On c-Hadamard Matrices

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Abstract: We reintroduce the problem of finding square ±1-matrices, denoted c-H(n), of order n, whose rows have non-zero inner product c. We obtain some necessary conditions for the existence of c-H(n) and provide a characterization in terms of SBIBD parameters. Several new c-H(n) constructions are given and new connections to Hadamard matrices and D-optimal designs are also explored.

Keywords: Bhaskar Rao Design, SBIBD, c-BRD, generalized balanced matrix, weighing designs, Hadamard matrix, excess, D-optimal.

1. Introduction

A *Hadamard matrix* of order 4n is a square ±1-matrix such that the inner product of any two rows is zero. A *Bhaskar Rao design (BRD)* is a {0, 1, -1}-matrix whose rows also have inner product zero [2,3] such that when the -1's are changed to +1's, the resulting matrix is the incidence matrix of a BIBD. Dey and Midha [6] extended the idea of BRD's to generalized balanced matrices, what the present authors termed *c-BRD's*. These are matrices with entries {0, 1, -1} whose inner products are a constant value c, an integer not necessarily zero but whose underlying matrix is still that of a BIBD. We showed that the necessary conditions are sufficient for the existence of all c-BRD(v,3,λ) [7,8,9]. Analogously, in [9] we defined a c-Hadamard matrix of order k and index c, denoted *c-H(k)*, to be a square matrix of 1's and -1's such that every row inner product is the integer c. Put another way, every X = c-H(n) satisfies $XX^T = (n-c)I + cJ$ where I is the identity matrix and J is the matrix of all ones. A Hadamard matrix is thus a 0-H(k). It is proved in [9] that if c-H(k) exists, then $c \equiv k \pmod 4$. This generalizes the well-known result that for every Hadamard matrix of order $k \geq 4$, $k \equiv 0 \pmod 4$. Raghavarao [15,16] with different terminology studied c-H(n) for c = 1, 2, and proved in any case that, necessarily, $c \geq -1$.

In Section 2 we give general necessary conditions on c and n, and we prove a new generalization to the Menon class of symmetric BIBD's. In Sections 3 and 4 we study the cases c = 1 and c = -1 especially. In Section 4 we apply these ideas to construct D-optimal r-by-r matrices for $r \equiv 2 \pmod 4$. Sections 5 and 6 deal with higher c values, and we show there are only finitely many c-H(n) if c+1 is an odd square. This last result also implies the non-existence of some symmetric designs.

For results on Hadamard matrices see [17] and the references therein. Any column sum of a Hadamard matrix is even, and the column sums are always congruent to each other mod 4 [1]. The sum of all the entries in a Hadamard matrix H is denoted $\sigma(H)$ and is called the excess of H. Best [1] showed that

$$n^2 2^{-n} \binom{n}{n/2} \leq \sigma(n) \leq n^{3/2}$$

where $\sigma(n) = \max \{\sigma(H) : H \text{ is a Hadamard matrix of order } n\}$. There are few results on the excess of a c-H(n). We return to the topic of excess briefly in Section 7.

2. Necessary conditions on c and n.

We now determine exactly which c's are allowed for a given n, by generalizing a well-known result for c = 0.

Theorem 1. *A necessary condition for the existence of c-H(n) is that $(n + cn - c)(n - c)^{n-1}$ be a square.*

Proof: Let X = c-H(n). Now $XX^T = (n - c)I + cJ$, and $|XX^T| = |X|^2$. Thus, one can see that

$$|X|^2 = n(n-c)^{n-1} + c(n-c)^{n-1} + \ldots + c(n-c)^{n-1},$$

and this reduces to the stated form.

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A consequence of Theorem 1 is that c must be ≥ -1, reproving Raghavarao's result. A consequence of the proof is that, if $c = -1$, then n must be odd, which also follows from the requirement that $c \equiv n \pmod 4$. The following theorem is implicit in Raghavarao [16] only for c = 1, 2. We give the general result.

Theorem 2. *Suppose X = c-H(n). If n is odd and $c \neq -1$, then* $n = \dfrac{c + y^2}{c + 1}$ *for some integer y. If n is even, then,*

$$n = \frac{c(c + 2) + \sqrt{4(c + 1)b^2 + c^4}}{2(c + 1)}$$

for some integer b.

