ON THE SMALLEST EDGE DEFINING SETS OF GRAPHS

S. AKBARI
G.B. KHOSROVSHAHI

Abstract

For a given sequence of nonincreasing numbers, $\mathbf{d} = (d_1, \ldots, d_n)$, a necessary and sufficient condition is presented to characterize $\mathbf{d}$ when its realization is a unique labelled simple graph. If G is a graph, we consider the subgraph G' of G with maximum edges which is uniquely determined with respect to its degree sequence. We call the set of $E(G)\backslash E(G')$ the smallest edge defining set of G. This definition coincides with the similar one in design theory.

1. Introduction

For a given set of four positive integers, v, k, t, λ, a t-(v, k, λ) design (or simply a t-design) is an ordered pair $(X, \mathcal{B})$ where X is a v-set and $\mathcal{B}$ is a collection of k-subsets of X (called blocks) such that every t-subset of X appears in exactly λ blocks.

Clearly every r-regular graph of order n, is a 1-$(n, 2, r)$ design where the edges are the blocks of the 1-design.

Let D be a given t-(v, k, λ) design and let S be a subset of blocks of D. Then we define

$$Ext(S) = \{ \mathcal{D} \mid \mathcal{D} \text{ is a } t\text{-}(v,k,\lambda) \text{ design } and\ S \subset \mathcal{D}\}.$$

If $Ext(S) = \{D\}$, then S is called a *defining set* for D and denoted by $d(D)$. A defining set $d(D)$ with minimum cardinality among the defining sets of D, is called smallest defining set for D, and is denoted by $d_s(D)$. For example, for a 2-$(7, 3, 1)$ design (Fano plane or the projective plane of order 2): $X = \{1, \dots, 7\}$,

$$\mathcal{B} = \{123, 145, 167, 246, 257, 347, 356\}.$$
$$d_s(D) = \{123, 145, 246\}.$$

We note that the $d_s(D)$ is not necessarily unique.

Let $G = (V, E)$ be a graph with $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$ of order n and size m, and for any $i, 1 \leq i \leq n$, the degree of v_i is denoted by d_i. The maximum and the minimum values of d_i of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Also, for any vertex v, the set of all vertices adjacent to $v, N(v)$, is called the neighborhood of v. The nonincreasing sequence $\mathbf{d} = (d_1, \dots, d_n)$ is called the *degree sequence of* G.

A nonincreasing sequence $\mathbf{d}$ is called *graphical* if there exists a graph G with degree sequence $\mathbf{d}$. All sequences in this paper, are graphical. From the other hand, suppose $\mathbf{d}$ is graphical and G is a graph with degree sequence $\mathbf{d}$, then G is called a *realization* of $\mathbf{d}$. If every two realizations of $\mathbf{d}$ are isomorphic, then $\mathbf{d}$ is called unigraphic [1].

In determining the smallest defining sets of a graph, one needs to characterize all those graphical sequences which have unique realization . We note that this uniqueness is not meant to be "up to isomorphism". To avoid any ambiguity, we give an example : Let $\mathbf{d} = (m, n, 1, \dots, 1)$, where m and n are natural numbers, be graphical. Then in the sense of [1] this $\mathbf{d}$ is unigraphic but in the sense of this paper it does not poses a unique realization.

An edge defining set for a graph G, $d(G)$, is a subset of E such that if any other graph $G' = (V, E')$ contains $d(G)$ having the same degree sequence as G, then $G' = G$. Similar to the case of designs, a defining set for a graph with smallest cardinality is called a smallest defining set for G and is denoted by $d_s(G)$.

It is easy to see that the definition of defining sets for graphs and designs coincide if the graph is regular.

