A Note on the Alon-Saks-Seymour coloring conjecture

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Abstract

Kahn(see [3]) reported that N. Alon, M. Saks and P. D. Seymour made the following conjecture. If the edge set of a graph G is the disjoint union of the edge sets of m complete bipartite graphs, then $\chi(G) \leq m+1$. The purpose of this paper is to provide a proof of this conjecture for $m \leq 4$ and $m \geq n-3$ where G has n vertices.

Let $\chi(G)$ denotes the chromatic number of a graph G, that is the fewest number of stable sets into which the vertices of G can be partitioned. The purpose of this note is to prove the following result which verifies a conjecture of Alon, Saks and Seymour in some cases (see [2] and [3]).

Theorem 1. *Let G be a graph on n vertices and assume that G is the edge disjoint union of m complete bipartite graphs. If $m \leq 4$ or $m \geq n-3$, then $\chi(G) \leq m+1$.*

If A and B are disjoint sets, then we denote by $K_{A,B}$ the complete bipartite graph with bipartition of vertices $\{A, B\}$. By $K_{A,B} \uplus K_{C,D}$, we mean the union of graphs $K_{A,B}$ and $K_{C,D}$ having no edges in common.

Lemma 2. *Let $G = K_{A,B} \uplus K_{C,D}$. Suppose $A \cap C \neq \emptyset$. Then $B \cap D = \emptyset$.*

Proof. Let a be a vertex in A and C. Suppose to the contrary that there is a vertex b in $B \cap D$. Then the edge connecting a and b belongs to both $K_{A,B}$ and $K_{C,D}$, a contradiction. □

Theorem 1 is a consequence of the following Lemmas.

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Lemma 3. *Let m be a positive integer and let $G = K_{A_1,B_1} \uplus \cdots \uplus K_{A_m,B_m}$. Suppose $A_1, \cdots, A_m$ are mutually disjoint. Then $\chi(G) \leq m + 1$.*

Proof. Consider the $m+1$ pariwise disjoint sets $A_1 \cup B_2 \cup B_3 \cup \cdots \cup B_m - B_1 - A_2 - \cdots - A_m$, $B_1 \cup A_2 - B_2 - A_3 - \cdots - A_m$, $(B_1 \cap B_2) \cup A_3 - B_3 - A_4 - \cdots - A_m$, $\cdots$, $(B_1 \cap \cdots \cap B_{m-1}) \cup A_m - B_m$ and $B_1 \cap \cdots \cap B_m$. As the first set is disjoint from $B_1, A_2, \cdots, A_m$, no edge connects two vertices of it. Hence one color suffices for its vertices. Similarly each of the above sets requires only one color. Since $A_1 \cup \cdots \cup A_m \cup B_1 \cup \cdots \cup B_m$ is the union of the above sets, $\chi(G) \leq m + 1$. □

The theorem is clearly true for $m = 1$, and so we start with $m = 2$.

Lemma 4. *Let $G = K_{A_1,B_1} \uplus K_{A_2,B_2}$. Then $\chi(G) \leq 3$.*

Proof. By Lemma 2, either $A_1 \cap A_2 = \emptyset$ or $B_1 \cap B_2 = \emptyset$. In any case the lemma follows by Lemma 3. □

Corollary 5. *Let G be a planar graph. If the edge set of a graph G is the disjoint union of the edge sets of m complete bipartite graphs, then $\chi(G) \leq m + 1$.*

Proof. As G is a planar graph, $\chi(G) \leq 4$. Hence the corollary follows easily from the above lemma. □

Lemma 6. *Let $G = K_{A_1,B_1} \uplus K_{A_2,B_2} \uplus K_{A_3,B_3}$. Then $\chi(G) \leq 4$.*

Proof. By Lemma 2 and Lemma 3, it is enough to consider the cases where exactly one pair of A_1, A_2 and A_3 intersects. We may assume that A_1 and A_2 intersect, but $A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$. Then B_1 is disjoint from B_2. Consider the partition $A_1 \cup B_2 \cup B_3 - B_1 - A_2 - A_3$, A_3, $B_1 \cup A_2 - A_1 - A_3$ and $A_1 \cap A_2$ of $A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3$. The lemma follows by a similar method as in the proof of Lemma 3. □

Lemma 7. *Let* $G = K_{A_1,B_1} \uplus \cdots \uplus K_{A_4,B_4}$. *Then* $\chi(G) \le 5$.

