Skolem arrays and Skolem labellings of ladder graphs

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Abstract

We introduce Skolem arrays, which are two-dimensional analogues of Skolem sequences. Skolem arrays are ladders which admit a Skolem labelling in the sense of [2]. We prove that they exist exactly for those integers $n \equiv 0$ or 1 (*mod* 4). In addition, we provide an exponential lower bound for the number of distinct Skolem arrays of a given order. Computational results are presented which give an exact count of the number of Skolem arrays up to order 16.

1 Introduction

A *Skolem sequence* of order n is an integer sequence of length $2n$ in which each of the integers $1, \ldots, n$ occurs twice, and, for each $1 \leq i \leq n$, the two occurrences of i are distance i apart. Skolem sequences and their generalizations have been actively studied by numerous authors; see [4] for a comprehensive survey of known results. Two of the fundamental results on Skolem sequences are the following.

Theorem 1 [5] *Skolem sequences of order n exist if and only if $n \equiv 0$ or 1 (mod 4).*

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Theorem 2 [1] *The number of distinct Skolem sequences of order n is $\geq 2^{\lfloor \frac{n}{3} \rfloor}$.*

In this article, we introduce *Skolem arrays*, which may be thought of as two-dimensional generalizations of Skolem sequences and prove analogues to Theorems 1 and 2. In [2] and [3], Mendelsohn and Shalaby studied Skolem labellings of several classes of graphs, including paths (Skolem labellings of paths amount to studying Skolem sequences), trees, and cycles. A $2 \times n$ Skolem array provides such a (weak) Skolem labelling for the Cartesian products $K_2 \square P_n$, i.e. *the ladder graphs.*

In Section 3 we solve the spectral problem for Skolem arrays: Skolem arrays of order n exist exactly for $n \equiv 0, 1 \ (mod\ 4)$. In Section 4, we give an exponential lower bound for the number of Skolem arrays of order n. Section 5 is devoted to the so-called split arrays; we show split arrays exist for all orders of the spectrum. In the last Section, we summarize the results of computational searches that produced the exact number of Skolem sequences of all orders up to 16 inclusive. We include an Appendix that provides a list of all distinct Skolem arrays up to order 5 inclusive.

2 Definitions

Definition 3 *Let A be a $2 \times n$ array. Then n is the* **order** *of A and the* **distance** *between two positions (a, b) and (c, d) of A is $|c - a| + |d - b|$.*

Definition 4 *A* **Skolem array** *is a $2 \times n$ array A in which each $i \in \{1, \ldots, n\}$ occurs in two positions of A which are distance i apart. The pair (i, i) is* **split** *if i appears in both rows; otherwise, (i, i) is* **nonsplit**. *For $1 \leq i \leq n$, we define $c_1(i)$ and $c_2(i)$ to be the column of the first and second instance of i, respectively, reading from left to right.*

By convention, the Skolem array of order 0 is the empty array. There is one Skolem array of order 1:

1
1

. It is easy to see that there are no Skolem arrays of order 2 or 3.

3	1	1	4
4	2	3	2

and

5	3	2	4	2
4	1	1	3	5

are Skolem arrays of orders 4 and 5, respectively. In the given array of order 5, $c_1(3) = 2$ and $c_2(5) = 5$.

The property of Definition 4 defining Skolem arrays may be restated as follows:

if (i, i) is split, then

$$c_2(i) = c_1(i) + (i - 1); \tag{1}$$

if (i, i) is non-split then

$$c_2(i) = c_1(i) + i. \tag{2}$$

3 The spectrum of Skolem arrays

Theorem 5 *A Skolem array of order n exists if and only if*

$$n \equiv 0 \text{ or } 1 \ (mod\ 4).$$

PROOF. We prove first that there are no Skolem arrays of order $n \equiv 2, 3$ $(mod\ 4)$.

