,\ldots,n_m}$	if $\exists n_i$ and n_j s.t. $n_i < n_j < 2n_i$	> 0	[16]
W_4		0	[6], [13]
W_n	$n \geq 6$, n even	2	This paper
W_n	$n \geq 5$, n odd	n	This paper
$H_{2,n}$	$n \geq 0$	0	[2]
$H_{m,n}$	$n > m \geq 3$	> 0	[14]
K_2		1	[14]
K_3		1	[14]
K_n	$n \geq 4$	n	[14]

* $r_j = f_{j+1} - 1$ where $\{f_j\}$ is the Fibonacci sequence with $f_1 = 1$, $f_2 = 2$.

3 Open Problems

The wheel is very close to another type of graph, the fan $F_n = P_n + K_1$ on $n+1$ vertices. A labelling is enough to provide an upper bound for the mod sum number of fans.

Theorem 3 *For n odd, $n \geq 5$, the mod sum number $\rho(F_n) \leq n$.*

The Fibonacci sequence can be used to construct a mod sum labelling of $F_n \cup \overline{K_n}$. Let the modulus be $z = (S_{n+1} - 1)a$ where $a > 4$ and label the center vertex with 1; next, label $v_1, v_2, \ldots, v_n \in V(F_n)$ with $1+a, 1+2a, 1+3a, \ldots, 1+S_n a$ where S_n is the nth Fibonacci number ($S_0 = 1$, $S_1 = 1$). Therefore, v_i receives $1 + S_i a$ as a label. Consequently, the isolated vertices are labeled $2+a, 2+2a, \ldots, 2+S_n a$. It is left to the reader to verify that the given labelling is, in fact, a mod sum labelling.
It is still an open problem to show the exact mod sum number for the fan.

Open Problem 2 *What is the mod sum number of the fan F_n?*

As shown in the table above, the case of complete bipartite graphs $K_{m,n}$ was almost completed by Wallace [16] with only one exception.

Open Problem 3 *What is the mod sum number of the bipartite graph $K_{m,n}$ when $2m \leq n \leq 3m - 3$ and n is odd?*

Wallace [16], has made the following conjecture concerning the difficulty of determining the mod sum number of a graph. The conjecture gains validity from the results of Cairnie and Edwards [3] who proved that the problems of finding cordial and
k-equitable labellings are NP-complete.

Conjecture 1 *Let $G = (V, E)$ be a graph; then, finding the mod sum number of G is NP-complete.*

Observing that the wheel W_n is as a 1-point suspension of the cycle C_n, it is natural, then, to consider t-point suspensions of C_n.

Open Problem 4 *What is the mod sum number of the t-point suspension of C_n for $t \geq 2$?*

In conclusion, we mention a broader extension of sum graphs. For any abelian group $(H, *)$, define a graph G to be a *H-sum graph* if there exists a labelling of the vertices of G with distinct non-identity elements of H such that two vertices v and u are adjacent if and only if $v * u$ appears as a label of some other vertex of G. Notice that a MSG is simply an Z_m-sum graph. All resolved and unresolved problems for sum graphs and mod sum graphs are applicable to the case of H-sum graphs.

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