Proof. Let $V = A_1 \cup \cdots \cup A_4 \cup B_1 \cup \cdots \cup B_4$. By Lemma 2 and Lemma 3, it is enough to consider the following cases.

Case (1) A_1 intersects with A_2, and $A_i \cap A_j = \emptyset$ if $(i,j) \ne (1,2)$: Then B_1 is disjoint from B_2. $A_1 \cup B_2 \cup B_3 \cup B_4 - B_1 - A_2 - A_3 - A_4$, A_3, A_4, $B_1 \cup A_2 - A_1 - A_3 - A_4$, $A_1 \cap A_2$ is a partition of V into 5 stable sets.

Case (2) A_1 intersects with A_2, A_3, and $A_i \cap A_j = \emptyset$ if $(i,j) \ne (1,2),(1,3)$: Then B_1 is disoint from B_2, B_3, and $B_1 \cup A_2 \cup A_3 \cup B_4 - A_1 - B_2 - B_3 - A_4$, A_4, $B_2 \cup B_3 - A_2 - A_3 - A_4$, $(A_1 - B_2 - A_3) \cup (A_2 \cap B_3 - B_1) - (B_3 - A_2)$, $(A_1 \cup B_2) \cap A_3$ is a partition of V into 5 stable sets.

Case (3) $A_1 \cap A_2 \ne \emptyset$, $A_3 \cap A_4 \ne \emptyset$, and $A_i \cap A_j = \emptyset$ if $(i,j) \ne (1,2),(3,4)$: Then $B_1 \cap B_2 = \emptyset$ and $B_3 \cap B_4 = \emptyset$. Consider the partition $(A_1 - A_2) \cup (B_2 - A_3 - A_4) \cup (B_3 - B_1 - A_2 - A_4) \cup (B_4 - B_1 - A_2 - A_3)$, $(A_2 - A_1) \cup (B_1 - A_3 - A_4)$, $(A_3 - B_4) \cup (A_4 - B_3)$, $A_1 \cap A_2$, $A_3 \cap B_4$, $B_3 \cap A_4$ of V into stable sets. As $A_3 \cap B_4$ or $B_3 \cap A_4$ is empty by Lemma 2, we have only five sets.

Case (4) Any two of A_1, A_2, A_3 intersect, but they are disjoint from A_4: Then B_1, B_2, B_3 are mutually disjoint, and $B_1 \cup A_2 \cup A_3 \cup B_4 - A_1 - B_2 - B_3 - A_4$, $A_1 \cup B_2 - A_2 - B_3 - A_4$, $(A_1 \cap A_2) \cup B_3 - A_3 - A_4$, A_4, $A_1 \cap A_2 \cap A_3$ is a partition of V into 5 stable sets.

Case (5) The only nonempty intersections among A_1, A_2, A_3, A_4 are $A_1 \cap A_2 \ne \emptyset$, $A_1 \cap A_3 \ne \emptyset$, $A_2 \cap A_4 \ne \emptyset$: Then $B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_4 = \emptyset$, and $A_1 \cup A_2 \cup B_3 \cup B_4 - B_1 - B_2 - A_3 - A_4$, $B_1 - A_4$, $A_4 - B_2$, $B_2 - A_3$, $A_3 - B_1$ is a partition of V into 5 stable sets. □

The theorem is clearly true for $m \ge n - 1$.

Lemma 8. *Let* $G = K_{A_1,B_1} \uplus \cdots \uplus K_{A_{n-2},B_{n-2}}$. *Then* $\chi(G) \le n - 1$.

Proof. Suppose $\chi(G) = n$. Then $G = K_n$, contradicting to the Graham-Pollak theorem (see [1]) which states that K_n cannot be partitioned into fewer than $n-1$ complete bipartite graphs. □

Lemma 9. *Let* $G = K_{A_1,B_1} \uplus \cdots \uplus K_{A_{n-3},B_{n-3}}$. *Then* $\chi(G) \leq n-2$.

Proof. Knowing that $\chi(G) \neq n$ suppose $\chi(G) = n-1$. Then $G \uplus K_{1,a} = K_n$ for some $a \leq n-1$, again contradicting to the Graham-Pollak theorem (see [1]). □

References

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