On the edge-residual number and the line completion number of a graph

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ABSTRACT. In this paper we introduce the edge-residual number $\rho(G)$ of a graph G. We give tight upper bounds for $\rho(G)$ in terms of the eigenvalues of the Laplacian matrix of the line graph of G. In addition, we investigate the relation between this novel parameter and the line completion number for dense graphs. We also compute the line completion number of complete bipartite graphs $K_{n,m}$ when either $m = n$ or both m and n are even numbers. This partially solves an open problem of Bagga, Beinecke and Varma [2].

1 Introduction

There are a lot of papers in the literature dealing with the concept of line graph and its generalizations. See, for instance, papers of Broersma and Hoede [5], Chartrand [7], Behzad [3], Harary and Norman [9] and Fiol and Lladó [8].

One of the most recent generalizations of this concept is the *super line graph* introduced by Bagga, Beineke and Varma [1, 2].

Given a graph G, the super line graph of index r, $\mathcal{L}_r(G)$, is defined as the graph whose vertices are the r-subsets of $E(G)$, and two vertices S and T are adjacent in $\mathcal{L}_r(G)$ if and only if there are edges $s \in S$ and $t \in T$ that share exactly one vertex in G. When $r = 1$ then $\mathcal{L}_1(G)$ is the usual line

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graph $L(G)$ of G. In [1, 2] the authors give several results concerning to this novel generalization. For example,

Theorem 1. ([1], Theorem 6) *Let G be a graph with q edges.*

a) *If G is a subgraph of H, then $\mathcal{L}_r(G)$ is an induced subgraph of $\mathcal{L}_r(H)$.*

b) *For $r < \frac{q}{2}$, $\mathcal{L}_r(G)$ is isomorphic to a subgraph of $\mathcal{L}_{r+1}(G)$.* □

Clearly, if two r-subsets of edges S and T are adjacent in $\mathcal{L}_r(G)$, then each pair of $(r+1)$-subsets S' and T' such that $S \subset S'$ and $T \subset T'$ are adjacent in $\mathcal{L}_{r+1}(G)$. Thus, if $\mathcal{L}_r(G)$ is a complete graph, then so is $\mathcal{L}_{r+1}(G)$. Note that, if G contains at least two incident edges, then $\mathcal{L}_{q-1}(G)$ is the complete graph K_q, where $q = |E(G)|$.

Motivated by the above facts, the authors of [1] define the *line completion number*, $lc(G)$ of a graph G with at least two incident edges as the minimum index r for which $\mathcal{L}_r(G)$ is a complete graph.

In this paper we introduce the *edge-residual number* $\rho(G)$ of a graph G and show its connection with $lc(G)$. The edge residual number of G is defined in Section 2, where tight spectral upper bounds are obtained. Next, following a previous work of the authors [10], we investigate the relationship between the edge-residual number and the line completion number for complete bipartite graphs and for dense graphs, i.e., graphs with minimum degree $\delta(G) \geq |V(G)|/2$. We obtain tight spectral upper bounds for $lc(G)$ and compute its value for complete graphs and complete t-partite graphs among other examples. Finally, we consider the computation of $lc(G)$ for bipartite complete graphs. This computation was proposed as an open problem in [2], where the authors suggest that a closed formula for this parameter is unlikely to exist. We obtain the value of $lc(K_{n,m})$ when either $n = m$ or both n and m are even numbers as a first step to solve this problem.

2 The edge-residual number

Let A be a subset of edges of graph G. We denote by $B_L(A)$ the set of edges of G which have at least one vertex in common with some edge in A. In other words, $B_L(A)$ is the set of vertices of the line graph $L(G)$ at distance at most one of a vertex in A. The *extertior* of A is $ex(A) = E(G) \backslash B_L(A)$. We define the *edge-residual number* $\rho(G)$ of G as

$$\rho(G) = \max_{A \subset E(G)} \{|A| \colon |ex(A)| \geq |A|\}.$$

The edge residual number and the line completion number of a graph are related by the following easy Lemma.

Lemma 2. *For a graph G,*

$$lc(G) \geq \rho(G) + 1.$$

Proof: Let $A \subset E(G)$ be a set of edges with cardinality $\rho(G)$ such that $|ex(A)| \geq |A|$. Let $A' \subset ex(A)$ with cardinality $|A|$. The pair of sets A, A' are two non adjacent vertices in $\mathcal{L}_{\rho(G)}(G)$, which is thus different from the complete graph. □

We shall derive a spectral upper bound for $\rho(G)$. A subset T of edges is an *edge bisector* of a simple connected graph G if $E(G)\backslash T$ can be partitioned into two subsets A, A' with the same cardinality such that $B_L(A) \cap A' = \emptyset$. Recall that the Laplacian matrix $\mathbf{L}$ of G is defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is the diagonal matrix of the degrees of the vertices in G and $\mathbf{A}$ is the adjacency matrix of G. We denote by $0 = \mu_0(G) \leq \mu_1(G) \leq \cdots \leq \mu_b(G)$ the eigenvalues of the Laplacian matrix of G. The following bound for the edge bisector of G was obtained in [11].

