PERFECT $\langle k, r\rangle$-LATIN SQUARES.

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ABSTRACT. A **perfect $\langle k, r\rangle$-latin square** $A = (a_{i,j})$ of order n with m elements is an $n \times n$ array in which each element occurs in each row and column, and the element $a_{i,j}$ occurs either k times in row i and r times in column j, or occurs r times in row i and k times in column j. In 1989, Cai, Kruskal, Liu and Shen studied the existence of perfect $\langle k, r\rangle$-latin squares. Here, a simpler construction of perfect $\langle k, r\rangle$-latin squares is given.

1. Introduction and Definitions.

Definition 1 (Perfect $\langle k, r\rangle$-latin square). *A* ***perfect $\langle k, r\rangle$-latin square*** *$A = (a_{i,j})$ of order n with m elements is an $n \times n$ array in which each element occurs in each row and column, and the element $a_{i,j}$ occurs either k times in row i and r times in column j, or occurs r times in row i and k times in column j.*

It was shown in [1] that the above definition implies $m = \frac{n(k+r)}{2kr}$. Further, it was shown that for any k, r and h that there exists a perfect $\langle k, r\rangle$-latin square of order $n = 2hkr(k + r)$. The goal here is to show how to construct a perfect $\langle k, r\rangle$-latin square of order n in a simpler manner than the construction presented in [1]. First a few definitions are needed.

Definition 2 ($\langle k, r\rangle$-pattern). *A* ***$\langle k, r\rangle$-pattern*** *$A = (a_{i,j})$ of order n with m elements is an $n \times n$ array in which each element occurs in each row and column, and if the cell (i, j) is not empty, then $a_{i,j}$ occurs either k times in row i and r times in column j, or occurs r times in row i and k times in column j.*

Row and column permutations of $\langle k, r\rangle$-patterns and perfect $\langle k, r\rangle$-latin squares are $\langle k, r\rangle$-patterns and perfect $\langle k, r\rangle$-latin squares respectively. Note also that if a $\langle k, r\rangle$-pattern has no empty cells, then it is a perfect $\langle k, r\rangle$-latin square.

Definition 3 (Disjoint). *Two $n \times n$ arrays are said to be disjoint if, whenever cell (i, j) of one is non-empty, then in the other array this cell is empty.*

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The definition of disjoint $\langle k,r\rangle$-pattern will be useful, as disjoint $\langle k,r\rangle$-patterns can be combined together to form new $\langle k,r\rangle$-patterns. If enough $\langle k,r\rangle$-patterns can be combined together, then a perfect $\langle k,r\rangle$-latin square will result. For this, a means to combine $\langle k,r\rangle$-patterns is needed.

Definition 4 ("+"). *Let $A=(a_{i,j})$ and $B=(b_{i,j})$, be two disjoint $\langle k,r\rangle$-patterns, define $A+B=(c_{i,j})$ so that $c_{i,j}=a_{i,j}$ if cell (i,j) of A is not empty, $c_{i,j}=b_{i,j}$ if cell (i,j) of B is not empty, and if cell (i,j) of A and B are both empty, then so is cell (i,j) of $A+B$. (Notice as A and B are disjoint then $a_{i,j}$ and $b_{i,j}$ cannot both be non-empty.)*

Next, some special arrays will be studied. These arrays have the property that they can be combined together to form disjoint $\langle k,r\rangle$-patterns. These disjoint $\langle k,r\rangle$-patterns can be further combined together to form perfect $\langle k,r\rangle$-latin squares.

Definition 5 (Type H array). *Define $H(n,k,p,(i_1,\dots,i_\lambda))$ (where $\lambda|n$ and $k|n$) to be an $n\times n$ array, where i_u, $1\le u\le\lambda$, is in cell $(\lambda t+u, p+s+\lambda t+u)$ with $0\le t\le\frac{n}{\lambda}-1$, $0\le s\le k-1$ and empty otherwise (where addition is taken on the residues $1,2,\dots,n$).*

Definition 6 (Type V array). *Define $V(n,k,p,(i_1,\dots,i_\lambda))$ (where $\lambda|n$ and $k|n$) to be an $n\times n$ array, where i_u, $1\le u\le\lambda$, is in cell $(s+\lambda t+u, p+k+\lambda t+u-1)$ with $0\le t\le\frac{n}{\lambda}-1$, $0\le s\le k-1$ and empty otherwise (where addition is taken on the residues $1,2,\dots,n$).*

Pictorially, think of a type H array as a diagonal strip of thickness k which is composed of horizontal strips. A type V array is similar, but instead of horizontal strips in the diagonal, it has vertical strips.

