A HIERARCHY OF COMPLETE ORTHOGONAL STRUCTURES

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ABSTRACT. We provide a hierarchy, linearly ordered by inclusion, describing various complete sets of combinatorial objects starting with complete sets of mutually orthogonal latin squares, generalizing to affine geometries and designs, frequency squares and hypercubes, and ending with (t, m, s)-nets.

1. AN OVERVIEW

In 1938 R.C. Bose [2] introduced what has become a fundamental and classical result in combinatorial design theory by demonstrating an equivalence between complete sets of mutually orthogonal latin squares (MOLS) of a given order and affine planes of the same order. In the last decade a number of results have generalized Bose's result to more complex combinatorial structures; see for example [14] or [15].

On one hand these generalizations have involved orthogonal frequency squares and orthogonal hypercubes which are natural but distinct generalizations of orthogonal latin squares, and on the other, affine geometries and affine resolvable designs which generalize affine planes. Working in the same time period from a very different perspective Niederreiter [25] introduced the concept of a (t, m, s)-net. Among the many useful properties (for example in the numerical integration of high dimensional integrals) exhibited by this latter structure is the ability to find examples which represent many of the above combinatorial objects. In this sense one might hope to establish (t, m, s)-nets as a general unifying structure. Towards this end, this paper is intended to clarify the relationships between the various combinatorial objects mentioned above by providing a linear ordering beginning with complete sets of MOLS and ending with (t, m, s)-nets.

In developing this hierarchy space considerations were a limiting parameter. In particular this forced the decision to restrict the system to a linear ordering involving complete sets rather than a more complex partially ordered structure which would include various partial designs such as Bruck nets, orthogonal arrays and transversal designs. However, as a future second step, the hierarchy provided here could be expanded to include such partial designs.

Even then certain relations may be open to interpretation and perhaps dispute. Affine geometries generalize affine planes by dimension and, on this basis, our system shows the former generalizing the latter. But all affine geometries are desarguesian which is not true of all affine planes.

The hierarchy given below holds for prime power orders. For example Level II, consisting of affine geometries, exists for dimension $d \geq 3$ if and only the order is a prime power. The remaining structures in Levels I-V exist for all prime powers but it is unknown whether any examples exist in nonprime power cases. In the case of complete sets of MOLS Mullen, [20], has nominated the question of their existence in the nonprime power case as a candidate for the "next Fermat problem".

It is not clear what the implications would be if one of these structures were found to exist in a nonprime power case. For example a complete set of MOLS of order m^h, for m a prime power, can always be used to construct a complete set of mutually orthogonal frequency squares (MOFS) of type $F(m^h; m^{h-1})$ by substitutions tothe symbols, [10]. But these substitutions correspond to blocks of an affine design whose existence, in turn, is also guaranteed by the fact m is a prime power. If for a particular value of m that is not a prime power, the MOLS existed, but not the required design, would a complete set of $F(m^h; m^{h-1})$ MOFS exist for that m? Specifically, suppose 99 MOLS of order 100 were discovered. Since it is known that 9 MOLS of order 10 do not exist and that a standard method would require these order 10 MOLS to construct a complete set of $F(100; 10)$ MOFS from the 99 MOLS of order 100, the existence of the MOFS would remain an open question.

Similarly to construct either a complete set of latin hypercubes or an affine design from a complete set of MOFS requires, in both cases, a second affine design whose points are the rows of the squares and blocks are certain collections of rows. If the MOFS existed but the second design did not, would one find a complete set of latin hypercubes or the affine design with matching parameters?

However, since in this paper all orders are assumed to be a prime power these issues do not arise.

The basic hierarchy is given in (1) below.

$$\begin{aligned} I.\ \ affine\ \ planes\ &\subset II.\ \ affine\ \ geometries \\ &\subset III.\ \ complete\ \ sets\ \ of\ \ frequency \\ &\qquad squares\ \ and\ \ rectangles \\ &\subset IV.\ \ complete\ \ sets\ \ of\ \ type\, 1\ \ hypercubes \\ &\subseteq V.\ \ complete\ \ sets\ \ of\ \ type\, 0\ \ hypercubes \\ &\qquad and\ \ affine\ \ designs \\ &\subset VI.\ \ (t,m,s)-nets \end{aligned} \tag{1}$$

The objects in Levels I and II of (1) are classical and well-known. Through the well-known connection between affine planes and complete sets of orthogonal latin squares (MOLS), Level I traces its origins back to Euler and his famous problem concerning the arrangements of 36 officers. A wide variety of relationships connecting such sets to other combinatorial structures are known, and some are given in Theorem 1 below.

