k-Geodomination in Graphs

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ABSTRACT

For an integer $k \geq 1$, a vertex v of a graph G is k-geodominated by a pair x, y of vertices in G if $d(x, y) = k$ and v lies on an $x - y$ geodesic of G. A set S of vertices of G is a k-geodominating set if each vertex v in $V - S$ is k-geodominated by some pair of distinct vertices of S. The minimum cardinality of a k-geodominating set of G is its k-geodomination number $g_k(G)$. A vertex v is openly k-geodominated by a pair x, y of distinct vertices in G if v is k-geodominated by x and y and $v \neq x, y$. A vertex v in G is a k-extreme vertex if v is not openly k-geodominated by any pair of vertices in G. A set S of vertices of G is an open k-geodominating set of G if for each vertex v of G, either (1) v is k-extreme and $v \in S$ or (2) v is openly k-geodominated by some pair of distinct vertices of S. The minimum cardinality of an open k-geodominating set in G is its open k-geodomination number $og_k(G)$. It is shown that each triple a, b, k of integers with $2 \leq a \leq b$ and $k \geq 2$ is realizable as the geodomination number and k-geodomination number of some tree. For each integer $k \geq 1$, we show that a pair (a, n) of integers is realizable as the k-geodomination number (open k-geodomination number) and order of some nontrivial connected graph if and only of $2 \leq a = n$ or $2 \leq a \leq n - k + 1$. We investigate how k-geodomination numbers are affected by adding a vertex. We show that if G is a nontrivial connected graph of diameter d with exactly ℓ k-extreme vertices, then $\max\{2, \ell\} \leq g_k(G) \leq og_k(G) \leq 3g_k(G) - 2\ell$ for every integer k with $2 \leq k \leq d$.

Key Words: geodomination, open geodomination, k-geodomination.

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1 Introduction

For vertices x and y in a connected graph $G = (V, E)$, the *distance* $d(x, y)$ is the length of a shortest $x-y$ path in G. The *diameter* of G is the maximum distance between any two vertices of G and is denoted by $\operatorname{diam} G$. An $x-y$ path of length $d(x, y)$ is called an $x-y$ *geodesic*. A vertex v is said to *lie in* an $x-y$ geodesic P if v is an internal vertex of P, that is, v is a vertex of P distinct from x and y. We refer to the book [4] for graph theory notation and terminology not described here. The *closed interval* $I[x, y]$ consists of x, y, and all vertices lying in some $x-y$ geodesic of G, while for $S \subseteq V$,

$$I[S] = \bigcup_{x,y \in S} I[x, y].$$

A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set.

The closed intervals in a connected graph G were studied and characterized by Nebeský [9, 10] and were also investigated extensively in the book by Mulder [7], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. The geodetic number of a graph was introduced in [1, 6] and further studied in [2]. It was shown in [6] that determining the geodetic number of a graph is an NP-hard problem. Closed intervals and geodetic numbers of digraphs were introduced and studied in [5]. Geodetic concepts were first studied from the point of view of domination by Chartrand, Harary, Swart, and Zhang in [3], where a pair x, y of vertices in a nontrivial connected graph G is said to *geodominate* a vertex v of G if $v \in I[x, y]$, that is, if either $v \in \{x, y\}$ or v lies in an $x-y$ geodesic of G. In [3], geodetic sets and the geodetic number were referred to as *geodominating sets* and *geodomination number* and it is this terminology that we adopt in this paper.

The *link* of a vertex v is the subgraph induced by its neighborhood. A vertex with a complete link is called *link-complete* (or *extreme*). In particular, every end-vertex in G is link-complete. Obviously, every link-complete vertex in a graph belongs to every geodominating set. In fact, if v is a link-complete vertex that lies on an $x-y$ geodesic, then $x = v$ or $y = v$. Hence every geodominating set of a graph contains all of its link-complete vertices. In [3], a pair x, y of distinct vertices of a graph G is said to *openly geodominate* a vertex v if v lies in an $x-y$ geodesic in G. A set S is an *open geodominating set* of G if for each vertex v, either (1) v is link-complete and $v \in S$, or (2) v is openly geodominated by some pair of vertices of S. An open geodominating set of minimum cardinality is an *og-set*, and this cardinality is the *open geodomination number* $og(G)$. The following observation appeared in [3].