It is worth noting that there are $k\frac{n}{\lambda}\lambda=kn$ non-empty cells in both the type V, and type H arrays.

Example 7. *The example of $H(12,2,0,(1,2,3))+V(12,2,6,(3,1,2))$ is given below:*

1	1					2	3					
	2	2					3	1				
		3	3					1	2			
			1	1					2	3		
				2	2					3	1	*type V array*
2					3	3					1	
2	3					1	1					
	3	1					2	2				
		1	2					3	3			
			2	3					1	1		
				3	1					2	2	*type H array*
3					1	2					3	

Observe that $H(12,2,0,(1,2,3))+V(12,2,6,(3,1,2))$ *is a* $\langle 2,1\rangle$*-pattern.*

This example shows a type V array and a type H array combine together to form a $\langle k,r\rangle$-pattern. These $\langle k,r\rangle$-patterns are important building blocks for perfect $\langle k,r\rangle$-latin squares.

2. A Perfect $\langle k,1\rangle$-latin square.

Here, an examination of a simpler problem, that of perfect $\langle k,1\rangle$-latin squares, will give an introduction to the techniques used for the problem of constructing perfect $\langle k,r\rangle$-latin squares. The idea of this section is to fit various type H and type V arrays together to get $\langle k,1\rangle$-patterns and then fit these $\langle k,1\rangle$-patterns together to get a perfect $\langle k,1\rangle$-latin square.

Notice that a $\langle k,1\rangle$-pattern must have the property that $2k(k+1)|n$ by the comments in the previous section. So by considering $n=2k(k+1)$ then the following is true.

Lemma 8.

$$H(2k(k+1),k,0,(0_1,0_2,...,0_{k+1}))+$$
$$V(2k(k+1),k,k(k+1),(0_{k+1},0_1,0_2,...,0_k))$$

is a $\langle k,1\rangle$*-pattern.*

Proof: Consider element 0_u, $1\le u\le k+1$. Element 0_u occurs in cells $((k+1)t+u,s+(k+1)t+u)$ of the type H array and cells $(s+(k+1)t+u+1,k(k+1)+k+(k+1)t+u)$ of the type V array, where $0\le s\le k-1$ and $0\le t\le 2k-1$. Next it is shown that this sum satisfies the conditions of a $\langle k,1\rangle$-pattern for the element 0_u.

0_u will occur in every row, as it occurs in rows $s+(k+1)t+u+1$, and also in rows $(k+1)t+u$, where $0\le s\le k-1$ and $0\le t\le 2k-1$. Similarly, 0_u will occur in every column.

Consider an element 0_u from the type H array in cell $(((k+1)t+u,s+(k+1)t+u)$. Then in this row, there will be k occurrences of 0_u; and in this column 1 occurrence of 0_u. The type V array cannot contribute to this row as $(k+1)t+u\not\equiv s+(k+1)t'+u+1 \pmod{k+1}$, for $0\le s\le k-1$ and $0\le t,t'\le 2k-1$. Further the type V array cannot contribute to this column as $k(k+1)+k+(k+1)t+u\not\equiv s+(k+1)t'+u \pmod{k+1}$ for $0\le s\le k-1$ and $0\le t,t'\le 2k-1$.

A similar argument can be made for the 0_u that are elements of a type V array.

Hence this is a $\langle k,1\rangle$-pattern.

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Corollary 9.

$$H(2k(k+1), k, p, (0_1, 0_2, ..., 0_{k+1})) +$$
$$V(2k(k+1), k, p+k(k+1), (0_{k+1}, 0_1, 0_2, ..., 0_k))$$

is a $\langle k, 1\rangle$-pattern.

Here it is worth observing that the non-empty cells of a type V array coincide with those of a type H array if and only if the arrays have the same n, k and p values. It follows that whether two arrays are disjoint depends only on the n, k and p values and not on the types of the arrays.

Lemma 10. *The array*

(1) $$H(2kr(k+r), k, ik, (i_1, ..., i_{k+r}))$$

is disjoint from

(2) $$H(2kr(k+r), k, jk, (j_1, ..., j_{k+r}))$$

for $i \neq j$ and $0 \leq i, j \leq k+r-1$.