2. MUTUALLY ORTHOGONAL LATIN AND FREQUENCY SQUARES

Theorem 1. *The existence of one of the following implies the existence of all others.*

(1) *a complete set of MOLS of order n;*
(2) *an affine plane of order n;*
(3) *a projective plane of order n;*
(4) *a transversal design of index 1, blocksize $n+1$, and groupsize n;*
(5) *an orthogonal array of strength 2, index 1, degree $n+1$, and order n.*

An affine plane satisfies the following system of axioms.

A1 There is a unique line joining any two points.

A2 Given a point P and a line ℓ not containing P, there is a unique line containing P and not intersecting ℓ.

A3 There exist three points not on a common line.

An affine geometry, $AG(d,n)$, of dimension d and order n may be constructed from the points and $(d-1)$-dimensional subspaces and cosets, ie the $(d-1)$-flats, of the d-dimensional vector space $(F_n)^d$ over F_n, the field of n elements. Recursively all such k-flats, $k = 0, 1, \ldots, d-1$ are themselves affine geometries of dimension k with 0-flats corresponding to points, 1-flats to lines, and 2-flats to planes. The points and lines in the 2-flats, obtained in this way, satisfy axioms A1, A2, and A3 above. In this sense it is natural to view affine geometries as a higher dimensional generalization of affine planes despite the earlier observation that all 2-flats of d-dimensional, $d \geq 3$, affine geometries, but not all affine planes, are desarguesian.

Frequency squares are squares in which each symbol i, $i = 1, 2, \ldots, m$ occurs λ_i times in each row and column. We will be concerned only with the case where $\lambda_1 = \lambda_2 = \cdots = \lambda_m$. Specifically for m and h positive integers with $m \geq 2$, an $F(m^h; m^{h-1})$ freque ncy square is an $m^h \times m^h$ array in which each of m symbols occurs exactly m^{h-1} times in each row and column. Two such frequency squares are orthogonal if each ordered pair of symbols occurs m^{2h-2} times when one square is superimposed on the other. A complete set of mutually orthogonal frequency squares (MOFS) of type $F(m^h; m^{h-1})$ has $(m^h - 1)^2/(m-1)$ members; see Hedayat, Raghavarao and Seiden [4]. Such complete sets are known to exist for m a prime power; see again [4]. The only other known example of complete sets of MOFS are those of type $F(4t; 2t)$ which exist whenever there exists a Hadamard matrix of order $4t$; see Morgan [18] where a generalization of this result is discussed. These MOFS will be of type $F(m^h; m^{h-1})$ only if $t = 2^k$ for some positive integer k in which case $4t$ will be a prime power.

We will represent an affine resolvable design with blocksize k and n blocks per parallel class as an $AD(k,n)$. With two parameters given, the remaining parameters of such a design are fixed. As a design the affine geometry $AG(d,n)$ is an $AD(n^{d-1}, n)$.

For m a prime power, a complete set of $F(m^h; m^{h-1})$ MOFS will, in certain cases, give the even-dimensioned affine geometry $AG(2h, m)$. In those cases in which such a complete set does not give the equivalent affine geometry, it will give an affine design whose parameters are consistent with that of the geometry. So as not to exclude odd-dimensioned affine geometries, and the affine designs whose parameters match these geometries, we consider as well complete sets of mutually orthogonal frequency rectangles (MOFR) of type $F(m^h, m^{h-1}; m^{h-2}, m^{h-1})$. These rectangles have m^h rows and m^{h-1} columns, and each of the m symbols occurs m^{h-2} times in each row and m^{h-1} times in each column. Two such rectangles are orthogonal if, upon superposition of one rectangle on the other, each ordered pair occurs m^{2h-3}

times. A complete set in this case has $(m^h-1)(m^{h-1}-1)/(m-1)$ members. In the interest of conciseness we will focus on the even-dimensional cases which correspond to square frequency objects. Nevertheless virtually all, if not all, results extend naturally to the odd-dimensional cases.