The sum of all the column labels of A is

$$\sum_{i=1}^{n}(c_1(i) + c_2(i)) = 2 \cdot \sum_{i=1}^{n} i = n(n+1). \tag{3}$$

Define $N = \{i : (i, i) \text{ is non-split in } A\}$, $S = \{i : (i, i) \text{ is split in } A\}$.

Using (1) and (2) plus $N \cap S = \emptyset$ and $N \cup S = \{1, \ldots, n\}$, we have that (3) equals

$$\begin{aligned}
& \sum_{i \in N} c_1(i) + \sum_{i \in N}(c_1(i) + i) + \sum_{i \in S} c_1(i) + \sum_{i \in S}(c_1(i) + (i-1)) \\
= \;& 2\sum_{i \in N} c_1(i) + 2\sum_{i \in S} c_1(i) + \sum_{i \in N} i + \sum_{i \in S}(i-1) \\
= \;& 2\sum_{i=1}^{n} c_1(i) + \sum_{i=1}^{n} i - |S| \\
= \;& 2\sum_{i=1}^{n} c_1(i) + \frac{n(n+1)}{2} - |S|.
\end{aligned}$$

Hence,

$$\sum_{i=1}^{n} c_1(i) = \frac{n(n+1) + 2|S|}{4} \in \mathbf{Z},$$

so

$$n(n+1) + 2|S| \equiv 0 \pmod 4\ . \tag{4}$$

Since the number of entries in the first row of A from non-split pairs must be even, $|S| \equiv n$ $(mod\ 2)$. If A is a Skolem array of order $n \equiv 2$ $(mod\ 4)$, then

$2|S| \equiv 0 \pmod 4$, so the left hand side of (4) becomes $2(3) + 0 \equiv 2 \pmod 4$ which is a contradiction. Similarly, if $n \equiv 3 \pmod 4$ then $2|S| \equiv 2 \pmod 4$ and so the left hand side of (4) becomes $3(4) + 2 \equiv 2 \pmod 4$, also a contradiction.

We complete the proof of the theorem by recursively constructing Skolem arrays of all orders $n \equiv 0$ and $1 \pmod 4$. Our induction on n begins with the Skolem arrays of order 0 and 1.

Assume that A is a Skolem array of order n. We construct a Skolem array of order $n+4$ by adding two columns

$n+4$	$n+2$
$n+3$	$n+1$

to the left of A and two columns

$n+3$	$n+2$
$n+1$	$n+4$

to the right. ∎

Corollary 6 *A ladder graph, $K_2 \square P_n$, has a Skolem labelling if and only if $n \equiv 0$ or 1 (mod 4).*

Although the spectrum of Skolem arrays is the same as the spectrum of Skolem sequences, there is no obvious one-one correspondence between the two classes. In fact, as n grows, there are apparently more Skolem sequences than Skolem arrays (see Section 6).

4 The number of Skolem arrays

In this section we give an exponential lower bound for the number of distinct Skolem arrays of order n. However, we must first develop ways of extending Skolem arrays.

A Skolem array A of order $2n$ is *vertically split* if each value $1, \ldots, 2n$ occurs in the first n columns. There are exactly four vertically split Skolem arrays of order 4:

4	2	3	2
3	1	1	4

3	1	1	4
4	2	3	2

2	3	2	4
4	1	1	3

4	1	1	3
2	3	2	4

Using the construction of Theorem 5 on a vertically split array A, it is easy to see that there are vertically split Skolem arrays of all orders $n \equiv 0 \pmod 4$.

Definition 7 *Let A be a Skolem array of order n. Let E be a $2 \times 2k$ vertically split array, for $k > 0$. Add n to each entry of E producing $E+n$. The* **extension** *of A by E, denoted $E(A)$, is the $2 \times (n+2k)$ Skolem array formed by appending the first k columns of $E+n$ to the left of A, and the rest to the right.*

Note that the new arrays created in the proof of Theorem 5 are merely extensions of A by $E = \begin{array}{|c|c|c|c|}\hline 4 & 2 & 3 & 2 \\ \hline 3 & 1 & 1 & 4 \\ \hline \end{array}$. In addition, an extension of a vertically split array A is also vertically split.