Proof: This follows by noting that there are elements in cells $((k+r)t+u, ik+s+(k+r)t+u)$ for $0 \leq t \leq 2kr-1$, $0 \leq s \leq k-1$ in array 1 and in cells $((k+r)t+u, jk+s+(k+r)t+u)$ for $0 \leq t \leq 2kr-1$, $0 \leq s \leq k-1$ in array 2. So given a particular row, $(k+r)t+u$, t and u become fixed. As $i \neq j$, the columns of non-empty cells in this row are distinct. Hence these two arrays are disjoint.

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In the following theorem and the subsequent results it is convenient to increment i_1 etc. along with i so that, in the following case, the resulting array is defined on the symbols $\{0_1, ..., 0_{k+1}, 1_1, ..., 1_{k+1}, ..., k_1, ..., k_{k+1}\}$.

Theorem 11.

$$\sum_{i=0}^{k}(H(2k(k+1), k, ik, (i_1, i_2, ..., i_{k+1})) +$$
$$V(2k(k+1), k, k(k+1)+ik, (i_{k+1}, i_1, i_2, ..., i_k)))$$

is a perfect $\langle k, 1\rangle$-latin square.

Proof: Taking $r=1$ in Lemma 10, ensures that this sum is well defined. The number of non-empty cells in a type V, or type H array is $kn = 2k^2(k+1)$. As there are $k+1$ type V arrays, and $k+1$ type H arrays, there are $4k^2(k+1)^2$ cells covered in this sum (which is the total number of cells in the array). Lemma 8 gives that this is a $\langle k, 1\rangle$-pattern and since there are no empty cells it is a perfect $\langle k, 1\rangle$-latin square.

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3. A Perfect $\langle k, r\rangle$-latin square.

This section proceeds much the same as Section 2, but investigates the more general case of $\langle k, r\rangle$-patterns.

It can easily be seen that if k and r are not relatively prime, then the problem can be solved for k' and r', ($k' = \frac{k}{\gcd(k,r)}$, $r' = \frac{r}{\gcd(k,r)}$) and expanded to solve the problem for k and r by Kronecker products. (See Corollary 16 for a more detailed description of this.)

Lemma 12. *Assume k and r are relatively prime. Choose j_l such that $j_l \equiv -l \pmod{k+r}$, $j_l \equiv 0 \pmod{k}$, and $j_l \equiv l \pmod{r}$ for $0 \le l \le r-1$. Set $j_l' = j_l + rk(k+r)$. Then*

$$\sum_{l=0}^{r-1}(H(2kr(k+r), k, j_l, (0_{k+r-l+1}, 0_{k+r-l+2}, ..., 0_{k+r}, 0_1, 0_2, ..., 0_{k+r-l}))$$
$$+V(2kr(k+r), k, j_l', (0_{k+1}, 0_{k+2}, ..., 0_{k+r}, 0_1, 0_2, ..., 0_k)))$$

is a $\langle k, r\rangle$-pattern.

Proof: First, observe that such j_l can be chosen from the Chinese Remainder Theorem, as r, k and $r+k$ are relatively prime. Also Lemma 10 guarantees that this sum is well defined.

Consider an element 0_u. It lies in cell: $((k+r)t+u+l, j_l+s+(k+r)t+u+l)$ from the type H array, and $(s+(k+r)t+u+r, j_l'+k+(k+r)t+u+r-1)$ from the type V array, $0 \le s \le k-1$, $0 \le t \le 2kr-1$ and $0 \le l \le r-1$.

Consider row $(k+r)t+u+l$. This contains k occurrences of 0_u, namely in cells, $((k+r)t+u+l, j_l+s+(k+r)t+u+l)$, from the type H array. No type V array can contribute a 0_u to this row as $s'+(k+r)t'+u+r \not\equiv (k+r)t+u+l \pmod{2kr(k+r)}$, for $0 \le s' \le k-1$ and $0 \le t, t' \le 2kr-1$. A type H array with a different l value cannot contribute to this row as $(k+r)t+u+l \not\equiv (k+r)t'+u+l' \pmod{k+r}$, for $0 \le t, t' \le 2kr-1$.