The initial result connecting frequency objects and geometries was provided by Mullen [19] who showed, implicitly, that any affine geometry $AG(2h, m)$, of even dimension, gives a complete set of MOFS of type $F(m^h; m^{h-1})$. This result motivated a number of others that clarified further the relationships between complete sets of MOFS and MOFR on the one hand, and affine geometries and affine resolvable designs on the other.

In [15], Laywine and Mullen gave an equivalence between affine designs of type $AD(m^{2h-1}, m)$ and complete sets of MOFS of type $F(m^h; m^{h-1})$ when the MOFS were derived by special substitutions to the symbols of a complete sets of MOLS. In doing so they gave a characterization of those sets of MOFS equivalent to an affine geometry. This characterization implied that

$$\text{Level II} \subseteq \text{Level III}.$$

Laywine [11] extended the conversion of MOFS to an $AD(m^{2h-1}, m)$ design to include all sets of complete MOFS of type $F(m^h; m^{h-1})$ for m a prime power rather than just those obtainable by substitutions to the symbols of MOLS. More recently the same author [13] showed when this conversion could be reversed, and in doing so, characterized those $AD(m^{2h-1}, m)$ designs that could be represented as a complete set of MOFS of type $F(m^h; m^{h-1})$. At the same time he exhibited an $AD(8, 2)$ design that could not be represented as a complete set of $F(4; 2)$ MOFS, thus providing an explicit proof that the class of complete sets of $F(m^h; m^{h-1})$ MOFS is a proper subset of the class of $AD(m^{2h-1}, m)$ affine designs. In terms of the hierarchy (1), this showed that

$$\text{Level III} \subset \text{Level } V.$$

Suchower, [29] and [30], has shown the analogous results connecting MOFR of type $F(m^h, m^{h-1}; m^{h-2}, m^{h-1})$ and the affine geometry $AG(2h-1, m)$. Similarly results connecting complete sets of $F(m^h, m^{h-1}; m^{h-2}, m^{h-1})$ MOFR with $AD(m^{2h-2}, m)$ affine designs can be obtained by a straightforward generalization of the results in [29] and [30].

A hypercube of dimension d, order n and type 0 is an $n \times n \times \cdots \times n$ array with n^d points and n distinct symbols such that each symbol occurs n^{d-1} times in the array. If any $1 \leq k \leq d-1$ coordinates are fixed, and if each symbol occurs n^{d-k-1} times in the subarray so defined, the hypercube is said to be of type k. It should be noted that any hypercube of type

k is also a hypercube of type $0 \le j < k$. Two hypercubes, regardless of type, are orthogonal if upon superposition of one on the other, each of the n^2 ordered pairs occurs n^{d-2} times. Since two orthogonal hypercubes of type 1 and dimension 2 are just orthogonal latin squares we will refer to type 1, and higher, hypercubes as latin hypercubes. A complete set of latin hypercubes has $(n^d - 1)/(n-1) - d$ members; see [16].

Consider the array

$$\begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{matrix}$$

In isolation one might view this array as an $F(4;2)$ frequency square, and, indeed, it is a member of a complete set of 9 orthogonal $F(4;2)$ MOFS. In fact it is a member of H_1, H_2 and H_3, the three non-isomorphic complete sets of $F(4;2)$ MOFS first given by Schwager, Federer and Raktoe [27] and shown below. Two sets of MOFS are isomorphic if the frequency squares of one set can be transformed into the frequency squares of the other by permutations on the symbols of some squares and a series of permutations on the rows and columns of all the squares in the set.

	1	2	3	4	5	6	7	8	9
	0011	0101	0110	0011	0011	0101	0101	0110	0110
H_1 :	0011	0101	0110	1100	1100	1010	1010	1001	1001
	1100	1010	1001	0011	1100	0101	1010	0110	1001
	1100	1010	1001	1100	0011	1010	0101	1001	0110
	0011	0101	0110	0011	0011	0101	0101	0110	0110
H_2 :	0011	0101	0110	1100	1100	1010	1010	1001	1001
	1100	1010	1001	0011	1100	0110	1001	0101	1010
	1100	1010	1001	1100	0011	1001	0110	1010	0101
	0011	0101	0110	0011	0011	0101	0101	0110	0110
H_3 :	0011	0101	0110	1100	1100	1010	1010	1001	1001
	1100	1010	1001	0101	1010	0110	1001	0011	1100
	1100	1010	1001	1010	0101	1001	0110	1100	0011

However written as

0	1		1	0
0	1		1	0
1	0		0	1
1	0		0	1

one might be more inclined to interpret the array as a 4-dimensional latin hypercube of order 2. As such it is a member of a complete set of 11 mutually orthogonal latin hypercubes given below.