In a Skolem array of order n there are only two possible locations for the two values of n: $(0,1)$ and $(1,n)$ or $(1,1)$ and $(0,n)$. In any vertically split array, the two values of 1 must be horizontally adjacent and located in the middle of the array. Therefore when extending a Skolem array A of order n by a vertically split E of order $2k$ to form $E(A)$ there are four possibilities for the positions of n and $n+1$:

$\cdots$	$n+1$	n	$\cdots$	$-$	$n+1$	$\cdots$
$\cdots$	$-$	$-$	$\cdots$	n	$-$	$\cdots$

,

plus the three others that result from horizontal and vertical reflections.

We note that if we interchange the right-most n and $n+1$ values in the array above, the resulting array is also Skolem:

$\cdots$	$n+1$	n	$\cdots$	$-$	n	$\cdots$
$\cdots$	$-$	$-$	$\cdots$	$n+1$	$-$	$\cdots$

.

The analogous interchanges work in the other three cases. We name this operation on extensions *switching* and use the notation $S(E(A))$ for a switch of the extension E of A. Note that $S(E(A))$ is vertically split if A is.

Lemma 8 *Let A_1 and A_2 be distinct Skolem arrays of order n; E_1 and E_2 distinct vertically split arrays. Then the following hold.*

1. *$E_i(A_j) = E_k(A_l)$ implies that $i = k$ and $j = l$ for all $i, j, k, l \in \{1, 2\}$.*
2. *For any switch S, $S(E_i(A_j)) \neq E_k(A_l)$ if $i \neq k$ or $j \neq l$.*
3. *$S(E_i(A_j)) = S(E_k(A_l))$ implies that $i = k$ and $j = l$ for all $i, j, k, l \in \{1, 2\}$.*

PROOF.(1) is immediate. For (3) simply reverse the switch and then use (1). (2) is immediate if $k \neq i$. If $k = i$, note that in $S(E_k(A_j))$ the value $n+1$ is in a position (a, b) with $b = k+1$ or $k+n$. However, in $E_k(A_l)$, $n+1$ is in a position (c, d) with $d = k$ or $k+n+1$. ∎

We now prove the main theorem of this section.

Theorem 9 *1. Let $d_y(n)$ be the number of vertically split Skolem arrays of order n. For $n \equiv 0$ (mod 4), $n > 4$,*

$$d_y(n) \geq 2^{\frac{3(n-4)}{4}} \cdot 12.$$

2. Let the number of distinct Skolem arrays of order n be $s(n)$. Then $s(4) = 8, s(5) = 24$ and for $n > 5$,

$$s(n) \geq \begin{cases} 2^{\frac{3n}{4}-2} \cdot 31 & \text{if } n \equiv 0 \text{ (mod 4);} \\ 2^{\frac{3(n-1)}{4}} \cdot 3 & \text{if } n \equiv 1 \text{ (mod 4).} \end{cases}$$

PROOF. (1) For $n > 4, n \equiv 0$ (*mod* 4), let A be a vertically split array of order $n-4$. The four vertically split arrays of size 4 gives rise to 4 extensions of A and switching in these extensions gives rise to 4 more extensions of A. These extensions of A are distinct by Lemma 8.

Hence,

$$d_y(n) \geq 8 d_y(n-4).$$

Using equation (2) and the fact that $d_y(8) = 96$, (1) follows.

For item (2) of the theorem, first suppose n is even. We first note that $s(4) = 8$ and $s(8) = 496$ (see Section 6 below). For $n > 8$ we proceed by induction on n. Assume the result is true for $n-4$ where $n > 8$ is fixed. Using the four vertically split Skolem arrays of order 4, we may extend the Skolem arrays of order $n-4$ to obtain

$$2^{\frac{3(n-4)}{4}-2} \cdot 4 \cdot 31$$

many distinct Skolem arrays of order n (using Lemma 8). Switching these Skolem arrays multiplies this number by 2 by Lemma 8, and the result follows.