A set of two disjoint collections of 2-subsets of V, denoted by T^+ and T^-, is called a *trade* if they are mutually vertex-balanced, i.e., the frequency of any vertex $v \in V$ appearing in T^+ is the same as in T^-. For example, let $V = \{v_1, \dots, v_6\}$, then

$$\begin{aligned} T^+ &= \{v_1v_2, v_3v_4, v_5v_6\}, \\ T^- &= \{v_2v_3, v_4v_5, v_1v_6\}, \end{aligned}$$

where $v_iv_j = \{v_i, v_j\}$ and $T = \{T^+, T^-\}$ is a trade. We note that if in a graph $G = (V, E)$ such that $T^+ \subset E$, then by exchanging T^+ by T^- in E, a graph with the same degree sequence of G is obtained. This action is called the *trading-off* G by T. If $|T^+| = |T^-| = 2$, then $T = \{T^+, T^-\}$ is called a minimal trade. In this paper, by a trade we shall mean the minimal trade. The concept of "trading-off" has already been utilized in the literature under different name as "switching" [1].

The concept of defining set for designs is rather complicated and not much is known about it. For a brief review on trades and defining sets, see [2].

2. Results

In this paper, through Theorem 1, we completely determine the smallest defining set for any arbitrary graph. In Theorem 1, by a graph, we shall mean a multigraph with no loop.

Theorem 1. Let G be a graph of size m, and let $t(G)$ denote the number of edges of a triangle in G (with possible multiple edges) with maximum number of edges. Then we have $|d_s(G)| = m - \max(\Delta(G), t(G))$.

Proof. Let S be an edge defining set of $G = (V, E)$. Then $E \backslash S$ does not contain $T^+(T^-)$ of any trade, and hence any two edges of $E \backslash S$ are adjacent. Therefore, regardless of repeated edges, $E \backslash S$ is either a triangle or a star, that is $K_{1,r}$. This completes the proof.

Remark 1. We note that for any $v \in V$, $S_v = E(G \backslash \{v\})$ is in fact a defining set.

In what follows we confine ourselves to simple graphs, then the definition of defining set slightly varies. *A restricted edge defining set* of G (simple) is a set of edges, S, such that if a simple graph G' contains S and has the same degree sequence of G, then $G = G'$. We denote this kind of defining set by $d_r(G)$ and likewise, $d_{rs}(G)$ is used for the smallest defining set of restricted kind.

In the following lemma, by trading-off, we mean: For a given graph $G = (V, E)$, and a trade $T = \{T^+, T^-\}$, suppose that $T^+ \subset E$ and $T^- \cap E = \emptyset$. Then by trading-off G by T, we obtain a simple graph on the same set of vertices and $T^- \cup (E \backslash T^+)$ as the set of edges.

Lemma 1. Let G and G' be two simple graphs of order n having a non-increasing degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$. Then G' is obtained from G by trading-off G by finite number of trades. For a proof, see [3, p.45].

Theorem 2. Suppose G is a simple graph of order n such that $\delta(G) \geq 1$. If $|d_{rs}(G)| = 0$, (i.e., there exists only one simple graph with degree sequence of G), then $\Delta(G) = n - 1$.

Proof. Let v be a vertex of G with maximum degree and let w be a vertex of G such that $w \notin N(v) \cup \{v\}$. Let $u \in N(w)$ and $u' \in N(v) - \{u\}$. If u' is not adjacent to u, then G contains $T^+ = \{vu', wu\}$ which is contradictory with $|d_{rs}(G)| = 0$. Therefore, u is adjacent to any $u' \in N(v) - \{u\}$. This implies that $\Delta(G) = d(v) < d(u)$, which is also a contradiction. Thus v is adjacent to all other vertices of G. This completes the proof.

Corollary. If G is a triangle-free graph of size m, then $|d_{rs}(G)| = m - \Delta(G)$.

Remark 3. Suppose G is a simple graph of size m. If H is a subgraph of G with maximum edges such that $|d_{rs}(H)| = 0$, then

$$|d_{rs}(G)| = m - |E(H)|.$$

Finally the following corollary is an immediate consequence of Theorem 2.

Corollary. Let G be a simple graph of size m, then

$$m - \binom{\Delta + 1}{2} \leq |d_{rs}(G)| \leq m - \Delta.$$

Note. In what follows, $H_k = (h_{ij})$ will denote a $k \times k$ lower triangular matrix defined as follows:

$$h_{ij} = \begin{cases} 1 & i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

Also, in the following theorem, H^t denotes the transpose of H.