The 9 MOFS of set H_1 together with the six canonical row and column squares

0000	0000	0000	0011	0101	0110
0000	1111	1111	0011	0101	0110
1111	0000	1111	0011	0101	0110
1111	1111	0000	0011	0101	0110

are equivalent to an $AD(8,2)$ design isomorphic to the geometry $AG(2,4)$. The 11, 4-dimensional latin MOHC of order 2

1	2	3	4	5	6	7	8	9	10	11
0011	0101	0110	0011	0011	0101	0101	0110	0110	0000	0110
0011	0101	0110	1100	1100	1010	1010	1001	1001	1111	0110
1100	1010	1001	0011	1100	0101	1010	0110	1001	1111	0110
1100	1010	1001	1100	0011	1010	0101	1001	0110	0000	0110

together with the 4 canonical hypercubes

0000	0000	0011	0101
0000	1111	0011	0101
1111	0000	0011	0101
1111	1111	0011	0101

are equivalent to the same design. In fact, viewed as orthogonal arrays, both sets consist of the same 15 arrays.

3. FREQUENCY SQUARES, RECTANGLES AND LATIN HYPERCUBES

A similar natural identification exists between odd-dimensioned latin hypercubes and the frequency rectangles we have considered. The natural question is then: For $h \geq 2$ is the class of complete sets of $F(m^h; m^{h-1})$ MOFS and $F(m^h, m^{h-1}; m^{h-1}, m^{h-2})$ MOFR equivalent to the class of complete sets of latin MOHC of order m and dimension $d \geq 2h-1$? In more intuitive terms are frequency squares, at least as far as membership in a complete set is concerned, simply "flattened" hypercubes which compensate for the restriction to dimension 2 by permitting greater flexibility in the repetition of symbols? If the classes are not equivalent

Does the existence of a member of Level III imply the existence of a member with matching parameters in Level IV, or, alternatively, could the Level IV member imply the existence of the Level III member?

While it is not difficult to show that hypercubes give a class of designs at least as large as MOFS, as done in the following theorem, a complete answer to these questions will be given in Theorem 7. We begin by showing that

$$\text{Level III} \subset \text{Level IV}.$$

Theorem 2. *For m a prime power the existence of a complete set of $(m^h - 1)^2/(m-1)$ MOFS of type $F(m^h; m^{h-1})$ implies the existence of a complete set of $(m^{2h} - 1)/(m-1) - 2h$ latin hypercubes of order m and dimension $2h$.*

Proof. Assign coordinates $x_1, x_2, \ldots, x_h$ and $y_1, y_2, \ldots, y_h$ to the rows and columns, respectively, of the frequency squares where $x_i, y_i \in F_m$, the field of m elements for $i = 1, \ldots, h$. Fixing any one of these $2h$ coordinates in any of the frequency squares defines a subarray consisting of m^{h-1} rows if the coordinate is one of $x_1, x_2, \ldots, x_h$ or m^{h-1} columns in the case of $y_1, y_2, \ldots, y_h$. Any such subarray contains each of the m symbols m^{2h-2} times so that each frequency square gives a latin hypercube. The orthogonality of the MOFS implies the orthogonality of the hypercubes.

It follows from Mullen's polynomial representation of MOFS [19] that there are exactly $(m^h - 1)/(m-1)$ distinct polynomials of the form $a_1x_1 + a_2x_2 + \cdots + a_hx_h$ such that none is a multiple of another, and no polynomial has all zero coefficients. Further, each of these polynomials represents a unique canonical row square. As well it follows from [12] and [16] that these same polynomials represent $(m^h - 1)/(m-1)$ orthogonal hypercubes of dimension $2h$, order m, and type at most $h-1$. Of these polynomials there are exactly h with a single nonzero coefficient, and these represent canonical type 0 hypercubes. The remaining polynomials represent latin or type 1 hypercubes; in fact it is shown in [16] that the type is exactly one less than the number of nonzero coefficients.

