Set-magic Labelings of Infinite Graphs

N. M. Singhi, G. R. Vijayakumar and N. Usha Devi

Abstract

A graph G without isolated vertices is said to be *set-magic* if its edges can be assigned distinct subsets of a set X such that for every vertex v of G, the union of the subsets assigned to the edges incident with v is X; such a set-assignment is called a *set-magic labeling* of G. In this note, we study infinite set-magic graphs and characterize infinite graphs G having set-magic labelings f such that $|f(e)| = 2$ for all $e \in E(G)$.

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Graphs considered in this note are without loops, multiple edges and isolated vertices. We refer to [4] for graph theoretic terminology. For set theoretic terminology and results, we refer to [2, 3, 6].

Let G be a graph. For any vertex v in G we denote its neighborhood by $N(v)$ and its degree by $\deg v$. If an edge e of G is incident with v, we sometimes write $e \sim v$. An edge with end-vertices x and y is denoted by xy. If an end-vertex of an edge e is of degree 1, then e is called a *pendant edge* of G. For any set X, let $P(X)$ be the set of all subsets of X and $P_0(X)$ be the set of all finite subsets of X.

Let $G = (V, E)$ be a graph and X be a set. A map $f : E \mapsto P(X)$ is said to be a *set-magic labeling* of G by X if f is injective and for all $v \in V$, $\bigcup_{e \sim v} f(e) = X$. A graph is said to be *set-magic* if it admits a set-magic labeling. This notion was introduced by Sedlaćek [5]. (His definition of set-magic labeling does not exclude isolated vertices; however graphs with isolated vertices which are set-magic according to his definition are trivial; any such graph can have at most one edge to which only the empty-set can be assigned.) He also has found out when a graph is set-magic. (His result—A graph G is set-magic if and only if G has at most one vertex of degree 1—is inaccurate. K_2 has two vertices of degree of 1; but it is set-magic.)

Theorem 1 *A graph G is set-magic if and only if it has at most one pendant edge.*

Proof. Let e be any edge of G. Assign $E(G)$ to e if it is a pendant edge; let $E(G) \setminus \{e\}$ be assigned to e otherwise. If G has at most one pendant edge, it is clear that this assignment is a set-magic labeling of G by $E(G)$.

Conversely, if G is set-magic, then obviously G has at most one pendant edge. ∎

The labeling given by the proof of Theorem 1 in the case of finite graphs G satisfies—as one may like to have—the following.

$$|f(e)| < \infty \quad \text{for all } e \in E(G). \qquad \cdots (1)$$

A natural question to ask is the following.

(A) When does an infinite graph G have a set-magic labeling f which satisfies (1)?

Sedlaćek has also constructed in [5] connected infinite graphs G with set-magic labelings f which satisfy (1). Such a labeling f for a complete graph, by its vertex-set can be given as follows: for any edge e, $f(e) = \{u, v\}$ where u and v are the end-vertices of e. (Finding infinite set-magic graphs G which cannot have set-magic labelings f for which (1) holds, is also not difficult; for example, it can be easily seen that no infinite path G has a set-magic labeling f which satisfies (1). A simple necessary condition for an infinite graph G to have a set-magic labeling f which satisfies (1) is that $\deg v$ is infinite for all $v \in V(G)$—see Lemma 5 for more details.)

In the case of infinite graphs G the labeling given by the proof of Theorem 1 satisfies the following. (Note that $E(G)$ is infinite, since G has no isolated vertices.)

$$|f(e)| \text{ is same for all } e \in E(G). \qquad \cdots (2)$$

It can be verified that a finite graph $G \neq K_2$ which has one pendant edge cannot have a set-magic labeling f by a finite set, which satisfies (2). (Of course it can have a set-magic labeling f by an infinite set, which satisfies (2). But this kind of labeling is quite uninteresting and unnatural.)

For infinite graphs G we can ask the following questions:

(B) When does G have a set-magic labeling f such that $|f(e)| = \eta$ for all $e \in E(G)$ where η is a positive integer?

(C) In particular, determine when G has a set-magic labeling f such that

$$|f(e)| = 2 \quad \text{for all } e \in E(G). \qquad \cdots (3)$$

In (C), if we replace the constant 2 by 1, then the answer is trivial: there is no such infinite graph. (Among finite graphs, K_2 is the only graph which has such a labeling.)

B. D. Acharya brought question (A) to the second author's attention when the latter was visiting Mehta Research Institute, Allahabad, India in 1984. This was settled in [7]. B. D. Acharya has also raised the following question in [1].

Determine the graphs (finite and infinite) which admit set-magic labelings f such that $|f(e)| = |f(e')|$ whenever e, e' are edges in the same (connected) component of G.

This question, as such, is answered by the preceding discussion; so instead of this, we can ask the following one:

(D) When does an infinite graph G have a set-magic labeling f which satisfies (1) such that $|f(e)| = |f(e')|$ whenever e, e' are edges in the same (connected) component of G?

In this note our aim is to settle the questions (B), (C) and (D). In the sequel, we need the following three set theoretic results.

Proposition 2 *If X is any infinite set, then $|P_0(X)| = |X|$.*

Proposition 3 *If $\{A_\alpha : \alpha \in I\}$ is a collection of finite sets where the indexing set I is infinite, then $\left|\bigcup_\alpha A_\alpha\right| \leq |I|$.*

Proposition 4 *If A and B are two sets such that one of them is infinite then $|A \cup B| = \max\{|A|, |B|\}$.*

For the proofs of Propositions 2, 3 and 4, the reader is referred to [2, 6].