Theorem 3. Let $\mathbf{d} = (d_1, \ldots, d_1, \ldots, d_k, \ldots, d_k)$ be a sequence of the natural numbers and for every $1 \leq j \leq k$, i_j is the number of occurrences of d_j in the sequence. If $\sum_{j=1}^{k} i_j = n$, then $\mathbf{d}$ is the degree sequence of a unique simple graph of order n if and only if

$$\left[\begin{array}{c|c} H_{\frac{k}{2}} & O \\ \hline O & -H^t_{\frac{k}{2}} \end{array}\right] \left[\begin{array}{c} i_1 \\ \vdots \\ i_{\frac{k}{2}} \\ i_{\frac{k}{2}+2} \\ \vdots \\ i_k \\ -(n-1) \end{array}\right] = \left[\begin{array}{c} d_k \\ d_{k-1} \\ \vdots \\ d_1 \end{array}\right], \text{for } k \text{ even,}$$

and

$$\left[\begin{array}{c|c} H_{\frac{k+1}{2}} & O \\ \hline O & -H^t_{\frac{k-1}{2}} \end{array}\right] \begin{bmatrix} i_1 \\ \vdots \\ i_{\frac{k+1}{2}} \\ i_{\frac{k+5}{2}} \\ \vdots \\ i_k \\ -(n-1) \end{bmatrix} = \begin{bmatrix} d_k \\ d_{k-1} \\ \vdots \\ d_{\frac{k+3}{2}} \\ d_{\frac{k+1}{2}+1} \\ d_{\frac{k-1}{2}} \\ \vdots \\ d_1 \end{bmatrix}, \text{for } k \text{ odd.}$$

Proof. We prove the theorem for k even, and for k odd the proof is similar.

Let G be the only graph with a degree sequence as stated in the statement of the theorem. First by induction on n, we show that

$$H_{\frac{k}{2}} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_{\frac{k}{2}} \end{bmatrix} = \begin{bmatrix} d_k \\ d_{k-1} \\ \vdots \\ d_{\frac{k}{2}+1} \end{bmatrix}, \text{ for } k \text{ even,}$$

and

$$H_{\frac{k+1}{2}} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_{\frac{k+1}{2}} \end{bmatrix} = \begin{bmatrix} d_k \\ d_{k-1} \\ \vdots \\ d_{\frac{k+3}{2}} \\ d_{\frac{k+1}{2}+1} \end{bmatrix}, \text{ for } k \text{ odd.}$$

For $n = 2$ or 3, the statement is a triviality. Based on Theorem 2, the vertices $v_1, \ldots, v_{i_1}$ are adjacent to all the vertices. Now, we omit these vertices and delete the edges of G which are incident to them. We name the remaining graph as G_1. The degree sequence of G_1 is as follows:

$$(\underbrace{d_2 - i_1, \ldots, d_2 - i_1}_{i_2}, \ldots, \underbrace{d_k - i_1, \ldots, d_k - i_1}_{i_k})$$

and this sequence uniquely determines a graph which is G_1, since otherwise there would exist two graphs with the same degree sequence as G. Now, if

$\delta(G_1) \geq 1$, then since $\Delta(G_1) = n - 1 - i_1$, hence $d(v_{i_1+1}) = n - 1 - i_1$ in G_1, and consequently $d(v_{i_1+1}) = n - 1$ in G, which is a contradiction since $d_2 < d_1$. Therefore, $\delta(G_1) = 0$ which in turn implies that $d_k = i_1$. Now by deleting the isolated vertices of G_1, we obtain a graph G_2 such that

$$|V(G_2)| = n - (i_1 + i_k).$$

Now by induction hypothesis, we have

$$H_{\frac{k-2}{2}} \begin{bmatrix} i_2 \\ i_3 \\ \vdots \\ i_{\frac{k}{2}} \end{bmatrix} = \begin{bmatrix} d_{k-1} - i_1 \\ d_{k-2} - i_1 \\ \vdots \\ d_{\frac{k}{2}+1} - i_1 \end{bmatrix}.$$

But by $d_k = i_1$, we obtain

$$H_{\frac{k}{2}} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_{\frac{k}{2}} \end{bmatrix} = \begin{bmatrix} d_k \\ d_{k-1} \\ \vdots \\ d_{\frac{k}{2}+1} \end{bmatrix}.$$

The proof for odd k's is similar.