Proof: Suppose n is odd. Recall $c \equiv n \pmod 4$. Then the exponent n - 1 in Theorem 1 is even. It follows that $n + cn - c = y^2$, for some integer y. Hence, $n = (c + y^2)/(c+1)$. Now suppose n is even. Then, by Theorem 1, $c \neq -1$, and for some b,

$$(n + cn - c)(n - c) = b^2$$
$$(1+c)n^2 - (c^2 + 2c)n + c^2 = b^2.$$

Solving for n, the result follows.

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In the even case, the plus sign in front of the radical may not be replaced by a negative sign since, in order that the numerator be positive, it follows that $c^2 > b^2$. From this and the second equation in the proof, it follows that $n < c(c + 2)/(c + 1)$, an impossibility. The following main theorem is new. Recall that a *regular* matrix has equal row (and column) sums.

Theorem 3. *Suppose $c \equiv n \pmod 4$. If there exists a symmetric balanced incomplete block design S = SBIBD(n, $(n \pm y)/2$, $(c + n \pm 2y)/4$), where $y^2 = n + cn - c$, then there exists X = c-H(n). Conversely, if a regular X = c-H(n) exists, then there is a symmetric S = SBIBD(n, $(n \pm y)/2$, $(c + n \pm 2y)/4$) where $y^2 = n + cn - c$.*

Proof: We hypothesize a symmetric BIBD, say S, with the parameters given, and want that when the 0's in S are replaced by minus ones, the result is X = c-H(n). Suppose we form X by replacing the zeros with minus ones. Suppose Row i and Row j are any rows of X, $i \neq j$, and suppose <Row i, Row j> = c. Let λ be the number of overlaps with +1 in each of Rows i and j. Let t denote the number of +1's in Row i that overlap -1's in Row j, and let u be the number of minus ones that overlap in each row. Then

$$c = \lambda + u - 2t, \text{ and}$$
$$n = \lambda + u + 2t.$$

Subtracting these gives

$$n - c = 4t.$$

But $k = \lambda + t$, by counting plus ones. So, $t = k - \lambda$. Thus, t (and hence c and u) is independent of the rows selected. These conditions on inner products give us

$$n - 4k + 4\lambda = c. \qquad (1)$$

Now, as $\lambda = k(k - 1)/(n - 1)$ for any symmetric BIBD, we substitute into *(1)* and get

$$n - 4k + \frac{4k(k-1)}{n-1} = c.$$

Now we clear the equation of fractions and solve for k.

$$4k^2 - 4nk + n^2 - (c+1)n + c = 0,$$
$$k = \frac{4n \pm 4\sqrt{(c+1)n - c}}{8},$$
$$k = \frac{n \pm y}{2}. \qquad (2)$$

Now substituting *(2)* into *(1)* and solving for λ we get

$$\lambda = \frac{c + n \pm 2y}{4}.$$

Conversely, by the regularity hypothesis we may assume there are k plus ones in each row (and column). Let us consider any two rows R_i and R_j, $i \neq j$. Let λ denote the number of plus ones in R_i that overlap plus ones in R_j. Let t denote the number of plus ones in R_i that overlap minus ones in R_j (t will also equal the number of minus ones in R_i that overlap plus ones in R_j since the number of ones in each row is by regularity the same, namely λ+t). Let u denote the number of minus ones that overlap in R_i and R_j. Since $\langle R_i, R_j \rangle = c$, we have

$$c = \lambda + u - 2t$$

$$n = \lambda + u + 2t.$$

Subtracting, we get

$$n - c = 4t.$$

This shows t is independent of i and j. Now $k = \lambda + t$. Thus

$$\lambda = k - (n - c)/4.$$

Thus, λ is independent of i and j as well. It follows that u is also independent of the rows chosen. Now form S from X by changing all minus ones to zeros. It follows by what we have done that

$$SS^T = (k - \lambda)I + \lambda J$$

from which we infer that $S = SBIBD(n, k, \lambda)$. For any symmetric BIBD, $\lambda = k(k - 1)/(n - 1)$. Into this we substitute the previous expression for λ just above, and, simplifying, we get

$$4k^2 - 4kn + n^2 - cn - n + c = 0.$$

Now, by hypothesis $X = c\text{-}H(n)$, and from Theorem 1 it follows that y is an integer where $y^2 = n + cn - c$. Thus,

$$4k^2 - 4kn + n^2 - y^2 = 0.$$

Solving for k we get $k = (n \pm y)/2$. From this, $\lambda = (n + c \pm 2y)/4$.