Observation 1.1 *Every geodominating set of a graph G contains every link-complete vertex of G. In particular, if the set W of link-complete vertices is a geodominating set of G, then W is the unique g-set and the unique og-set of G and so $g(G) = og(G) = |W|$.*

For a graph G and an integer $k \geq 1$, a vertex v of G is *k-geodominated* by a pair x, y of distinct vertices in G if v is geodominated by x, y and $d(x, y) = k$. A set S of vertices of G is a *k-geodominating set* of G if each vertex v in $V - S$ is k-geodominated by some pair of distinct vertices of S. The minimum cardinality of a k-geodominating set of G is its *k-geodomination number* $g_k(G)$. A k-geodomination set of cardinality $g_k(G)$ is called a g_k-set of G. If a vertex v is k-geodominated by a pair x, y of vertices in G and $v \neq x, y$, then v is said to be *openly k-geodominated* by x and y. A vertex v is a *k-extreme vertex* if v is not openly k-geodominated by any pair of distinct vertices of G. A set S of vertices of G is an *open k-geodominating set* if for each vertex v of G, either (1) v is k-extreme and $v \in S$, or (2) v is openly k-geodominated by some pair of distinct vertices of S. The minimum cardinality of an open k-geodominating set of G is its *open k-geodomination number* $og_k(G)$. An open k-geodominating set of cardinality $og_k(G)$ is an *og_k-set*. The following observation is useful.

Observation 1.2 *For an integer $k \geq 1$, every k-geodominating set of a graph G contains every k-extreme vertex of G. In particular, if the set W of k-extreme vertices is a k-geodominating set of G, then W is the unique g_k-set and the unique og_k-set of G and so $g_k(G) = og_k(G) = |W|$.*

Consider the graph G of Figure 1, which has diameter 6. The vertices u and v are the only two vertices in G with $d(u, v) = 6$. Since the vertex x does not lie in the $u - v$ geodesic in G, it follows that x is a 6-extreme vertex of G. Moreover, the set $S = \{u, v, w, x\}$ of all 6-extreme vertices of G is a 6-geodominating set and so $g_6(G) = og_6(G) = 4$ by Observation 1.2.

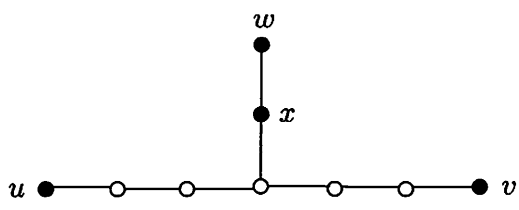

Figure 1: An example to illustrate Observation 1.2

For every graph G, the vertex set V is always both a k-geodominating set and an open k-geodominating set of G for all $k \geq 1$. Hence $g_k(G)$

and $og_k(G)$ exist for every graph G. Moreover, if G is a connected graph of order n and diameter d, where $d < k$, then every vertex of G is k-extreme. Thus V is the only k-geodominating set of G as well as the only open k-geodominating set in G, implying that $g_k(G) = og_k(G) = n$ for all $k > \text{diam}\, G$. Certainly, every open geodominating set is a geodominating set and, for each $k \geq 1$, every k-geodominating set is a geodominating set. These observations give the following.

Proposition 1.3 *Let G be a graph of order $n \geq 2$ and let $k \geq 1$ an integer. Then*

(a) $g(G) \leq og(G)$ *and* $g_k(G) \leq og_k(G)$,

(b) $g(G) \leq g_k(G)$ *and* $og(G) \leq og_k(G)$, *and*

(c) $2 \leq g(G), og(G), g_k(G), og_k(G) \leq n$.

These concepts are illustrated using the graph G of Figure 2a, where the solid circles in the graph indicate a g-set in Figure 2b, an og-set in Figure 2c, a g_2-set in Figure 2d, and an og_2-set in Figure 2e. Hence, $g(G) = 2$, $og(G) = 4$, $g_2(G) = 3$, and $og_2(G) = 5$. Thus, these four geodomination parameters are distinct for the graph G of Figure 2a.

Obviously, a geodominating set of a disconnected graph is the union of geodominating sets of its components. Hence it suffices to consider connected graphs only.

2 Geodomination and k-Geodomination

Certainly, if a connected G is not complete, then $\text{diam}\, G \geq 2$. Moreover, if G is a connected graph of order n and diameter 2, then $g_k(G) = n$ for all k with $k \geq 1$ and $k \neq 2$ by Observation 1.2. We show next that $g_2(G) = g(G)$ for every a connected graph G of diameter 2.