Now since the degree sequence

$$(\underbrace{d_1, \ldots, d_1}_{i_l}, \ldots, \underbrace{d_k, \ldots, d_k}_{i_k})$$

defines a unique simple graph G, hence the degree sequence

$$(\underbrace{n-1-d_k, \ldots, n-1-d_k}_{i_k}, \ldots, \underbrace{n-1-d_1, \ldots, n-1-d_1}_{i_1})$$

determines the unique simple graph $\overline{G}$ (complement of G). Since $d_1 = n-1$, therefore, we have

$$H_{\frac{(k-1)+1}{2}} \begin{bmatrix} i_k \\ i_{k-1} \\ \vdots \\ i_{\frac{k}{2}+1} \end{bmatrix} = \begin{bmatrix} n-1-d_2 \\ n-1-d_3 \\ \vdots \\ n-1-d_{\frac{k}{2}+1} \end{bmatrix},$$

and hence,

$$-H_{\frac{k}{2}} \begin{bmatrix} -(n-1) \\ i_k \\ \vdots \\ i_{\frac{k}{2}+2} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{\frac{k}{2}} \end{bmatrix}.$$

Therefore, we obtain

$$-H^t_{\frac{k}{2}} \begin{bmatrix} i_{\frac{k}{2}+2} \\ i_{\frac{k}{2}+3} \\ \vdots \\ i_k \\ -(n-1) \end{bmatrix} = \begin{bmatrix} d_{\frac{k}{2}} \\ d_{\frac{k}{2}-1} \\ \vdots \\ d_1 \end{bmatrix}.$$

It follows that

$$\left[\begin{array}{c|c} H_{\frac{k}{2}} & O \\ \hline O & -H^t_{\frac{k}{2}} \end{array}\right] \begin{bmatrix} i_1 \\ \vdots \\ i_{\frac{k}{2}} \\ i_{\frac{k}{2}+2} \\ \vdots \\ i_k \\ -(n-1) \end{bmatrix} = \begin{bmatrix} d_k \\ d_{k-1} \\ \vdots \\ d_1 \end{bmatrix}.$$

Now, we prove the second half of the theorem. Suppose that k is even. We prove the assertion by induction on n. For $n = 2$, the statement is clear. By assumption we have

$$\left[\begin{array}{c|c} H_{\frac{k}{2}} & O \\ \hline O & -H^t_{\frac{k}{2}} \end{array}\right] \begin{bmatrix} i_1 \\ \vdots \\ i_{\frac{k}{2}} \\ i_{\frac{k}{2}+2} \\ \vdots \\ i_k \\ -(n-1) \end{bmatrix} = \begin{bmatrix} d_k \\ d_{k-1} \\ \vdots \\ d_1 \end{bmatrix}.$$

From the above equality, we conclude that $d_1 = n-1$ and $d_k = i_1$. Now consider the following sequence

$$(\underbrace{d_2 - i_1, \ldots, d_2 - i_1}_{i_2}, \ldots, \underbrace{d_{k-1} - i_1, \ldots, d_{k-1} - i_1}_{i_{k-1}}).$$