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Theorem 3 is a generalization of the well-known result for $c = 0$ [**12**, p.371]: A Menon design $SBIBD(4u^2, 2u^2 - u, u^2 - u)$ exists if and only if a regular $0\text{-}H(4u^2)$ exists.

Theorem 4. *Suppose n is even, $c \equiv n \pmod 4$ and $y = (n + cn - c)^{1/2}$ is an integer. Then the symmetric design $(v, k, \lambda) = (n, (n-y)/2, (c+n-2y)/4)$ fails to exist if $n - c$ is not a perfect square.*

Proof: Suppose the design exists but $n - c$ is not a square. By Theorem 3, the c-H(n) exists, but by Theorem 1, the c-H(n) fails to exist.

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Theorem 4 is an unusual version of the well-known Schutzenburger's Theorem or the Bruck-Ryser-Chowla Theorem [**11**] for even n, i.e., that when v is even, $k - \lambda$ is a square (and of course, here $n - c = 4(k - \lambda)$ from the proof of Theorem 3). For applications of Theorem 4: there is no symmetric BIBD for the parameters (134, 57, 24) or (162, 70, 30) although the conditions of Theorem 3 are met - here $c = 2$ and in each case y is an integer, but $n - 2$ is not a square.

We include Table 1 and Table 2 at the end showing the current state of knowledge for c = 1 to c = 17, in each case, for small k, listing the line in [12] which confirms the SBIBD. When n = c, the corresponding design is J, the matrix of all 1's. *[In Table 1 and Table 2, we actually list parameters for the design complemantary to J which consists of all zeros since we use the minus sign in the formulas from Theorem 3 which correspond to k and λ.]* When c = n - 4, the design is the identity matrix I. UNK means the design was previously considered (i.e., listed in [12]), the necessary conditions are met, but the existence is still unknown. NL means the necessary conditions are met but the design is not listed, i.e., not known to exist or parameters too large for inclusion in [12]. DNE means the necessary conditions are met, but the design was previously indicated in [12] as not existing.

DNE BCR, used for odd n only, means the necessary conditions herein are met but the design does not exist because the parameters do not satisfy the diophantine equation required by the Bruck-Ryser-Chowla Theorem [11], (abbreviated alphabetically as BCR).

For all n, we examined values beyond n = 3500 such that y in Theorem 3 is an integer. Of course, when y is an integer, k and λ will also be integers. For each odd n, there are many admissible parameter sets. For even n, however, it is further necessary that n - c be a square. Admissible sets of parameters for even n are thus rare. We list odd and even c separately with a column for n-c in the latter case. For brevity, we list odd c,n only up to the first few cases too large to be in [12]. For even n, however, we list all cases for which the necessary conditions are met up to n = 10000. Thus, for the even case, designs not listed do not exist bcause the necessary conditions do not hold, and we use ?? to indicate the c-H(n) and corresponding design might exist.

3. The connection between c = 0 and c = -1.

Row inner products are unaffected by interchanges of rows or interchanges of columns or, when c = 0, by multiplying a row or column by -1. Similarly, a c-H(n) with c non-zero remains a c-H(n) if a column is multiplied by -1 but not when a row is multiplied by -1. Two c-Hadamard matrices are called equivalent if one is obtained from the other by a sequence of such operations which leaves row inner products unaffected. Every 0-Hadamard matrix is equivalent to one in standard form, i.e., such that the first row and first column have only +1 in each entry. Every c-H(k) is equivalent to a c-H(k) with all 1's in the first row. Such a c-H(n) is said to be normalized.