Consider column $j_{l_0}+s+(k+r)t+u+l_0$. There are r occurrences of 0_u here, namely, in cell $((k+r)t_l+u+l, j_l+s+(k+r)t_l+u+l)$ from the type H array, for each l, and appropriate values of t_l. (Notice that $j_l+l \equiv 0 \pmod{k+r}$, hence t_l' can be chosen such that $j_l+s+(k+r)t_l+u+l = j_{l_0}+s+(k+r)t+u+l_0$.) The type V array does not contribute to this sum because $j_l+s+(k+r)t+u+l \not\equiv j_{l'}'+k+(k+r)t'+u-1'+r \pmod{k+r}$.

Similarly, checking rows $s+(k+r)t+u+r$ and columns $j_l'+k+(k+r)t+u-1$ gives that 0_u occurs r and k times respectively.

Lastly, 0_u will occur in every row. To see this, consider a row a. If $a-u \equiv 0, 1, ..., r-1 \pmod{k+r}$, then it will be found in a type H array of row value $(k+r)t+u+l$. If $a-u \equiv r, r+1, ..., r+k-1 \pmod{k+r}$, then it will be found in a type V array of row value $s+(k+r)t+u+r$ with $s \equiv a-u-r \pmod{k+r}$. Similarly, it can be shown that 0_u will occur in every column.

Hence this is a $\langle k, r \rangle$-pattern.

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Corollary 13. *Assume that k and r are relatively prime. Select j_l as before. Then*

$$\sum_{l=0}^{r-1}(H(2kr(k+r),k,p+j_l,(0_{k+r-l+1},0_{k+r-l+2},...,0_{k+r},0_1,0_2,...,0_{k-l})) \\ +V(2kr(k+r),k,p+j'_l,(0_{k+1},0_{k+2},...,0_{k+r},0_1,0_2,...,0_k)))$$

is a $\langle k, r \rangle$-pattern.

The following theorem is equivalent to Theorem 3.1 of [1].

Theorem 14. *Assume k and r are relatively prime. Pick j_l, j'_l as above. Then:*

$$\sum_{i=0}^{k+r-1}\sum_{l=0}^{r-1}(H(2kr(k+r),k,ikr+j_l,(i_{k+r-l+1},i_{k+r-l+2},...,i_{k+r}, \\ i_1,i_2,...,i_{k+r-l}))+V(2kr(k+r),k,ikr+j'_l,(i_{k+1},i_{k+2},...,i_{k+r}, \\ i_1,i_2,...,i_k)))$$

is a perfect $\langle k, r \rangle$-latin square.

Proof: From Lemma 10 this sum is well defined.

There are $r(k+r)$ type H arrays, and $r(k+r)$ type V arrays. There are $2kr(k+r)k$ cells covered by each of the type H and type V arrays. Hence there are $4k^2r^2(k+r)^2$ cells covered in total, which is equal to the total number of cells in the $2kr(k+r) \times 2kr(k+r)$ array. Lemma 12 gives that this is a $\langle k, r \rangle$-pattern and since there are no empty cells it is a perfect $\langle k, r \rangle$-latin square.

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Example 15. *Consider $k = 3$, $r = 2$. Then $j_0 = 0$, $j_1 = 4$, $j'_0 = 10$ and $j'_1 = 14$. So taking i_u to be the $i(k+r)+u$ letter of the alphabet, the perfect*