One can easily see that

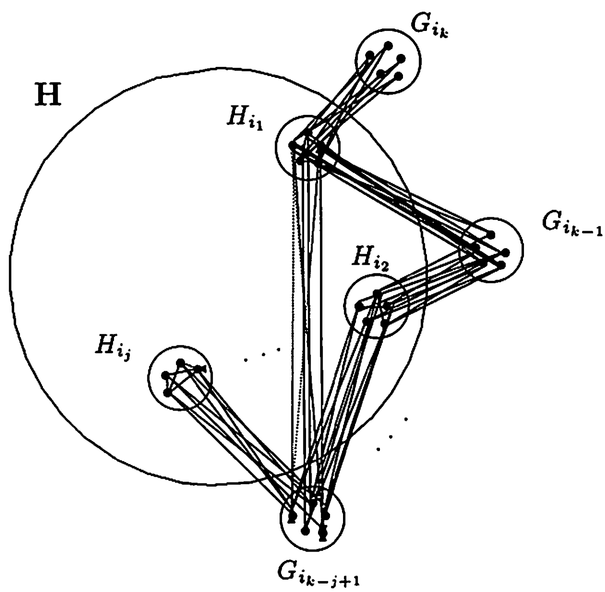
$$\left[\begin{array}{c|c} H_{\frac{k-2}{2}} & O \\ \hline O & -H^t_{\frac{k-2}{2}} \end{array}\right] \begin{bmatrix} i_2 \\ \vdots \\ i_{\frac{k}{2}} \\ i_{\frac{k}{2}+2} \\ \vdots \\ i_{k-1} \\ -(n-1-(i_1+i_k)) \end{bmatrix} = \begin{bmatrix} d_{k-1} - i_1 \\ d_{k-2} - i_1 \\ \vdots \\ d_2 - i_1 \end{bmatrix}.$$

By induction hypothesis, the above degree sequence determines a unique graph G_1. Therefore, G must be uniquely defined by its degree sequence.

Remark 2. Suppose that G is a simple graph of order n with the following degree sequence:

$$(\underbrace{d_1, \ldots, d_1}_{i_1}, \ldots, \underbrace{d_k, \ldots, d_k}_{i_k}),$$

where k is even and $|d_{rs}(G)| = 0$. Then G is constructed as follows: Let H denote $K_{i_1+\ldots+i_{\frac{k}{2}}}$. We add i_k vertices to H and join all of them to i_1 fixed vertices of H. Denote these i_k vertices by graph G_{i_k} and induced complete subgraph on i_1 vertices of H by graph H_{i_1}. Now add i_{k-1} vertices to the new graph and join them to all the vertices of H_{i_1}. Also we consider i_2 vertices of $V(H)\backslash V(H_{i_1})$ and join them to all of i_{k-1} vertices. We denote these i_{k-1} vertices by graph $G_{i_{k-1}}$ and the induced complete subgraph on i_2 vertices of H by H_{i_2}. Similarly, for any $j, 1 \leq j \leq \frac{k}{2}$, $G_{i_{k-j+1}}$ and H_{i_j} are defined, see the following figure.



Clearly, the order of the above mentioned graph is $\sum_{j=1}^{k} i_j = n$, and for any j, $1 \le j \le \frac{k}{2}$, the degrees of any vertices of H_{i_j} and $G_{i_{k-j+1}}$ are $n - 1 - \sum_{t=0}^{j-2} i_{k-t}$ and $\sum_{t=1}^{j} i_t$, respectively.
By Theorem 3, we have

$$n - 1 - \sum_{t=0}^{j-2} i_{k-t} = d_j \quad \text{and} \quad \sum_{t=1}^{j} i_t = d_{k-j+1}.$$

Consequently, the degree sequence of this graph is identical with the degree sequence of G. It can be easily verified that this graph does not contain any trade. For k odd, the construction is similar.

Acknowledgement. The authors are indebted to the Institute for Studies in Theoretical Physics and Mathematics for support. The first author wishes to express his appreciation to the Research Council of the Sharif University of Technology for support. Also the first author is indebted to the National Research Council of I. R. Iran (NRCI) as a National Research Project under the grant number 2546.

References

1. Shuo-Yen R. Li, *Graphic sequences with unique realization*, J. Comb. Theory, Ser. (B) **19**, 42-46 (1975).

2. Street A. P., *Trades and defining sets*, the CRC Handbook of Combinatorial Designs (C.J. Colbourn and J.H. Dinitz, eds.), CRC Press, Boca Raton, 474-477, (1996).

3. West D. B., *Introduction to Graph Theory*, Prentice-Hall, Inc, (1996).

S. AKBARI, S_AKBARI@MATH.SHARIF.AC.IR
Department of Mathematical Sciences
Sharif University of Technology
P. O. BOX 11365-9415, Tehran , Iran
G. B. KHOSROVSHAHI, REZAGBK@IPM.AC.IR
Department of Mathematics, University of Tehran and
Institute for Studies in Theoretical Physics and Mathematics
P. O. Box 19395-5746, Tehran, Iran