Proposition 2.1 *If G is a connected graph of diameter 2, then $g_2(G) = g(G)$.*

Proof. Since $g(G) \leq g_2(G)$ by Proposition 1.3, we need only show that $g_2(G) \leq g(G)$, that is, that every g-set in G is a g_2-set. Let S be a g-set of G. If $S = V$, then $g(G) = g_2(G) = |V|$ by Proposition 1.3. So we may assume that $S \neq V$. Let $v \in V - S$. Since S is a g-set, it follows that v is geodominated by some $x, y \in S$. So v lies in an $x-y$ geodesic in G. Because $\text{diam}\, G = 2$, it follows that $d(x, y) = 2$. Thus S is a 2-geodominating set of G and so S is a g_2-set by Proposition 1.3. ■

Certainly, the condition that the graph G in Proposition 2.1 has diameter 2 is necessary as the graph G of Figure 2 shows. Also, note that the converse of Proposition 2.1 is not true. For example, in the graph G of

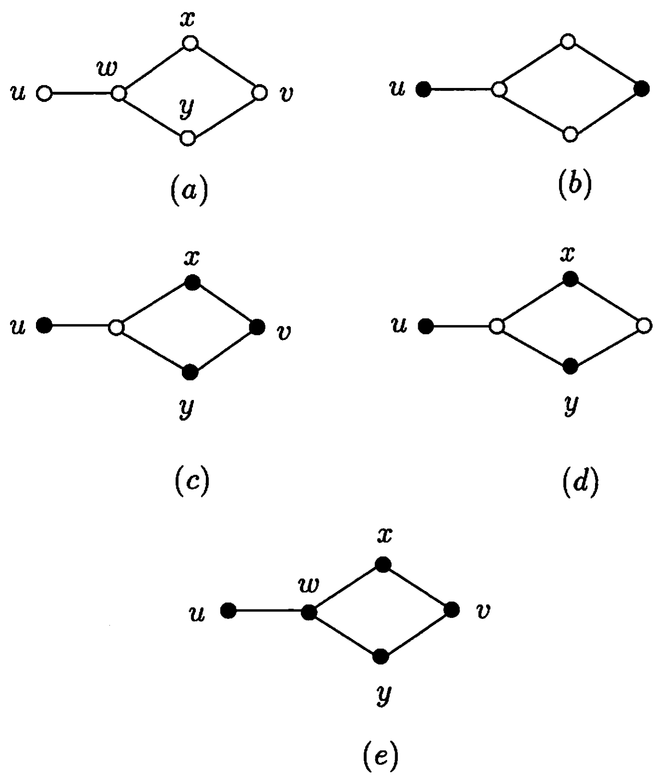

Figure 2: Geodomination parameters

Figure 3, the set consisting of the four end-vertices of G is a g-set as well as a g_2-set. Thus $g_2(G) = g(G)$, but $\operatorname{diam} G = 3$. This example can be extended to show that, for each integer $d \geq 1$, there exists a connected graph G of diameter d such that $g_2(G) = g(G)$.

By Proposition 2.1, if G is a connected graph of diameter 2, then $g_2(G) = g(G)$. However, in general, $g_k(G) \neq g(G)$ for an integer k with $2 \leq k \leq \operatorname{diam} G$. As an example, we consider those connected graphs with diameter at least 3 and geodomination number 2.

Proposition 2.2 *Let G be a connected graph of order $n \geq 3$, with* $\operatorname{diam} G \geq 3$ *and $g(G) = 2$ and let $k \geq 1$ be an integer. Then $g_k(G) = g(G)$ if and only if $k =$* $\operatorname{diam} G$.

Proof. Let $\operatorname{diam} G = d$ and let $S = \{x, y\}$ be a g-set of G. Then x and y are antipodal, that is, $d(x, y) = d$. Thus S is a d-geodominating set and

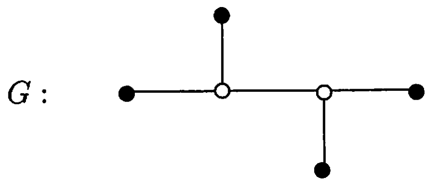

Figure 3: A graph G of diameter 3 with $g_2(G) = g(G)$

so $g_d(G) = 2$. It remains to verify the converse. If $k = 1$ or $k > d$, then V is the only geodominating set of G and so $g_k(G) = n \geq 3 > g(G)$. So we assume that $2 \leq k \leq d-1$. Note that every k-geodominating set of G is also a geodominating set. Since $g(G) = 2$, every g-set S of G contains two antipodal vertices and so S is a d-geodominating set. Since $k < d$, it follows that S is not a k-geodominating set. Thus no 2-element subset of V is a k-geodominating set. Therefore, $g_k(G) \geq 3 > g(G)$ for all $2 \leq k \leq d-1$. ∎

We have seen that $2 \leq g(G) \leq g_k(G)$ for every connected graph G and every integer $k \geq 1$. If $k = 1$, then $g(G) = g_k(G) = |V|$ for every connected graph G. Thus if a and b are integers with $2 \leq a < b$, then there exists no graph G with $g(G) = a$ and $g_1(G) = b$. On the other hand, next we show that for $k \geq 2$, every pair a, b of integers with $2 \leq a \leq b$ is realizable as the geodomination number and k-geodomination number for some tree. In order to do this, we first state the k-geodomination numbers of paths P_n of order $n \geq 3$ for all $k \geq 1$. We omit the proof since it is routine.