The following result gives a necessary condition for an infinite graph G to have a set-magic labeling which satisfies (1). (This was proved in [7] as a part of its main theorem; here we prove it again for the sake of completeness.)

Lemma 5 *If an infinite graph G has a set-magic labeling f which satisfies* (1) *then*

$$\deg v = |V(G)| \quad \textit{for all } v \in V(G). \qquad \cdots (4)$$

Proof. Let f be a set-magic labeling of an infinite graph G by a set X, which satisfies (1). Then for any $v \in V(G)$,

$$\begin{aligned} \deg v &\leq |V(G)| \\ &= \Big|\bigcup_{xy \in E(G)} \{x, y\}\Big| \quad \text{(since } G \text{ has no isolated vertices)} \\ &\leq |E(G)| \quad \text{(by Proposition 3)} \\ &\leq |P_0(X)| \quad \text{(since } f(E(G)) \subseteq P_0(X) \text{ and } f \text{ is injective)} \end{aligned}$$

$$= |X| \quad \text{(by Proposition 2)}$$
$$= \left|\bigcup_{e\sim v} f(e)\right|$$
$$\leq \deg v \quad \text{(by Proposition 3).}$$

Thus (4) holds. ∎

Now let us state the main theorem of this note.

Theorem 6 *Any infinite graph G which satisfies* (4) *has a set-magic labeling f which satisfies* (3).

To prove this theorem, we need one more set theoretic result.

Lemma 7 *Any non-empty set X can be endowed with a well-ordering $\leq$ such that for all $a \in X$, $\left|\{x \in X : x < a\}\right| < |X|$.*

Proof. Let $\preccurlyeq$ be a well-ordering of X. For any $a \in X$, let $S_a = \{x \in X : x \prec a\}$. If $|S_a| < |X|$ for all $a \in X$, then $\preccurlyeq$ has the required property; so assume that $\{a \in X : |S_a| = |X|\}$ is non-empty; let α be the smallest element of this set. Since $|S_\alpha| = |X|$, there is a bijection $\theta : X \mapsto S_\alpha$. Now define a relation $\leq$ on X as follows: for any $x, y \in X$, $x \leq y \iff \theta(x) \preccurlyeq \theta(y)$. It is easy to see that $\leq$ is a well-ordering on X with the required property. ∎

Proof of Theorem 6. By Lemma 7, we can define a well-ordering $\leq$ on $V(G)$ such that

$$\left|\{x \in V(G) : x < a\}\right| < |V(G)| \quad \text{for all } a \in V(G). \qquad \cdots (5)$$

For any $v \in V(G)$, let $T(v) = \{x \in N(v) : x > v\}$ and $R(v) = \{x \in V(G) : x > v\}$. It is easy to see by Proposition 4 and (5) that $|T(v)| = |V(G)| = |R(v)|$; therefore there exist bijective maps $\phi_v : T(v) \mapsto V(G)$ and $\psi_v : V(G) \mapsto R(v)$. Let us denote for any $x \in T(v)$, its image under ϕ_v by $\phi(v,x)$ and for any $x \in V(G)$, its image under ψ_v by $\psi(v,x)$.

Now define a map $f : E(G) \mapsto P(V(G))$ as follows. For any $uv \in E(G)$ with $u < v$, $f(uv) = \{\phi(u,v), \psi(\phi(u,v),u)\}$. Let us show that f is a set-magic labeling of G.

Suppose $uv, xy \in E(G)$ such that $f(uv) = f(xy)$; assume that $u < v$ and $x < y$. Then by the construction of ψ,

$$\phi(u,v) < \psi(\phi(u,v),u) \quad \text{and} \quad \phi(x,y) < \psi(\phi(x,y),x).$$

Therefore

$$\phi(u,v) = \phi(x,y) \quad \text{and} \quad \psi(\phi(u,v),u) = \psi(\phi(x,y),x). \qquad \cdots (6)$$

Since $\psi_{\phi(u,v)}$ is injective, it follows from (6) that $u = x$; this implies again by (6) that $v = y$, since ϕ_u is injective. Thus it follows that f is injective.

Next let $u \in V(G)$. Let us show that $\bigcup_{e\sim u} f(e) = V(G)$. Let $w \in V(G)$. Since ϕ_u is surjective, for some $v \in T(u)$, $\phi(u,v) = w$ whence $w \in f(uv) \subseteq \bigcup_{e\sim u} f(e)$. Thus we have $\bigcup_{e\sim u} f(e) = V(G)$. This completes the proof. ∎

Now let us summarize what we have done so far:

Theorem 8 *For an infinite graph $G = (V, E)$, the following are equivalent.*

(i) *G has a set-magic labeling f such that $|f(e)| < \infty$ for all $e \in E$.*

(ii) *G has a set-magic labeling f such that $|f(e)| < \infty$ for all $e \in E$ and $|f(e)| = |f(e')|$ whenever e, e' are edges in the same connected component of G.*

(iii) *G has a set-magic labeling f such that $|f(e)| = \eta$ for all $e \in E$ where η is a positive integer.*

(iv) *G has a set-magic labeling f such that $|f(e)| = 2$ for all $e \in E$.*

(v) *For all $v \in V$, $\deg v = |V|$.*

Proof. Theorem 6 is (v) $\Rightarrow$ (iv). (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is obvious. Lemma 5 is (i) $\Rightarrow$ (v). ∎

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N. M. Singhi and G. R. Vijayakumar
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba
Mumbai 400 005
India

N. Usha Devi
Mannar Thirumalai Naickar College
Pasumalai
Madurai 625 004
Tamil Nadu
India