If n is odd, the result follows in a similar fashion from the fact that $s(5) = 24$ and $s(9) = 1336$ (see Section 6 below). ∎

5 Split arrays

In the previous section, we examined the vertically split arrays; that is, those arrays where each entry occurred in both the first and last half of the columns. Now we consider what happens if each entry must occur in each row. First, we show that at least half of the pairs in any Skolem array must be split.

Theorem 10 *If A is a Skolem array of order n, then A has at least $\lceil \frac{n}{2} \rceil$ split pairs.*

PROOF. Let A be a Skolem array. From (4) in the proof of Theorem 5,

$$\sum_{i=1}^{n} c_1(i) = \frac{1}{4}(n(n+1) + 2|S|).$$

The sum of the column numbers is minimized when the left-most positions of the array are used. If n is even, this means that the left $\frac{n}{2}$ are occupied. If n is odd, this means that the $\frac{n-1}{2}$ columns are entirely occupied, and there is an entry in the $\frac{n+1}{2}$ column. Hence

$$\sum_{i=1}^{n} c_1(i) \geq \begin{cases} 2\sum_{i=1}^{\frac{n}{2}} i = \frac{n(n+2)}{4} & \text{if } n \text{ is even;} \\ 2\sum_{i=1}^{\frac{n-1}{2}} i + \frac{(n+1)}{2} = \frac{(n+1)^2}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Therefore,

$$|S| \geq \lceil \frac{n}{2} \rceil.$$

∎

A Skolem array A is *split* if every row pair (i, i) is split. For example,

2	3	1	4
4	2	1	3

is split.

Theorem 11 *There exists a split Skolem array of every order n, $n \equiv 0, 1$ (mod 4).*

PROOF. Note that in any Skolem array, there are precisely four possible locations for the entries n and $n-1$, namely,

n	...	$n-1$	-
$n-1$	...	-	n

and its 3 images under reflections. Thus, the existence of a single split array gives split arrays with n and $n-1$ in all possible locations.

The unique Skolem array of order 1 is split.

Let $n \equiv 0$ or 1 (*mod* 4). Assume there exist split Skolem arrays for all orders m, $m \equiv 0$ or 1 (*mod* 4), $m < n$. We construct a split Skolem array of order n.

Let $k = \lfloor \frac{n+1}{3} \rfloor + 1$. Place *only* the entries $k+1, \ldots, n$ as follows:

$$n-2i \text{ in } (1,i+1) \text{ and } (2,n-i)$$

$$n-2i-1 \text{ in } (2,i+1) \text{ and } (1,n-i-1),$$

where $i = 0,\dots,\lfloor\frac{n-k-1}{2}\rfloor$. If n and k have the same parity, we have

n	$\cdots$	$k+2$	$\cdots$	$k+1$	$k+3$	$\cdots$	$-$
$n-1$	$\cdots$	$k+1$	$\cdots$	$-$	$k+2$	$\cdots$	n

if they have opposite parity,

n	$\cdots$	$k+3$	$k+1$	$\cdots$	$k+2$	$\cdots$	$-$
$n-1$	$\cdots$	$k+2$	$-$	$\cdots$	$k+1$	$\cdots$	n

Put k in $(1,n)$ and $(2,n-k+1)$.

Case 1. Suppose $n \equiv 0$ or 9 $(mod\ 12)$. Then $k \equiv 1$ or 0 $(mod\ 4)$, respectively.

n	$\cdots$	$k+3$	$k+1$	$\cdots$	$-$	$k+2$	$\cdots$	k
$n-1$	$\cdots$	$k+2$	$-$	$\cdots$	k	$k+1$	$\cdots$	n

Consider the sub-array consisting of k columns $\frac{n-k+1}{2},\dots,n-k+1$. It contains precisely two occupied positions $(1,1)$ and $(2,k)$. Since $k \equiv 0$ or 1 $(mod\ 4)$, $k < n$, there exists an split array A of order k with k in these two positions. Use the remaining entries in A to fill the rest of the subarray.