Lemma 2.3 *Let P_n be a path of order $n \geq 3$. Then $g_1(P_n) = n$ and*

$$g_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

For each integer k with $3 \leq k \leq n-2$,

$$g_k(P_n) = \left\lfloor \frac{n}{k} \right\rfloor + \ell,$$

where

$$\ell = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{k} \\ 2 & \text{if } n \equiv 0, 2 \pmod{k} \\ 3 & \text{otherwise.} \end{cases}$$

Moreover, $g_{n-1}(P_n) = 2$ and $g_k(P_n) = n$ for all $k \geq n$.

Theorem 2.4 *Let $k \geq 2$ be an integer. For each pair a, b of integers with $2 \leq a \leq b$, there exists a tree T with $g(T) = a$ and $g_k(T) = b$.*

Proof. First, assume that $b = a$. For $k = 2$, let T be the star $K_{1,a}$. Since $\operatorname{diam} T = 2$, it then follows from Proposition 2.1 that $g_2(T) = g(T) = a$. So we may assume that $k \geq 3$. Let T be obtained from the path P_k : $v_1, v_2, \cdots, v_k$ by joining the $a-1$ new vertices $u_1, u_2, \cdots, u_{a-1}$ to v_1. Since the set $\{v_k, u_1, u_2, \cdots, u_{a-1}\}$ of the end-vertices is a k-geodominating set of T, we obtain $g(T) = g_k(T) = a$ by Observations 1.1 and 1.2. Next we assume that $a < b$. We consider two cases.

Case 1. $b = a + 1$. For each i with $1 \leq i \leq a$, let $T_i : v_{i1}, v_{i2}, \cdots, v_{ik}$ be a copy of the path P_k of order k. Then the tree T is obtained from the trees T_i $(1 \leq i \leq a)$ by joining a new vertex v to each vertex v_{i1} for all $1 \leq i \leq a$. Let $S = \{v_{1k}, v_{2k}, \cdots, v_{ak}\}$ be the set of the end-vertices of T. Then $g(T) = |S| = a$ by Observation 1.1. Certainly, S is not a k-geodominating set of T as $d(x, y) = 2k$ for all distinct $x, y \in S$. Since $S' = S \cup \{v\}$ is a k-geodominating set of T, it follows that S' is a g_k-set of T. Thus $g_k(T) = a + 1 = b$.

Case 2. $b = a + j$ for some $j \geq 2$. By Lemma 2.3, the path P_{jk} : $v_1, v_2, \cdots, v_{jk}$ of order jk has the k-geodomination number $j + 2$. If $a = 2$, then $g(P_{jk}) = 2$ and so P_{jk} has the desired property. So we assume that $a \geq 3$. Let T be obtained from P_{jk} by joining the $a - 2$ new vertices $u_1, u_2, \cdots, u_{a-2}$ to v_2. Again, T has a end-vertices and so $g(T) = a$. Moreover, $g_k(T) = (j + 2) + (a - 2) = j + a = b$. ∎

It was shown in [2] that for each pair a, n of integers with $2 \leq a \leq n$, there exists a connected graph G of order n and geodomination number a. However, it is not true for k-geodomination number. Next we determine, for each integer $k \geq 1$ those pairs (a, n) of positive integers that are realizable as the k-geodomination number and order of some nontrivial connected graph.

Theorem 2.5 *Let $k \geq 1$ be an integer. A pair (a, n) of integers is realizable as the k-geodomination number and order of some nontrivial connected graph if and only if $2 \leq a = n$ or $2 \leq a \leq n - k + 1$.*

Proof. We first show that if G is a connected graph of order $n \geq 2$, then either $g_k(G) = n$ or $2 \leq g_k(G) \leq n - k + 1$. Assume that $g_k(G) < n$. Let S be a g_k-set of G. Thus $|S| < n$. So there is a vertex $v \in V - S$ such that v is k-geodominated by some pair x, y of distinct vertices in S, where $x \neq v$ and $y \neq v$. Hence v lies in an $x - y$ geodesic $P : x = v_0, v_1, \cdots, v_k = y$ in G, where $k \geq 2$. Since then $S' = V - \{v_1, v_2, \cdots, v_{k-1}\}$ is a k-geodominating set of G, it follows that $g_k(G) \leq |S'| = n - k + 1$.