$\langle k,r\rangle$*-latin square above is realized as:*

```
a a a v w x f f f e e e k k k j j j p p p o o o u u u t t t b c d y y y g h i b c d l m n g h i q r s l m n v w x q r s
t b b b w x y g g g a a a l l l f f f q q q k k k v v v p p p c d e u u u h i j c d e m n o h i j r s t m n o w x y r s
t p c c c x y u h h h b b b m m m g g g r r r l l l w w w q q q d e a v v v i j f d e a n o k i j f s t p n o k x y u s
t p q d d d y u v i i i c c c n n n h h h s s s m m m x x x r r r e a b w w w j f g e a b o k l j f g t p q o k l y u v
w p q r e e e u v w j j j d d d o o o i i i t t t n n n y y y s s s a b c x x x f g h a b c k l m f g h p q r k l m u v
w x q r s a a a v w x f f f e e e k k k j j j p p p o o o u u u t t t b c d y y y g h i b c d l m n g h i q r s l m n v
w x y r s t b b b w x y g g g a a a l l l f f f q q q k k k v v v p p p c d e u u u h i j c d e m n o h i j r s t m n o
k x y u s t p c c c x y u h h h b b b m m m g g g r r r l l l w w w q q q d e a v v v i j f d e a n o k i j f s t p n o
k l y u v t p q d d d y u v i i i c c c n n n h h h s s s m m m x x x r r r e a b w w w j f g e a b o k l j f g t p q o
k l m u v w p q r e e e u v w j j j d d d o o o i i i t t t n n n y y y s s s a b c x x x f g h a b c k l m f g h p q r
s l m n v w x q r s a a a v w x f f f e e e k k k j j j p p p o o o u u u t t t b c d y y y g h i b c d l m n g h i q r
s t m n o w x y r s t b b b w x y g g g a a a l l l f f f q q q k k k v v v p p p c d e u u u h i j c d e m n o h i j r
s t p n o k x y u s t p c c c x y u h h h b b b m m m g g g r r r l l l w w w q q q d e a v v v i j f d e a n o k i j f
g t p q o k l y u v t p q d d d y u v i i i c c c n n n h h h s s s m m m x x x r r r e a b w w w j f g e a b o k l j f
g h p q r k l m u v w p q r e e e u v w j j j d d d o o o i i i t t t n n n y y y s s s a b c x x x f g h a b c k l m f
g h i q r s l m n v w x q r s a a a v w x f f f e e e k k k j j j p p p o o o u u u t t t b c d y y y g h i b c d l m n
o h i j r s t m n o w x y r s t b b b w x y g g g a a a l l l f f f q q q k k k v v v p p p c d e u u u h i j c d e m n
o k i j f s t p n o k x y u s t p c c c x y u h h h b b b m m m g g g r r r l l l w w w q q q d e a v v v i j f d e a n
o k l j f g t p q o k l y u v t p q d d d y u v i i i c c c n n n h h h s s s m m m x x x r r r e a b w w w j f g e a b
c k l m f g h p q r k l m u v w p q r e e e u v w j j j d d d o o o i i i t t t n n n y y y s s s a b c x x x f g h a b
c d l m n g h i q r s l m n v w x q r s a a a v w x f f f e e e k k k j j j p p p o o o u u u t t t b c d y y y g h i b
c d e m n o h i j r s t m n o w x y r s t b b b w x y g g g a a a l l l f f f q q q k k k v v v p p p c d e u u u h i j
f d e a n o k i j f s t p n o k x y u s t p c c c x y u h h h b b b m m m g g g r r r l l l w w w q q q d e a v v v i j
f g e a b o k l j f g t p q o k l y u v t p q d d d y u v i i i c c c n n n h h h s s s m m m x x x r r r e a b w w w j
f g h a b c k l m f g h p q r k l m u v w p q r e e e u v w j j j d d d o o o i i i t t t n n n y y y s s s a b c x x x
y g h i b c d l m n g h i q r s l m n v w x q r s a a a v w x f f f e e e k k k j j j p p p o o o u u u t t t b c d y y
u u h i j c d e m n o h i j r s t m n o w x y r s t b b b w x y g g g a a a l l l f f f q q q k k k v v v p p p c d e u
v v v i j f d e a n o k i j f s t p n o k x y u s t p c c c x y u h h h b b b m m m g g g r r r l l l w w w q q q d e a
b w w w j f g e a b o k l j f g t p q o k l y u v t p q d d d y u v i i i c c c n n n h h h s s s m m m x x x r r r e a
b c x x x f g h a b c k l m f g h p q r k l m u v w p q r e e e u v w j j j d d d o o o i i i t t t n n n y y y s s s a
b c d y y y g h i b c d l m n g h i q r s l m n v w x q r s a a a v w x f f f e e e k k k j j j p p p o o o u u u t t t
p c d e u u u h i j c d e m n o h i j r s t m n o w x y r s t b b b w x y g g g a a a l l l f f f q q q k k k v v v p p