Theorem 3. ([11], Theorem) *Let T be an edge bisector of a connected simple graph G with m edges. Then,*

$$|T| \geq m\frac{\mu_1(L(G))}{\mu_b(L(G))},$$

where $L(G)$ is the line graph of G. □

From the above result we can deduce the following.

Theorem 4. *Let G be a connected simple graph with m edges. Then.*

$$\rho(G) \leq \frac{m}{2}\left(1 - \frac{\mu_1(L(G))}{\mu_b(L(G))}\right). \tag{1}$$

The bound is tight.

Proof: Let A be a subset of edges of G of cardinality $\rho(G)$. Let $A' \subset ex(A)$ such that $|A'| = |A|$. Then, $T = E(G)\backslash(A \cup A')$ is an edge bisector of G and, by Theorem 3,

$$|T| = m - 2\rho(G) \geq m\frac{\mu_1(L(G))}{\mu_b(L(G))},$$

which implies inequality (1).

To see that the above bound is tight, take $G = K_{n,n}$. We have $\mu_1(L(K_{n,n})$ $= n$ and $\mu_b(K_{n,n}) = 2n$. By (1), $\rho(K_{n,n}) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. On the other hand, let W_1 be a subset of one of the stable sets of $K_{n,n}$ with $\lfloor \frac{n}{2} \rfloor$ vertices and W_2 a subset of the other stable set with $\lceil \frac{n}{2} \rceil$ vertices. Then the set A of edges of the subgraph of $K_{n,n}$ induced by $W_1 \cup W_2$ and its exterior $ex(A)$ have both cardinality $\left\lfloor \frac{n^2}{4} \right\rfloor$. Therefore, $\rho(K_{n,n}) = \left\lfloor \frac{n^2}{4} \right\rfloor$. □

3 The line completion number of dense graphs

In this Section we show that, for dense graphs, that is, graphs of order n and minimum degree $\delta(G) \geq n/2$, there is equality in Lemma 2. This fact is used to obtain spectral bounds for the line completion number and actual values of this parameter for specific classes of graphs.

We use the following notation. For a set A of edges of a graph G, we denote by $\langle A \rangle$ the set of vertices which are incident to some edge in A. Let X, Y be two sets of vertices. We denote by $G[X]$ the subgraph of G induced by X and $e(X,Y)$ denotes the edges of G which join one vertex in X with one vertex in Y.

Theorem 5. *Let G be a graph of order $n > 2$ and minimum degree $\delta(G) \geq n/2$. Then,*

$$lc(G) = \rho(G) + 1.$$

Proof: By Lemma 2, it suffices to show that $lc(G) \leq \rho(G) + 1$.

Set $r = \rho(G) + 1$ and suppose that $\mathcal{L}_r(G)$ is not the complete graph. We show that this assumption leads to a contradiction.

Let m be the smallest integer such that there are two non adjacent vertices A, A' in $\mathcal{L}_r(G)$ such that $|A \cap A'| = m$.

Note that $m = 0$ implies $A' \cap B_L(A) = \emptyset$. Therefore $A' \subset ex(A)$. But then $|ex(A)| \geq |A'| = |A| > \rho(G)$ contradicting the definition of the edge residual number. Hence, $m > 0$. Note that since A and A' are nonadjacent in $\mathcal{L}_r(G)$, then $A \cap A'$ must consist of m independent edges. Let $X = \langle A \backslash A' \rangle$ and $Y = \langle A' \backslash A \rangle$.

Suppose that there is an edge $e \in E(G) \backslash (A \cup A' \cup e(\langle A \rangle, \langle A' \rangle))$. Then the two end vertices of e are either contained in $V(G) \backslash \langle A \rangle$ or in $V(G) \backslash \langle A' \rangle$. We may assume that $e \in V(G) \backslash \langle A' \rangle$. Let $u \in A \cap A'$. Then $(A \cup \{e\} \backslash \{u\})$ and A' are two non adjacent vertices in $\mathcal{L}_r(G)$ whose intersection has cardinality less than m, a contradiction. Hence,

$$E(G) = A \cup A' \cup e(\langle A \rangle, \langle A' \rangle).$$

In particular, since $\delta(G) > 0$, the set of vertices of the graph is the disjoint union $V(G) = X \cup C \cup Y$, where $C = \langle A \cap A' \rangle$. Since the graph is connected then $e(C,X) \cup e(C,Y) \neq \emptyset$. Moreover, $e(C,X) \cap A = e(C,Y) \cap A' = \emptyset$.