Similarly, in the context of MOFS, polynomials of the form $b_1y_1 + b_2y_2 + \cdots + b_hy_h$ represent a set of canonical column squares, and in the context of hypercubes, a set of h canonical type 0 hypercubes together with $(m^h - 1)/(m-1) - h$ latin hypercubes.

Combining the F-squares and canonical row and column squares repesented by polynomials with two nonzero coefficients gives the required

$$\frac{(m^h-1)^2}{m-1} + 2\left(\frac{m^h-1}{m-1} - h\right) = \frac{m^{2h}-1}{m-1} - 2h$$

latin hypercubes. ■

In summary a complete set of $(m^h-1)^2/(m-1)$ MOFS of type $F(m^h; m^{h-1})$ together with sets of $(m^h-1)/(m-1)$ canonical row squares and $(m^h-1)/(m-1)$ canonical column squares can be converted to an $AD(m^{2h-1}, m)$ design or to a complete set of $(m^{2h}-1)/(m-1) - 2h$ orthogonal latin hypercubes of dimension $2h$ and $2h$ canonical type 0 hypercubes.

Corollary 3. *If an $AD(m^{2h-1}, m)$ design can be derived from a complete set of $(m^h-1)^2/(m-1)$ MOFS of type $F(m^h; m^{h-1})$ using the above construction, the same design can be obtained from a complete set of $(m^{2h}-1)/(m-1) - 2h$ latin MOHC of order m and dimension $2h$.*

4. FREQUENCY SQUARES AND AFFINE DESIGNS

Bhat and Shrikhande [1] introduced the *characteristic* of an affine design, and used it to distinguish between nonisomorphic designs with the same parameters. The characteristic is given by the number of *special $\mu-$tuples* in an $AD(k, n)$ where a special $\mu-$tuple is a set of $\mu = k/n$ points contained in $n+1$ blocks of that $AD(k, n)$. Laywine [13] extended this notion of special $\mu-$tuples to that of hyperspecial $\mu-$tuples and to orthogonal classes of special and hyperspecial $\mu-$tuples in order to characterize those affine designs equivalent to a complete set of MOFS. The next theorem from [13] provides that characterization.

Theorem 4. *For m a prime power and h a positive integer, the existence of an $AD(m^{2h-1}, m)$ with two orthogonal classes of hyperspecial m^h-tuples is equivalent to the existence of a complete set of $F(m^h; m^{h-1})$ MOFS.*

Theorem 4 implies that, where parameters of the MOFS and affine design are compatible, the class of complete sets of MOFS is properly included in the class of affine designs. In [13] this was explicitly demonstrated using the five existing nonisomorphic $AD(8, 2)$ designs provides by Bhat and Shrikhande. Three of these were identified with H_1, H_2 and H_3, the three complete sets of $F(4; 2)$ MOFS displayed in Section 2, and the remaining two were shown to lack the necessary orthogonal classes of special 4-tuples to be represented as MOFS. Hence

$$\text{Level III} \subset \text{Level V}.$$

5. HYPERCUBES AND AFFINE DESIGNS

In this section we will examine relationships between complete sets of type 0 hypercubes, type 1 or latin hypercubes, and affine designs. We begin by stating an equivalence between type 0 hypercubes and affine designs. This equivalence follows from Theorem 3.2 of Shrikhande [28].

Theorem 5. *The existence of a complete set of $(n^d - 1)/(n - 1)$ type 0 hypercubes of dimension d and prime power order n is equivalent to the existence of an $AD(n^{d-1}, n)$.*

Suppose we wish to restrict the complete set from the most general case of type 0 hypercubes, as above, to the case of d canonical hypercubes and $(n^d - 1)/(n - 1) - d$ of at least type 1. In the canonical hypercube $C_k, k = 1, 2, \ldots, d$ symbol s occurs only in all the points in the subarray determined by setting coordinate $x_k = s$ for $s = 0, 1, \ldots, n-1$. In a type $j = 1, \ldots, d - 1$ hypercube, every subarray determined by fixing j coordinates contains each symbol n^{d-j-1} times. Then, if the set of $(n^d - 1)/(n - 1)$ hypercubes contains d canonical members, Theorem 6 shows the rest must be type 1 (and higher) so that the set is latin.