Suppose H is a 0-Hadamard matrix of order 4n in standard form. Then every row after the first has 2n plus ones and 2n minus ones; otherwise, the row inner product of Row x with Row 1 would fail to be zero, for $2 \leq x \leq 4n$. Now, if we delete the first row and first column of any standard form 0-H(n) forming a matrix, say X, then the inner product of each pair of rows of X is -1. Thus, in X we have created a (-1)-H(4n-1). Conversely, suppose a (-1)-H(4n-1) exists; then we construct a 0-H(4n) as follows. Add a beginning column of 1's to the (-1)-H(4n-1). We have a matrix with 4n columns and 4n-1 rows and each pair of rows is orthogonal. By a theorem of Shrikhande and Bhagwandas **[18]**, we can complete the matrix to a 0-H(4n). This proves the following characterization of all (-1)-H(k) matrices.

Theorem 5. *A 0-H(4n) exists if and only if a (-1)-H(4n-1) exists.*

The *Hadamard Conjecture* is that there exists 0-H(4n) for every n = 1, 2, ... The existence of (-1)-H(4n-1) is thus equivalent to the existence of 0-H(4n). At present, 0-H(4n) exist for all 4n up to 424 according to **[17]**, and of course for infinitely many other multiples of 4.

4. The case c = 1 and D-Optimal Matrices.

We would like to determine if a construction like that in the previous section for c = -1 exists in the case c = 1. Curiously, one can find a natural construction, but the situation for c = 1 turns out to be as different as possible from that of c = -1 since the natural construction leads to only one example.

A 1-H(5) is easily constructed by placing minus ones on the diagonal of J_5, the 5-by-5 matrix of all ones. A 0-H(4) can be similarly constructed. The constant diagonal of minus ones with plus ones elsewhere leads to the next two easily proved theorems.

Theorem 6. *If X =1-H(5) is in standard form (i.e., Row 1 consists of all*

1's), then there is a column with four minus ones. (There is one equivalence class for 1-H(5).)

Theorem 7. *A k-H(k+4) exists for every $k \geq -1$.*
Proof: Let $H = J - 2I$.

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Now we investigate the other key feature referred to above - constant row sums. Any matrix for 0-H(4) is equivalent to one with -1 on the main diagonal and 1 elsewhere. These have constant row sum of 2. Such matrices give our natural construction.

Theorem 8. *If $A = 0\text{-}H(4n)$ satisfies $\sum_{j=1}^{4n} a_{ij} = 2$ for each i, then the matrix*

$$\begin{array}{c|c} -1 & 1 \ \cdots \ 1 \\ \hline 1 & \\ 1 & A \\ \cdots & \\ 1 & \end{array}.$$

gives a 1-H(n).

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A 1-H(9) can not be formed by the method of Theorem 8. From Theorem 1, if $A = 1\text{-}H(n)$, then, since $\det(A) = (n-1)^{(n-1)/2}\sqrt{2n-1}$ is an integer, $2n - 1$ is a square, say $(2q + 1)^2$. Then $n = 2q^2 + 2q + 1$, and 9 is not of this form. *This new argument is stronger - it shows, in fact, that a 1-H(9) does not exist at all.* Admissible n for 1-H(n) are recognizable as the hypotenuses of a class of primitive pythagorean triples $(2q + 1, 2q^2 + 2q, 2q^2 + 2q + 1)$. The question arises: are there other examples with constant row sum 2 which would lead to other 1-H(n)? In fact, the useful idea of constant row sums, critical in the study of excess, has been studied quite fully and will answer our question. In the Menon class of SBIBD's mentioned just after Theorem 3, one can say: **[17, p.525]** *Constant row sums occur in 0-H(v) if the SBIBD($4N^2$, $2N^2 \pm N$, $N^2 \pm N$) exists and $v = 4n^2$. Conversely, constant row sums can occur in 0-H(v) only if $v = 4N^2$ and the constant sum is $\pm 2N$.* This shows that, surprisingly, constant row sum of 2 can occur only for N = 1 and v = 4.