Suppose that there are two independent edges $e \in e(C,X)$ and $e' \in e(C,Y)$. Let $u, u' \in A \cap A'$ be incident with e and e' respectively (we may have $u = u'$). Then the sets $(A \backslash \{u\}) \cup \{e\}$ and $(A' \backslash \{u'\}) \cup \{e'\}$ are non adjacent vertices of $\mathcal{L}_r(G)$ and their intersection has cardinality less than m, again a contradiction. Hence, either all vertices in $e(C,X) \cup e(C,Y)$ share one vertex in C or one of the two sets $e(C,X)$ and $e(C,Y)$ is the empty set. Since $|C| \geq 2$, the first possibility implies that $\delta(G) = 1$. Therefore, we mav assume that $e(C, Y) = \emptyset$. This implies that the neighborhood of each vertex in Y is contained in $V(G)\backslash C$. Since $\delta(G) \geq n/2$ we have $|C| < n/2$. Therefore, for each vertex $c \in C$ we have $|e(\{c\}, X)| \geq 2$.

Suppose that there is an edge $e = \{c_1, c_2\} \in E(G[C])\backslash(A \cap A')$. Let u_1, u_2 be the edges of $A \cap A'$ incident with c_1 and c_2 respectively. Let e_1, e_2 be two edges in $e(\{c_1\}, X)$. Then the sets $(A \cup \{e_1, e_2\})\backslash\{u_1, u_2\}$ and $(A' \cup \{e\}\backslash\{u_1\}$ intersect in less than m edges of G and are non adjacent in $\mathcal{L}_r(G)$, a contradiction. Hence $E(G[C]) = A \cap A'$. In particular, for each $c \in C$, $|e(\{c\}, X)| \geq n/2$.

Since $e(C, Y) = \emptyset$, there is at least one edge $e = \{x, y\} \in e(X, Y)$, with $x \in X$ and $y \in Y$. Let $D = e(\{x\}, X)$ and $u = \{c_1, c_2\} \in A \cup A'$. We have $|D| \leq |X| - 1 < n - 3 \leq |D'| = |e(\{c_1, c_2\}, X)|$. Let D'' be a subset of D' with cardinality $|D|$. Then the sets $(A \cap D'')\backslash D$ and $(A' \cup \{e\})\backslash\{u\}$ are non adjacent in $\mathcal{L}_r(G)$ and intersect in less than m edges of G. This contradiction completes the proof. □

The-following example shows that the condition on the minimum degree of the graph in the above Theorem is close to be tight.

Example: Let G be the disjoint union of K_{m+1} and K_{m-1}. The minimum degree of the graph is $\delta(G) = m - 2 = \frac{n}{2} - 2$. Denote by V_1 the set of vertices that generate the subgraph isomorphic to K_{m+1} and denote by V_2 the set of remaining vertices in graph G. Then it is easily checked that $\rho(G) = |E(G[V_2])|$.

On other hand, let $V_1^* = V_1\backslash\{x, y\}$ for some pair of vertices $x, y \in V_1$. Then the sets of edges $A = E(G[V_2]) \cup \{x, y\}$ and $B = E(G[V_1^*]) \cup \{x, y\}$ are non adjacent vertices in $\mathcal{L}_{\rho(G)-1}(G)$. Therefore,

$$lc(G) \geq \rho(G) + 2.$$

□

As a consequence of Theorems 4 and 5 we obtain the following spectral upper bound for the line completion number of a dense graph.

Theorem 6. *Let G be a dense graph with m edges. Then,*

$$lc(G) \leq \frac{m}{2}\left(1 - \frac{\mu_1(L(G))}{\mu_b(L(G))}\right) + 1.$$

Example: Let G be the graph obtained by adding an edge to $K_3 \times K_2$ which does not produce multiple edges. Then G is a dense graph and $lc(G) = \rho(G) + 1 > 3$. On the other hand, $\mu_1(L(G)) = 2.5617$ and $\mu_{10}(L(G)) = 7.4728$. By applying the above Theorem, we have $lc(G) \leq 4.28$. Therefore, $lc(G) = 4$.