For the converse, we show that for every pair (a, n) of integers such that either $2 \leq a = n$ or $2 \leq a \leq n - k + 1$, there is a connected graph G of order n with $g_k(G) = a$. For $a = n \geq 2$, the complete graph K_n has the desired properties. So we may assume that $2 \leq a \leq n-k+1$. Let G be the graph obtained from the path $P_{k+1} : v_0, v_1, \cdots, v_k$ of order $k+1$ by (1) adding the $a-2$ new vertices $u_1, u_2, \cdots, u_{a-2}$ and joining each of u_i $(1 \leq i \leq a-2)$ to v_{k-1} and (2) adding the $n-k-a+1$ new vertices $w_1, w_2, \cdots, w_{n-k-a+1}$ and joining each of w_j $(1 \leq j \leq n-k-a+1)$ to both v_1 and v_3. The graph G is shown in Figure 4. Then the order of G is n. Since the set $\{v_0, v_k, u_1, u_2, \cdots, u_{a-2}\}$ of k-extreme vertices of G is a k-geodominating set, it follows that $g_k(G) = a$. ■

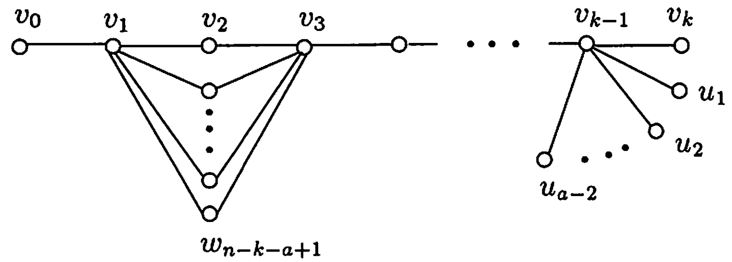

Figure 4: A graph G of order n with $g_k(G) = a$

Arguments similar to the ones used in the proofs of Theorems 2.4 and 2.5 give the realization results for the geodomination number and open k-geodomination number of a graph for all $k \geq 2$. So we only state these facts in the next two results and omit their proofs.

Theorem 2.6 *Let $k \geq 2$ be an integer. For each pair a, b of integers with $2 \leq a \leq b$, there exists a tree T with $g(T) = a$ and $og_k(T) = b$.*

Theorem 2.7 *Let $k \geq 1$ be an integer. A pair (a, n) of integers is realizable as the open k-geodomination number and order of some nontrivial connected graph if and only if $2 \leq a = n$ or $2 \leq a \leq n-k+1$.*

3 How k-Geodomination Numbers Are Affected by Adding a Vertex

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. So we consider how the k-geodomination number of a connected graph is affected by the addition of a single vertex (at least one edge incident with this vertex). It was shown in [8] that if a pendant edge is added to a connected graph, then the geodomination number of the resulting graph can stay the same or increase by at most 1, but cannot decrease. We state this fact as follows.

Theorem 3.1 *If G' is a graph obtained by adding a pendant edge to a connected graph G, then $g(G) \leq g(G') \leq g(G) + 1$.*

We now consider how the k-geodomination number of a connected graph is affected by the addition of a pendant edge. Let G' be the graph obtained from a connected graph G by adding a pendant edge uv, where u is not a vertex of G and v is a vertex of G. If S is a g_k-set of G, then $S \cup \{u\}$ is a k-geodominating set of G', implying that $g_k(G') \leq |S \cup \{u\}| = g_k(G) + 1$. Thus, we obtain an upper bound for $g_k(G')$ in terms of $g_k(G)$ similar to the one described in Theorem 3.1.

Proposition 3.2 *If G' is a graph obtained by adding a pendant edge to a connected graph G, then $g_k(G') \leq g_k(G) + 1$.*

Note that it is possible that $g_k(G') = g_k(G) + i$ for each $i \in \{0, 1\}$ in Proposition 3.2. To illustrate this fact, consider the graphs G, G', and G'' of Figure 5, where G' is obtained from G by adding the pendant edge v_3x and G'' is obtained from G' by adding the pendant edge xy. The solid circles in each of the graphs G, G', and G'' indicate a g_k-set in that graph. Hence $g_k(G) = a$, $g_k(G') = g_k(G)+1 = a+1$, and $g_k(G'') = g_k(G') = a+1$.