A few 1-H(4k+1) and 2-H(4k+2) were exhibited by Raghavarao **[15]** (see also **[16, pp.318-319]**): a 1-H(n) for for n = 5, 13, and 25, corresponding to $y^2 = 3^2, 5^2$, and 7^2; and 2-H(6). These were studied in the context of various optimality conditions for weighing designs. One can apply the 1-H(n) to D-optimal {1, -1}-matrices, as shown by Yang **[22]**. Suppose n ≡ 2 (mod 4), and A and B are commuting matrices with elements 1, -1 such that

$$AA^T + BB^T = (n-2)I_{n/2} + 2J_{n/2}.$$

Then the n-by-n matrix $Y = \begin{bmatrix} A & B \\ -B^T & A^T \end{bmatrix}$ has maximum determinant among all n-by-n ±1 matrices. These are called D-optimal, and their construction is an ongoing problem; see **[10]** and **[20]**. A list of known D-optimal designs of order n < 200 is given by Koukouvinos **[10]**. Now suppose X = 1-H(n/2). Then the Y constructed as above with X = A = B is D-optimal. For example, the (61, 25, 10)-design gives a 1-H(61) which in turn gives a D-optimal design for n = 122 **[14]**.

There are infinitely many n such that a 1-H(n) might exist as for any odd square, y^2, just take $n = (1 + y^2)/2$, from Theorem 2. In fact, Whiteman **[20]**, using that, for any prime power q, an SBIBD, say A, exists, with

$$A = \mathrm{SBIBD}(2q^h+2q^{h-1}+\ldots+2q+1,\ q^h,\ (q^h-q^{h-1})/2),$$

formed such a matrix A with h = 2, $v = 2q^2 + 2q + 1$, $k = q^2$, $\lambda = (q^2 - q)/2$. Then he replaced the 0's in A by -1's to get X. It follows that $XX^T = 4(k - \lambda)I + (v - 4(k - \lambda))J = 2(q^2 + q)I + J = (n - 1)I + J$. That is, X is a 1-H(n). Then the D constructed as above is D-optimal.

The family of examples used by Whiteman is due to Brouwer **[5]**. In this same family, the case $q = 2^2$ gives the example SBIBD(41, 16, 6) which corresponds to a 1-H(41); see **[4,19,21]**. When q = 5, one gets SBIBD(61, 25, 10).

Theorem 9. [20] *There are infinitely many 1-H(n) where $n = 2q^2 + 2q + 1$ for q = 2, 4, or an odd prime power.*

■

5. c odd, $c \geq 3$.

Theorem 10. *Suppose there exists $S = SBIBD(q^2 + q + 1, (q^2 - q)/2, (q^2 - 3q+2)/4)$, with $q \equiv 1 \pmod 4$. Then a 3-H(n) exists with $n = q^2 + q + 1$.*

Proof: Let X be the matrix obtained by replacing 0's in S by -1. Then X is a 3-H(n) since, on the one hand, $4(k - \lambda) = 4[(q^2 - q)/2 - (q^2 - 3q + 2)/4] = q^2 + q - 2 = q^2 + q + 1 - 3$, and on the other,

$$XX^T = 4(k - \lambda) I + (n - 4(k-\lambda)) J = (n - 3)I + 3J.$$

■

These designs are known to exist for q = 1, 5 (n = 3, 31 in Table 1), and include half of the admissible parameter sets for c = 3.

Theorem 11. *Suppose c and q are odd and suppose there exists*

$S = SBIBD((c + 1)q^2 + 2q + 1, [(c + 1)q^2 - (c-1)q]/2, [(c + 1)q^2 - 2cq + c - 1]/4)$.

Then there exists X = c-H(n) derived from S.

Proof: First, y is an integer since $n + cn - c = (c + 1)^2q^2 + 2(c + 1)q + 1 = [(c + 1)q + 1]^2 = y^2$. Also,

$$n - y = (c + 1)q^2 + 2q + 1 - [(c + 1)q + 1] = (c + 1)q^2 + 2q + 1 - cq - q - 1$$

$$= (c + 1)q^2 + q - cq = 2k.$$

Thus, $k = (n - y)/2$. Next,

$$(n + c - 2y) = (c + 1)q^2 + 2q + 1 + c - 2[(c + 1)q + 1] = (c + 1)q^2 + 2q + 1 + c - 2cq - 2q - 2$$

$$= (c + 1)q2 - 2cq + c - 1 = 4\lambda.$$

Hence, $\lambda = (n + c - 2y)/4$. Finally,

$$4(k - \lambda) = 4\{(n - y)/2 - (n + c - 2y)/4\} = n - c.$$

Thus $XX^T = (n - c)I + cJ$, and X is a c-H(n).