When G is a regular graph, the upper bound of Theorem 6 can be simplified. For regular graphs it is a well-known fact (see e.g. Biggs [4]) that

$$\mu_1(L(G)) = \mu_1(G) \text{ and } \mu_b(L(G)) = 2d.$$

□

Thus Theorem 6 can be stated in the following way.

Theorem 7. *Let G be a dense regular graph on n vertices and degree d. Let $\mu_1 = \mu_1(G)$ the second smallest eigenvalue of the Laplacian of G. Then,*

$$lc(G) \leq \frac{n}{8}(2d - \mu_1) + 1.$$

□

As an application of the above Theorem we get the following results. Denote by $K_{t[r]}$ the complete t-partite graph, with each partite set of order r. Part (i) of the following Theorem was obtained in [2].

Theorem 8. *For $n \geq 2$ and $k \geq 1$,*

(i) $lc(K_n) = \frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right) + 1$,

(ii) $lc(K_{t[2k]}) = \frac{k^2 t(t-1)}{2} + 1$.

Proof:

(i) Suppose first that $n = 2m$ is an even number. For the complete graph K_{2m} we have $\mu_1(K_{2m}) = 2m$. Then, by Theorem 7, $\rho(K_{2m}) \leq \frac{n(n-2)}{8}$. On the other hand, the set of edges in the subgraph induced by a subset of m vertices has cardinality $\binom{m}{2} = \frac{n(n-2)}{8}$ and leaves $\binom{m}{2}$ edges in the exterior. Therefore, $\rho(K_{2m}) = \frac{n(n-2)}{8}$ and

$$lc(K_{2m}) = \frac{n(n-2)}{8} + 1.$$

Let $n = 2m+1$ be an odd number. The exterior of the set A of edges generated by a set of m vertices of K_{2m+1} has more than $|A|$ edges, so that $\rho(K_{2m+1}) \geq \frac{m(m-1)}{2}$.

Suppose that $\rho(K_{2m+1}) > \frac{m(m-1)}{2}$. Let $A \subset E(K_{2m+1})$ such that $|A| = \rho(K_{2m+1})$. Since $|B_L(A)| + |A| \leq |B_L(A)| + |ex(A)| = m(2m+1)$, we have

$$|B_L(A)| - |A| < m(m+2).$$

On the other hand, let $a = |\langle A\rangle|$ and $a' = |\langle ext(A)\rangle|$. We have $\min\{a, a'\} \geq m+1$ and $|B_L(A)|-|A| \geq a\cdot a' \geq (m+1)^2 > m(m+2)$, a contradiction. Therefore $\rho(K_{2m+1}) = \frac{m(m-1)}{2}$ and

$$lc(K_{2m+1}) = \frac{m(m-1)}{2} + 1.$$

(ii) For the complete t-partite graph $K_{t[2k]}$ we have $\mu_1(K_{t[2k]}) = 2k(t-1)$. By Theorem 7, $\rho(K_{t[2k]}) \leq \frac{k^2t(t-1)}{2}$. Let X consist of k vertices in each stable set of the t-partite graph and let A the set of edges in the subgraph induced by X. Then $|A| = |ex(A)| = \frac{k^2t(t-1)}{2}$. Therefore $\rho(K_{t[2k]}) \geq \frac{k^2t(t-1)}{2}$. By combining the two inequalities, Theorem 5 gives $lc(K_{t[2k]}) = \frac{k^2t(t-1)}{2} + 1$. □

The following Proposition is another example of application of Theorem 7.

Proposition 9. *Let K_n be the complete graph on $n \geq 2$ vertices. Then,*

$$lc(K_n \times K_2) = \frac{n(n-1)}{2} + 1.$$

Proof: The Laplacian eigenvalues of $K_n \times K_2$ can be obtained by all possible sums of the respective Laplacian eigenvalues of K_n and K_2. Since $\mu_1(K_2) = 2$ and $\mu_1(K_n) = n \geq 2$, then $\mu_1(K_n \times K_2) = 2$. Thus, by Theorem 7, $\rho(K_n \times K_2) \leq \frac{n(n-1)}{2} = |E(K_n)|$. Let A be the set of those edges generated by one of the copies of K_n, so $|A| = |E(K_n)| = \frac{n(n-1)}{2}$. Therefore, $\rho(K_n \times K_2) = \frac{n(n-1)}{2}$ and $lc(K_n \times K_2) = \frac{n(n-1)}{2} + 1$. □

4 Complete bipartite graphs

In [2], the authors consider the determination of the line completion number of complete bipartite graphs as an open problem. They suggest that there might not be a global formula of this parameter for complete bipartite graphs because it depends on the parity of n and m as well as its relative sizes.