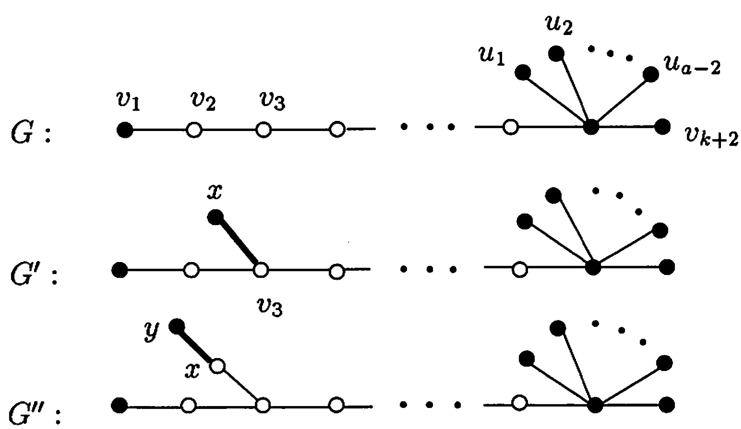

Figure 5: Graphs G, G', and G'' with $g_k(G) = a$, $g_k(G') = a + 1$, and $g_k(G'') = a + 1$

We have seen in Theorem 3.1 that if G' is a graph obtained by adding a pendant edge to a connected graph G, then $g(G) \leq g(G')$. However, in general, it is not true for k-geodomination number. In fact, the addition of a pendant edge to a connected graph G can produce a graph whose k-geodomination number is strictly smaller than that of G. We saw that if G is a connected graph of order n and diameter d and $k > d$ is an integer, then every vertex of G is k-extreme and so $g_k(G) = n$. However, adding a pendant edge to G may increase the diameter of the graph and then decrease the k-geodomination number. For example, let $G = C_{2p} : v_1, v_2, \cdots, v_{2p}, v_1$ for some integer $p \geq 2$. Since $\operatorname{diam} G = p$, it follows that $g_{p+1}(G) = 2p$. Let G' be the graph obtained from G by adding the pendant edge uv_{p+1}. Then $\operatorname{diam} G' = p + 1$ and v_1 and u are antipodal vertices of G'. Since the set $\{v_1, u\}$ is a $(p+1)$-geodominating set in G', it follows that $g_{p+1}(G') = 2$. Moreover, in the case when k is less than the diameter of a graph G, the addition of a pendant edge to G can also produce a graph whose k-geodomination number is strictly smaller than that of G. For example, consider the graphs G and G' of Figure 6, where G' is obtained from G by adding the pendant edge uv_9. Since the set $\{v_0, v_4, v_5, v_6, v_9, x, y\}$ is a g_5-set of G, it follows that $g_5(G) = 7$. On the other hand, the set $\{u, v_0, v_5, x, y\}$ is a g_5-set of G' and so $g_5(G') = 5$. In fact, this example can be extended to show that, in the case when k is less than the diameter of a graph G, the addition of a pendant edge to a connected graph G can result in a graph whose k-geodomination number is significantly smaller than that of G.

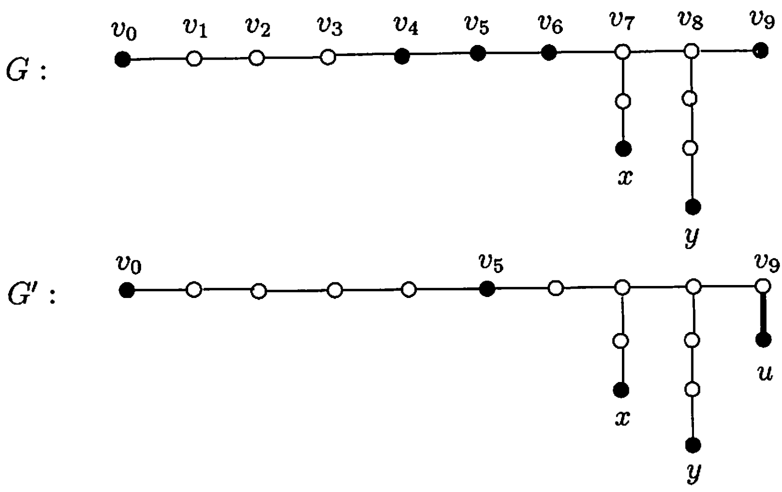

Figure 6: Graphs G and G' with $g_5(G) = 7$, and $g_5(G') = 5$

Moreover, the addition of a vertex v to a connected graph G such that more than one edge is incident with v can result in a graph whose k-geodomination number is same as G, or is significantly larger or smaller than that of G, where $2 \leq k \leq \text{diam}\, G$.