■

For example, if c = 1, we get the examples due to Whiteman and Brouwer, and for the case c = 5, it follows that q = 1, 3, 5 give n = 9, 61, 161 in Table 1. [In fact, c may be even in the theorem. Analysis shows that $c \equiv n \equiv 0 \pmod 4$ and $q \equiv 1 \pmod 4$ may occur in the theorem - see Table 2: whenever c = 4j and n = 4j + 4, or when c = 12 and q = 5.]

6. c even, $c \geq 4$.

For even c, admissible parameter sets must satisfy two quadratic equations and hence are more sparse than for odd c. But the case $c = 8$ seems exceptionally sparse (see Table 2). This is no accident as the two trivial designs 8-H(8) and 8-H(12) are all there are for $c = 8$.

Theorem 12. *If c is even and if c + 1 is a square, then there are only finitely many c-H(n) associated with symmetric designs as in Theorem 3.*

Proof: We have $y^2 = (c+1)n - c$ and $x^2 = n - c$ from Theorems 3 and 4. Combining gives

$$y^2 = (c+1)n - c = (c+1)(x^2+c) - c = (c+1)x^2 + c^2,$$
$$y^2 = v^2 + c^2$$

where $v^2 = (c+1)x^2$. Now as is well-known, there are only finitely many Pythagorean triples with a fixed leg size c since there are only finitely many integer factorizations of $c^2 = (y+v)(y-v)$.

■

For $c = 8$, write $64 = y^2 - v^2 = (y+v)(y-v)$. Only integer factorizations of 64 giving an integer pair (y,v) are relevant. As may be checked, $64 \times 1 = (y+v)(y-v)$ gives $y + v = 64$, $y - v = 1$, and $y = 32.5$, not an integer. Similarly, 32×2 gives $y = 17$ - but y is necessarily even (as n and c are even). 16×4 gives $y = 10$ and $n = 12$. Finally 8×8 gives $x = 0$ and $n = 8$. As there are no more factorizations of 64, the only two examples of 8-H(n) are the obvious (trivial) ones.

For $c = 24$, there are only three possible designs, the two obvious ones ($c = n = 24$, and $c + 4 = n = 28$) and $(v, k, \lambda) = (220, 73, 24)$. In the latter case, $y = 74$.

Theorem 13. *Suppose $c = 4m$ and $4m + 1$ is a square. Suppose $\frac{m^4 + 2m^2 + m + 1}{4m + 1} = 4N$. If an SBIBD exists with parameters $(v, k, \lambda) = (4N, 2N - m^2 - 1, N - m^2 + m - 1)$, then the associated X = c-H(4N) exists.*

■

With $m = 6$ in the theorem, we get the example for $c = 24$.

7. Variation of excess.

We start this section with an application of Theorem 1.

Theorem 14. *Suppose $c \geq 1$, and $X = c\text{-}H(n)$ is in normalized form. Then $\sigma(X)$ is a square.*

Proof: After the first row, each row sum is c. So $\sigma(X) = n + (n - 1)c$. By Theorem 1, $\sigma(X) = y^2$.

■

What can one say in general about the distribution of the excess? Is it possible that every multiple of 4 between $\pm\sigma(H)$ will occur for some member in the equivalence class of a given H? As a partial answer to this open question, we have the following result.

Theorem 15. *Every 0-H(4n) is equivalent to a Hadamard matrix, say H_x such that $\sigma(H_x) = 4x$ for some x with $-n \leq x \leq n$.*

Proof: Suppose 0-H(4n) is a normalized form Hadamard matrix. Its excess, the sum of all the entries, is 4n since after the first row of all ones, each row sums to zero. Consider the left-most column. It contains only plus ones. Multiply this left-most column by -1. Now $\sigma(H) = -4n$ (we lost 4n plus ones and gained 4n negative ones). Now each row after the first has row-sum of -2. In general, multiplying a column by -1 changes each row sum by 2 or -2. Multiplying a row by -1 changes each column sum by 2 or -2. Starting with the matrix with excess of -4n, suppose we next multiply each row by -1 starting with the bottom row. Then the excess gains 4 over the previous excess after each multiplication because each row-sum switches from -2 to 2. In this way the value of $\sigma(H)$ equals every multiple of 4 from -4n to 4n. Note the excess will be exactly zero after the bottom n rows have been multiplied by -1.