Theorem 6. *If d members of a set of $(n^d - 1)/(n - 1)$, d-dimensional type 0 hypercubes of order n are canonical the remaining hypercubes are latin.*

Proof. Let the canonical hypercube C_k be superimposed on any noncanonical hypercube H. Upon superimposition of the hypercubes orthogonality requires that in the array $x_k = s$ the ordered pair (s, t), for any $t = 0, 1, \ldots, n - 1$, will appear n^{d-2} times. In turn this implies that any symbol t occurs n^{d-2} times in this particular subarray of H. By repeating for all $s = 0, 1, \ldots, n - 1$ and then for all $k = 1, 2, \ldots, d$ it follows that all noncanonical hypercubes are latin. ∎

Lemma 5.5 of Niederreiter [25] showed that any set of $n + 1$ orthogonal squares of order n could be transformed into $n - 1$ MOLS together with a canonical row and a canonical column square. It follows from Theorem 6 that an analogous transformation can be done for $d > 2$ if one can find d canonical hypercubes among the $(n^d - 1)/(n - 1)$ type 0 hypercubes, or, equivalently, find d parallel classes among the $(n^d - 1)/(n - 1)$ classes of an $AD(n^{d-1}, n)$ such that the intersection of any $k = 1, 2, \ldots, d$ nonparallel blocks from these classes contains exactly n^{d-k} points. If the d coordinates are defined so that the levels of each corresponds to the occurrences of the symbols of one of the d canonical classes, then each point will be given uniquely by the coordinates $(x_1, x_2, \ldots, x_d)$. Mullen and Whittle [24] defined a set of $s \geq d$ hypercubes of dimension d to be d-orthogonal if superimposition of any subset of d of the hypercubes gave each d-tuple exactly once. Clearly our condition on the d special classes corresponds exactly to the notion of d-orthogonality defined by Mullen and Whittle in the case where $s = d$.

A sufficient condition for the existence of d such classes is the existence of $d-1$ independent prime classes as introduced and studied by Kimberley [7] and Mavron [17]. Prime classes intersect all other classes in the design in a manner that resembles the intersections of hyperplanes in an affine geometry. We require classes where only their mutual intersections resemble those of prime classes and hyperplanes. It is unknown whether such a set of d classes can always be found in an $AD(n^{d-1}, n)$, i.e. in a set of $(n^d - 1)/(n-1)$ type 0 hypercubes of dimension d and order n, so that we can only conclude that

$$\text{Level IV} \subseteq \text{Level V}.$$

This uncertainy suggests the following:

Open Question. *Is the existence of every set of $(n^d - 1)/(n-1)$ type 0 hypercubes of dimension d and order n equivalent to the existence of d canonical and $(n^d-1)/(n-1)-d$ latin hypercubes of dimension d and order n?*

In the case of the two nonisomorphic $AD(8,2)$ designs, neither of which could be represented by a complete set of $F(4;2)$ MOFS, a computer search revealed that both contained many collections of 4 classes with the intersection properties permitting them to be canonical hypercubes. A similar computer search of a number of $AD(16,2)$ designs revealed in each design many collections of 5 canonical classes. While the search of $AD(16,2)$ designs was far from exhaustive, the designs chosen were those which one would expect to have fewer special μ-tuples and so probably less likely to have canonical classes. Combining Theorem 2 and the example of the last two of five nonisomorphic $AD(8,2)$ designs gives:

Theorem 7. *The class of complete sets of MOFS of type $F(m^h; m^{h-1})$ is a proper subset of the class of complete sets of latin hypercubes of dimension $2h$ and order m.*

For the sake of completeness we close this section by mentioning two other notions of orthogonality for hypercubes that have been discussed by Höhler in [6] and Morgan [18]. In [6] Höhler considers a stronger notion of orthogonality and shows that for $d \geq 3$, a complete set of $(n-1)^{d-1}$ Höhler hypercubes of dimension d exists if and only if n is a prime power (when $d = 2$, this notion reduces to the usual notion of MOLS). In [18] Morgan discusses yet another notion of orthogonality, called *equiorthogonality*, and shows that if m is a prime power, then there exist complete sets of $(n-1)^d/(m-1)$ equiorthogonal frequency hypercubes of dimension $d \geq 2$ based upon m symbols. While both of these notions are generalizations of the usual concept of a hypercube, to date no general relationship has been established between either of these and (t,m,s)-nets other than in the d=2 case. Accordingly these objects will not be discussed further in this paper.