Theorem 3.3 *For every pair k, N of positive integers with $k \geq 2$, there exist*

(a) *a connected graph T_0 with $\text{diam}\, T_0 \geq k$ and a graph G_0 such that G_0 is obtained from T_0 by adding a vertex v with more than one edge incident with v and $g_k(T_0) = g_k(G_0)$,*

(b) *a connected graph T_1 with $\text{diam}\, T_1 \geq k$ and a graph G_1 such that G_1 is obtained from T_1 by adding a vertex v with more than one edge incident with v and $g_k(T_1) - g_k(G_1) = N$,*

(c) *a connected graph T_2 with $\text{diam}\, T_2 \geq k$ and a graph G_2 such that G_2 is obtained from T_2 by adding a vertex v with more than one edge incident with v and $g_k(G_2) - g_k(T_2) = N$.*

Proof. First we verify (a). Let T_0 be the path $P_{ak+1} : v_0, v_1, v_2, \cdots, v_{ka}$ of order $ak + 1$. By Proposition 2.3, $g_k(T_0) = a + 1$. Now the graph G_0 is obtained from T_0 by adding a new vertex u and joining u to v_0, v_2, and v_{ik+1} for all $1 \leq i \leq a - 1$. The graph G_0 is shown in Figure 7 for $k = 4$ and $a = 3$. It can be verified that the set $\{v_0\} \cup \{v_{ik} : 1 \leq k \leq a\}$ is a g_k-set of G_0 and so $g_k(G_0) = a + 1$ as well.

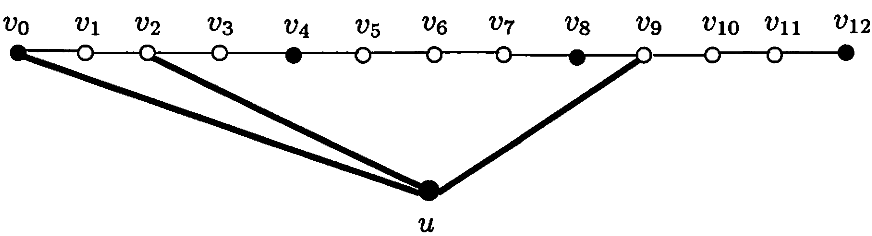

Figure 7: The graph G_0 for $k = 4$ and $a = 3$

Next we verify (b). For each integer i with $1 \leq i \leq N$, let $F_i : v_{i1}, v_{i2}, \cdots, v_{i,k-1}$ be a path of order $k - 1$. Then the graph T_1 is obtained from the paths F_i by adding a new vertex v and joining v to each of the vertices v_{i1} for all $1 \leq i \leq N$. It can be verified that the set $\{v_{i,k-1} : 1 \leq i \leq N\} \cup \{v_{11}, v_{21}\}$ is a g_k-set of G and so $g_k(T_1) = N + 2$.

Let G_1 be obtained from T_1 by adding a new vertex u and joining u to each of the end-vertices $v_{i,k-1}$ for all $1 \leq i \leq N$. Then $d(u,v) = k$. Moreover, $\{u,v\}$ is a k-geodominating set of G_1 and so $g_k(G_1) = 2$. Therefore, $g_k(T_1) - g_k(G_1) = (N+2) - 2 = N$.

Finally, we verify (c). Let the graph T_2 be obtained from the $N+2$ paths F_i $(1 \leq i \leq N+2)$ described in (b) by adding a new vertex v and joining v to each of the vertices v_{i1} for all $1 \leq i \leq N+2$. Thus $g_k(T_2) = 2$. Now let G_2 be obtained from T_2 by adding a new vertex w and joining w to both u and v. The graphs T_2 and G_2 are shown in Figure 8 for $k = 4$ and $N = 2$.

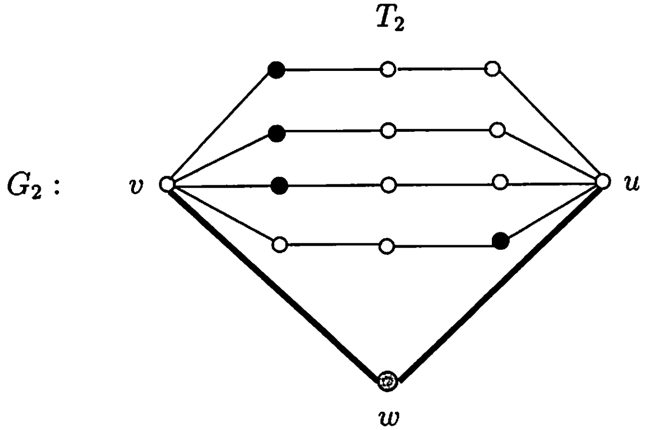

Figure 8: The graph G_2 for $k = 4$ and $N = 2$

It can be verified that the set $\{v_{i1} : 1 \leq i \leq N+1\} \cup \{v_{N+2,k-1}\}$ is a g_k-set of G_2 and so $g_k(G_2) = N+2$. The solid vertices in the graph G_2 of Figure 8 indicate a g_4-set described above and so $g_4(G_2) = 4$. Therefore, $g_k(G_2) - g_k(T_2) = N$. ■

The addition of a pendant edge to a connected graph has similar affect on k-geodomination number of a graph. An argument similar to one given proceeding Proposition 3.2 shows that if G' is a graph obtained by adding a pendant edge to a connected graph G, then $og_k(G') \leq og_k(G) + 1$. Also, it is possible that $og_k(G') < og_k(G)$. Moreover, the addition of a vertex v to a connected graph G such that more than one edge is incident with v can result in a graph whose open k-geodomination number is same as G, or is significantly larger or smaller than that of G. The proof of the next result is similar to the proof of Theorems 3.3 and is therefore omitted.