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Acknowledgement: The authors wish to thank Malcolm Greig for supporting calculations in Table 1 and a helpful reading of the manuscript.

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Table 1: Design Parameters for c-H(n), c odd										
c = 1						c = 9				
n	y	k	λ	Comment		n	y	k	λ	Comment
1	1	0	0	J		9	9	0	0	J
5	3	1	0	I		13	11	1	0	I
13	5	4	1	3		37	19	9	2	39
25	7	9	3	40		45	21	12	3	84
41	9	16	6	172		85	29	28	9	UNK 568
61	11	25	10	443		97	31	33	11	UNK 779
85	13	36	15	UNK 976		153	39	57	21	NL
113	15	49	21	[5]		169	41	64	24	NL
145	17	64	28	[5]						
						c=11				
c = 3						11	11	0	0	J
3	3	0	0	J		15	13	1	0	I
7	5	1	0	I		31	19	6	1	12
31	11	10	3	54		71	29	21	6	310
43	13	15	5	DNE 141		103	35	34	11	UNK 813
91	19	36	14	DNE 361		115	37	39	13	DNE 1073
111	21	45	18	NL		155	43	56	20	NL
183	27	78	33	DNE BCR		235	53	91	35	DNE BCR
c = 5						c=13				
5	5	0	0	J		13	13	0	0	J
9	7	1	0	I		17	15	1	0	I
21	11	5	1	6		53	27	13	3	DNE 96
29	13	8	2	DNE 28		61	29	16	4	170
49	17	16	5	171		121	41	40	13	1170
61	19	21	7	DNE 311		133	43	45	15	NL
89	23	33	12	DNE 780		217	55	81	30	NL
105	25	40	15	UNK 1171						
141	29	56	22	DNE BCR		c = 15				
161	31	65	26	DNE BCR		15	15	0	0	J
						19	17	1	0	I
c = 7						139	47	46	15	NL
7	7	0	0	J		151	49	51	17	NL
11	9	1	0	I						
67	23	22	7	DNE 338		c=17				
79	25	27	9	513		17	17	0	0	J
191	39	76	30	NL		21	19	1	0	I
211	41	85	34	DNE BCR		69	35	17	4	185
						77	37	20	5	DNE 271
						157	53	52	17	DNE BCR
						169	55	57	19	NL

Table 2: Design Parameters for c-H(n), c even

	n	y	sqrt(n-c)	k	λ	Comment
c=2	2	2	0	0	0	J
	6	4	2	1	0	I
	66	14	8	26	10	[Trung, 21]
	902	52	30	425	200	??
c=4	4	4	0	0	0	J
	8	6	2	1	0	I
	40	14	6	13	4	97
	260	36	16	112	48	??
	1768	8836	42	837	396	??
c=6	6	6	0	0	0	J
	10	8	2	1	0	I
	70	22	8	24	8	407
	330	48	18	141	60	??
	1606	106	40	750	350	??
c=8	8	8	0	0	0	J
	12	10	2	1	0	I
c=10	10	10	0	0	0	J
	14	12	2	1	0	I
	266	54	16	106	42	??
	910	100	30	405	180	??
	3146	186	56	1480	696	??
c=12	12	12	0	0	0	J
	16	14	2	1	0	I
	112	38	10	37	12	UNK 995
	336	66	18	135	54	??
	1036	116	32	460	204	??
c=14	14	14	0	0	0	J
	18	16	2	1	0	I
	78	34	8	22	6	337
	210	56	14	77	28	??
	590	94	24	248	104	??
	4370	256	66	2057	968	??
c=16	16	16	0	0	0	J
	20	18	2	1	0	I
	160	52	12	54	18	??
	416	84	20	166	66	??