6. (t,m,s)-NETS

We now give the definition for the fundamental concept of a (t,m,s)-net in base n, as first discussed by Niederreiter in [25]. Let $s \geq 1$ be a fixed integer. For an integer $n \geq 2$, an *elementary interval in base n* is an interval of the form

$$E = \prod_{i=1}^{s} [a_i n^{-d_i}, (a_i + 1) n^{-d_i}),$$

with integers $d_i \geq 0$ and integers $0 \leq a_i < n^{d_i}$ for $1 \leq i \leq s$. For integers $0 \leq t \leq m$, a (t,m,s)*-net in base n* is a point set of n^m points in $[0,1)^s$ such that every elementary interval E in base n of volume n^{t-m} contains exactly n^t points of the point set. We refer to [25] for a systematic development of the theory and various constructions of nets and to [25,26] for applications of nets to various areas of numerical analysis. Paper [21] presents a brief survey of several combinatorial constructions of nets.

For surveys of constructions of (t,m,s)-nets we refer to [22] and to Clayman, Lawrence, Mullen, Niederreiter and Sloane [3] which provides an updated table of net parameters as well as a brief summary of various methods of net construction. These tables can be found on the Journal of Combinatorial Designs web page located at http://www.emba.uvm.edu/~colbourn/jcd/T table-base2.html.

The following results provide connections between (t,m,s)-nets and complete orthogonal structures discussed earlier in this paper. In [25] Niederreiter proved

Theorem 8. *Let $n \geq 2$ be an integer. Then there exists a complete set of $n+1$ mutually orthogonal squares of order n if and only if there exists a $(0,2,n+1)$-net in base n.*

Theorem 8 was generalized by Mullen and Whittle [24] to

Theorem 9. *Let $n \geq 2$, $t \geq 0$ be integers. Then there exists a complete set of $(n^{t+2}-1)/(n-1)$ hypercubes of dimension $t+2$, order n and type 0 if and only if there exists a $(t, t+2, (n^{t+2}-1)/(n-1))$-net in base n.*

We also note from Laywine, Mullen and Whittle [16] that complete sets of MOLS and hypercubes of order n are conjectured to exist if and only if n is a prime power.

In Level IV it was shown that complete sets of type 0 hypercubes are equivalent to certain affine designs so that the Theorems 8 and 9 could be restated in terms of (t, m, s)-nets and affine designs.

In [23] it was shown by Mullen and Schmid that the existence of a $(t, t+k, s)$-net is equivalent to the existence of a set of strongly orthogonal hypercubes. Strongly orthogonal hypercubes are hypercubes whose elements are vectors, and whose orthogonality generalizes the usual notion of pairwise orthogonality. We refer to [23] for the definition, as well as a proof of

Theorem 10. *Let $s \geq 1$, $n \geq 2$, $t \geq 0$, $k \geq 2$ be integers. Then there exist s strongly orthogonal hypercubes of dimension $t+k$, order n, and strength k if and only if there exists a $(t, t+k, s)$-net in base n.*

Lastly we demonstrate the construction of a net that is not equivalent to a complete set of type 0 hypercubes, thus showing that Level V in (1) is strictly contained in Level VI.

Table 1 of [23] provides a complete set of 7 strongly orthogonal hypercubes of dimension 4, order 2, and strength 3. Since each of these hypercubes has exactly 4 occurrences of four symbols all 7 are type 0. By Theorem 10, these hypercubes are equivalent to a $(1, 4, 7)$-net in base $n = 2$. But they are not pairwise orthogonal since superimposition of one of the 7 hypercubes on any one of the remaining 6 produces 8 ordered pairs, each of which occurs twice, and leaves 8 other ordered pairs unrepresented. Hence this $(1, 4, 7)$-net in base 2 is not equivalent to a complete set of orthogonal hypercubes of type 0.

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