q q d e a v v v i j f d e a n o k i j f s t p n o k x y u s t p c c c x y u h h h b b b m m m g g g r r r l l l w w w q
r r r e a b w w w j f g e a b o k l j f g t p q o k l y u v t p q d d d y u v i i i c c c n n n h h h s s s m m m x x x
y s s s a b c x x x f g h a b c k l m f g h p q r k l m u v w p q r e e e u v w j j j d d d o o o i i i t t t n n n y y
u u t t t b c d y y y g h i b c d l m n g h i q r s l m n v w x q r s a a a v w x f f f e e e k k k j j j p p p o o o u
v v v p p p c d e u u u h i j c d e m n o h i j r s t m n o w x y r s t b b b w x y g g g a a a l l l f f f q q q k k k
l w w w q q q d e a v v v i j f d e a n o k i j f s t p n o k x y u s t p c c c x y u h h h b b b m m m g g g r r r l l
m m x x x r r r e a b w w w j f g e a b o k l j f g t p q o k l y u v t p q d d d y u v i i i c c c n n n h h h s s s m
n n n y y y s s s a b c x x x f g h a b c k l m f g h p q r k l m u v w p q r e e e u v w j j j d d d o o o i i i t t t
p o o o u u u t t t b c d y y y g h i b c d l m n g h i q r s l m n v w x q r s a a a v w x f f f e e e k k k j j j p p
q q k k k v v v p p p c d e u u u h i j c d e m n o h i j r s t m n o w x y r s t b b b w x y g g g a a a l l l f f f q
r r r l l l w w w q q q d e a v v v i j f d e a n o k i j f s t p n o k x y u s t p c c c x y u h h h b b b m m m g g g
h s s s m m m x x x r r r e a b w w w j f g e a b o k l j f g t p q o k l y u v t p q d d d y u v i i i c c c n n n h h
i i t t t n n n y y y s s s a b c x x x f g h a b c k l m f g h p q r k l m u v w p q r e e e u v w j j j d d d o o o i
j j j p p p o o o u u u t t t b c d y y y g h i b c d l m n g h i q r s l m n v w x q r s a a a v w x f f f e e e k k k
l f f f q q q k k k v v v p p p c d e u u u h i j c d e m n o h i j r s t m n o w x y r s t b b b w x y g g g a a a l l
m m g g g r r r l l l w w w q q q d e a v v v i j f d e a n o k i j f s t p n o k x y u s t p c c c x y u h h h b b b m
n n n h h h s s s m m m x x x r r r e a b w w w j f g e a b o k l j f g t p q o k l y u v t p q d d d y u v i i i c c c
d o o o i i i t t t n n n y y y s s s a b c x x x f g h a b c k l m f g h p q r k l m u v w p q r e e e u v w j j j d d
e e k k k j j j p p p o o o u u u t t t b c d y y y g h i b c d l m n g h i q r s l m n v w x q r s a a a v w x f f f e
a a a l l l f f f q q q k k k v v v p p p c d e u u u h i j c d e m n o h i j r s t m n o w x y r s t b b b w x y g g g
h b b b m m m g g g r r r l l l w w w q q q d e a v v v i j f d e a n o k i j f s t p n o k x y u s t p c c c x y u h h
i i c c c n n n h h h s s s m m m x x x r r r e a b w w w j f g e a b o k l j f g t p q o k l y u v t p q d d d y u v i
j j j d d d o o o i i i t t t n n n y y y s s s a b c x x x f g h a b c k l m f g h p q r k l m u v w p q r e e e u v w
x f f f e e e k k k j j j p p p o o o u u u t t t b c d y y y g h i b c d l m n g h i q r s l m n v w x q r s a a a v w
x y g g g a a a l l l f f f q q q k k k v v v p p p c d e u u u h i j c d e m n o h i j r s t m n o w x y r s t b b b w
x y u h h h b b b m m m g g g r r r l l l w w w q q q d e a v v v i j f d e a n o k i j f s t p n o k x y u s t p c c c
d y u v i i i c c c n n n h h h s s s m m m x x x r r r e a b w w w j f g e a b o k l j f g t p q o k l y u v t p q d d
e e u v w j j j d d d o o o i i i t t t n n n y y y s s s a b c x x x f g h a b c k l m f g h p q r k l m u v w p q r e
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The following Corollary is equivalent to Corollary 3.3 of [1].

Corollary 16. *For k and r, Let $g = \gcd(k,r)$, and $k' = k/g$ and $r' = r/g$. Then there exists a perfect $\langle k,r\rangle$-latin square of order $N = 2hgk'r'(k'+r')$*

Proof: Consider a perfect $\langle k',r'\rangle$-latin square A, of order $2k'r'(k'+r')$ (by Theorem 14). Taking the Kronecker product of A with $B = (b_{i,j})_{j,i=1}^{g}$,