Case 2. Suppose $n \equiv 1$ or 4 $(mod\ 12)$. Then $k \equiv 1$ or 2 $(mod\ 4)$ respectively. We have

n	$\cdots$	$k+2$	$-$	$\cdots$	$-$	$k+1$	$k+3$	$-$	k
$n-1$	$\cdots$	$k+1$	$-$	$\cdots$	$-$	k	$k+2$	$-$	n

Since $k-1 \equiv 0$ or 1 $(mod\ 4)$, the $k-1$ empty columns from $\frac{n-k}{2}+1$ to $n-k$ can be filled with a split array of order $k-1$.

Case 3. Finally, suppose $n \equiv 5$ or 8 $(mod\ 12)$. Then $k \equiv 3$ or 0 $(mod\ 4)$, respectively. We have

n	$\cdots$	$k+2$	$\cdots$	$-$	$k+1$	$k+3$	$\cdots$	k
$n-1$	$\cdots$	$k+1$	$\cdots$	k	$-$	$k+2$	$\cdots$	n

Consider the subarray consisting of the $k+1$ columns $\frac{n-k}{2}$ to $n-k+2$. Since $k+1 \equiv 0$ or 1 $(mod\ 4)$, there exists an split array A of order $k+1$ with $k+1$

in positions (2,1) and $(1, k)$ and with k in (1,1) and $(2, k)$ (clearly, $k + 1 < n$). Use A without those entries to fill the rest of the subarray. ∎

We do not know a lower bound for the number of distinct split Skolem arrays of order n.

6 Computational results

The number of distinct Skolem sequences, $\sigma(n)$, for small n is given in the following table (see Table 43.20 of [4]).

n	1	4	5	8	9	12	13
$\sigma(n)$	1	6	10	504	2656	455936	3040560

Computer searches (using a C++ program written by the second author) have yielded the following data ($d_x(n)$ is the number of distinct split Skolem arrays of order n).

n	1	4	5	8	9	12	13	16
$s(n)$	1	8	24	496	1336	54272	173440	10177712
$d_y(n)$	0	4	0	96	0	4000	0	270368
$d_x(n)$	1	4	4	32	96	992	2512	50512

7 More general arrays

Skolem arrays may be generalized to include m by n arrays of multiplicity λ (where λ denotes the number of pairs of each of the labels). For example, the following

2	2	2	2
3	1	1	3
3	1	1	3

is a 3 by 4 Skolem array with $\lambda = 2$. Another variation is to allow empty cells. For example, the following

3	1	4
2	1	3
4	2	*

is a 3 by 3 Skolem array with $\lambda = 1$ that has the minimum number of possible empty cells. It is an interesting problem (that we do not address) to investigate these more general Skolem arrays and in particular, classify their spectrums.

References

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8 Appendix

We include a computer-generated list of all the Skolem arrays up to order 5. A list of all Skolem arrays up to order 8 has been generated, and is available on request from the authors. For brevity, our list is up to reflection. We use an ordered pair notation for Skolem arrays, with a "$*$" representing a position on the second row. For example,

2	3	1	4
4	2	1	3

is written $(3,3^*),(1,2^*),(2,4^*),(4,1^*)$.

$n = 1$:

$(1,1^*)$

$n = 4$:

$(2^*,3^*),(2,4),(3,1^*),(1,4^*)$

$(2,2^*),(4,3^*),(3,1^*),(1,4^*)$

$n = 5$:

$(3,3^*),(2^*,4^*),(2,5),(4,1^*),(1,5^*)$

$(3^*,4^*),(3,2^*),(2,5),(4,1^*),(1,5^*)$

$(2^*,3^*),(3,4^*),(2,5),(4,1^*),(1,5^*)$

$(2^*,3^*),(3,5),(2,4^*),(4,1^*),(1,5^*)$

$(2,3),(2^*,4^*),(5,3^*),(4,1^*),(1,5^*)$

$(2,2^*),(3,4^*),(5,3^*),(4,1^*),(1,5^*)$