Theorem 3.4 *For every pair k, N of positive integers with $k \geq 2$, there exist*

(a) *a connected graph T_0 with* $\operatorname{diam} T_0 \geq k$ *and a graph G_0 such that G_0 is obtained from T_0 by adding a vertex v with more than one edge incident with v and $og_k(T_0) = og_k(G_0)$*

(b) *a connected graph T_1 with* $\operatorname{diam} T_1 \geq k$ *and a graph G_1 such that G_1 is obtained from T_1 by adding a vertex v with more than one edge incident with v and $og_k(T_1) - og_k(G_1) = N$,*

(c) *a connected graph T_2 with* $\operatorname{diam} T_2 \geq k$ *and a graph G_2 such that G_2 is obtained from T_2 by adding a vertex v with more than one edge incident with v and $og_k(G_2) - og_k(T_2) = N$.*

4 Bounds for k-Geodomination Numbers

In this section, we present upper and lower bounds for $g_k(G)$ and $og_k(G)$ in terms of the number of k-extreme vertices in G, which are similar to the bounds for $g(G)$ and $og(G)$ established in [3].

Theorem 4.1 *If G is a nontrivial connected graph of diameter d with exactly ℓ k-extreme vertices and $2 \leq k \leq d$ is an integer, then*

$$\max\{2, \ell\} \leq g_k(G) \leq og_k(G) \leq 3g_k(G) - 2\ell.$$

Proof. Certainly, $g_k(G) \geq 2$ for every nontrivial connected graph G. Since every k-extreme vertex belongs to every k-geodominating set of G by Observation 1.2, it follows that $\max\{2, \ell\} \leq g_k(G)$. By Proposition 1.3, $g_k(G) \leq og_k(G)$ for every nontrivial connected graph G. Thus it remains to verify the upper bound for $og_k(G)$. Let $g_k(G) = p$. If $\ell = p$, then $\ell \geq 2$ and the result follows from Observation 1.2. Thus we may assume that $\ell < p$. We consider two cases.

Case 1. $\ell \neq 0$. Let $S = \{u_1, u_2, \cdots, u_\ell\}$ be the set of k-extreme vertices of G and let $T = S \cup \{v_1, v_2, \cdots, v_{p-\ell}\}$ be a g_k-set of G. For each j with $1 \leq j \leq p - \ell$, since v_j is not a k-extreme vertex, v_j lies in some $v_{j1} - v_{j2}$ geodesic for some vertices v_{j1}, v_{j2} in G with $d(v_{j1}, v_{j2}) = k$. Let $T' = T \cup \{v_{j1}, v_{j2} : 1 \leq j \leq p - \ell\}$. We show that T' is an open k-geodominating set. It suffices to show that every vertex $v \in V - S$ is openly k-geodominated by two vertices of T'. Assume first that $v \notin T$. Since T is a k-geodominating set of G, it follows that v is k-geodominated by some pair x, y of vertices in T, which implies that v is openly k-geodominated by a pair of vertices in T'. If $v \in T$ and $v = v_j$ for some j with $1 \leq j \leq p - \ell$, then v is openly k-geodominated by v_{j1} and v_{j2}. Hence T' is an open k-geodominating set in G. Therefore,

$$og_k(G) \leq |T'| \leq p + 2(p - \ell) = 3p - 2\ell = 3g_k(G) - 2\ell.$$

Case 2. $\ell = 0$. the proof follows in an identical manner with $T = \{v_1, v_2, \cdots, v_p\}$ in Case 1 and is therefore omitted. ∎

It was shown in [3] that if a nontrivial connected graph G contains no link-complete vertices, then $og(G) \geq 4$. Since $og(G) \leq og_k(G)$ for all $k \geq 1$ and a link-complete vertex is a k-extreme vertex, it follows that $og_k(G) \geq 4$ for every nontrivial connected graph G containing no k-extreme vertices. Note that $og(C_{2k}) = og_k(C_{2k}) = 4$ for all cycles C_{2k} of order $2k$ with $k \geq 2$. Therefore, by Theorem 4.1 we obtain the following bounds for the open k-geodomination number of a graph G without k-extreme vertices.

Corollary 4.2 *Let $k \geq 2$ be an integer. For every connected graph G without k-extreme vertices,*

$$\max\{g_k(G), 4\} \leq og_k(G) \leq 3g_k(G).$$

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