In this section we analyze the problem of computing $lc(K_{n,m})$ when either $n = m$ or n and m are both even numbers as a first step for solving this problem.

Theorem 10. *Let $K_{n,m}$ the complete bipartite graph. Then,*

$$lc(K_{n,m}) = \rho(K_{n,m}) + 1.$$

Proof: By Lemma 2, it suffices to show that $lc(G) \leq \rho(G) + 1 = r$.

As in the proof of Theorem 5, suppose that there are two non adjacent vertices in $\mathcal{L}_r(G)$. Let m be the smallest integer such that $m = |A \cap A'|$ for a pair A, A' of non adjacent vertices in $\mathcal{L}_r(G)$.

If $m = 0$, then $A' \subset ex(A)$ and $|ex(A)| \geq |A| > \rho(G)$, which contradicts the definition of ρ. Thus $m > 0$.

Let $X = \langle A \backslash A' \rangle$, $Y = \langle A' \backslash A \rangle$ and $u \in A \cap A'$.

Since there is at least one edge in each of the sets $A \backslash A'$ and $A' \backslash A$ and the graph is complete bipartite, then there is a pair of independent edges e, e' such that $e \in e(\langle u \rangle, X)$ and $e' \in e(\langle u \rangle, Y)$. Thus, the sets of edges $(A \cup \{e\}) \backslash \{u\}$ and $(A' \cup \{e'\}) \backslash \{u\}$ are non adjacent vertices in $\mathcal{L}_r(G)$ and its intersection has less that m elements, which lead to a contradiction and the result follows. □

By using the above Theorem, we compute the line completion number of $K_{n,m}$ for several pairs of values of n and m. For our purposes we define the function $odd(n)$ as 1 if n is odd and 0 if n is even.

Theorem 11. *Let $1 \leq n \leq m$. If $n = m$ or n and m are even numbers, then*

$$lc(K_{n,m}) = \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{m}{2} \right\rfloor + odd(n \cdot m) \right) + 1.$$

Proof: We consider two cases.

Case 1. $n = m$.

If $n = m$ is an even number the result follows from part (ii) of Theorem 8. If $n = m = 2k + 1$ is an odd number, then we have $\mu_1(K_{n,n}) = n$. By Theorem 7, $\rho(K_{n,n}) \leq \left\lfloor \frac{n^2}{4} \right\rfloor = k^2 + k$. Let X and Y be the bipartition sets of $K_{n,n}$ and let A be the set of edges generated by the first k vertices of X and the first $k+1$ vertices of Y. Clearly, $|A| = |ex(A)| k^2 + k$. Therefore, $\rho(K_{n,n}) = \left\lfloor \frac{n^2}{4} \right\rfloor = k^2 + k$ and

$$lc(K_{n,n}) = k^2 + k + 1,$$

as claimed.

Case 2. $n = 2k \leq 2q = m$.

Let $X = 2k$ and $Y = 2q$ be the bipartition sets of $K_{2k,2q}$. Let A_1 be the set of edges generated by the first k vertices of X and the first q vertices of Y. We have $|ex(A_1)| = |A_1| kq$ and therefore $\rho(K_{2k,2q}) \geq kq$.

Suppose that $\rho(K_{2k,2q}) > kq$. Let A be a set of vertices such that $|A| = \rho(G)$ and $|ex(A)| \geq |A|$. Let $Z_1 = \langle A \rangle$ and $Z_2 = \langle ex(A) \rangle$, $X_i = Z_i \cap X$ and $Y_i = Z_i \cap Y$, $i = 1, 2$. We have

$$kq < |A| \leq |X_1| \cdot |Y_1| \text{ and } kq < |ex(A)| \leq |X_2| \cdot |Y_2|.$$

Since $|B_L(A)| + |ex(A)| = 4kq$, we have $|B_L(A)| - |A| < 2kq$. On the other hand, the above inequalities imply

$$|B_L(A)| - |A| \geq |X_1| \cdot |Y_1| + |X_2| \cdot |Y_2| \geq 2\sqrt{|X_1| \cdot |Y_1||X_2| \cdot |Y_2|} > 2kq,$$

a contradiction. Therefore, $\rho(K_{2k,2q}) = kq$ and $\rho(K_{2k,2q}) = kq + 1$. □

We conjecture that the formula above is true for all bipartite complete graphs.

Conjecture 12 $lc(K_{n,m}) = \lfloor \frac{n}{2} \rfloor \left(\lfloor \frac{m}{2} \rfloor + odd(n \cdot m) \right) + 1$, $1 \leq n \leq m$. □

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