Non-isomorphic Minimal Colorings of K_{4n+3}

V Vijayalakshmi *
Department of Mathematics, University of Bombay,
Vidyanagari, Bombay - 400098, India.
e-mail: *vijaya@math.mu.ac.in*

Abstract

We prove that the number of nonisomorphic minimal 2-colorings of the edges of K_{4n+3} is at least $2n$ less than the number of nonisomorphic minimal 2-colorings of the edges of K_{4n+2}, where n is a nonnegative integer. Harary explicitly gave all the nonisomorphic minimal 2-colorings of the edges of K_6 . In this paper, we give all the nonisomorphic minimal 2-colorings of the edges of K_7.

1 Introduction and background results

Definition 1.1 *If a graph G with 2-coloring of the edges has the minimum number of monochromatic triangles then that coloring of G is said to be a* **minimal coloring**. *We denote by* $M(K_3, G)$ *the number of monochromatic triangles in a minimal coloring of* G.

A.W.Goodman [1] has proved the following result regarding $M(K_3, K_n)$, where K_n is the complete graph on n vertices. The same result was proved by Sauve [3] in a more elegant and simple way using the method of weights.

*Supported by CSIR fellowship, India.

Proposition 1.2 [**A. W. Goodman**]

$$\begin{aligned} M(K_3, K_n) &= \frac{1}{3}\, t\,(t-1)\,(t-2) && \textit{if } n = 2t, \\ &= \frac{2}{3}\, t\,(t-1)\,(4t+1) && \textit{if } n = 4t+1, \\ &= \frac{2}{3}\, t\,(t+1)\,(4t-1) && \textit{if } n = 4t+3. \end{aligned}$$

Definition 1.3 *Let G be a graph in which the edges are colored with two colors, say, red and blue. Let v be any vertex of G. We define the* **degree pair** *of the vertex v as (s,t) where s is the number of red edges incident at v and t is the number of blue edges incident at v.*

Sauve [3] has proved the following result about the degree pairs of the vertices in a minimal coloring of K_n.

Proposition 1.4 [**L. Sauve**] *A 2-coloring of the edges of K_n is a minimal coloring if and only if the degree pair of*

(1) *any vertex is $(t, t-1)$ or $(t-1, t)$, when $n = 2t$;*

(2) *any vertex is $(2t, 2t)$, when $n = 4t+1$;*

(3) *$4t+2$ vertices are $(2t+1, 2t+1)$ and the degree pair of one exceptional vertex is $(2t, 2t+2)$ or $(2t+2, 2t)$, when $n = 4t+3$.*

Definition 1.5 *Let G be a graph. Suppose $C_1(G)$ and $C_2(G)$ are two 2-coloring of the edges of G. $C_1(G)$ and $C_2(G)$ are said to be* **non-isomorphic** *if and only if the graph defined by one color of $C_1(G)$ is not isomorphic to the graph defined by either color of $C_2(G)$.*

2 2-Colorings of K_{4n+3}

In this section we prove that in any minimal coloring of K_{4n+3}, if we remove a suitable vertex we get a minimal coloring of K_{4n+2} and that a minimal coloring of K_{4n+2} can be extended to a minimal coloring of K_{4n+3} in at most one way. We construct all the 4 non-isomorphic colorings of K_7 using the non-isomorphic minimal colorings of K_6 given by Harary [2].

B. Radhakrishnan Nair and A. Vijayakumar [4] have proved the following theorem about the number of monochromatic triangles incident at a vertex v in any 2-edge coloring of a graph G. Their theorem, in our language is as follows.

Theorem 2.1 *Let G be a complete graph on n vertices in which the edges are colored with two colors, say red and blue and r be the number of red edges in G. For any vertex v of G, let $d(v)$ = the number of red edges incident at v, $N(v)$ = the set of all vertices which are joined to v by red color and $T(v)$ = the number of monochromatic triangles which are incident at v. Then*

$$T(v) = \sum_{u \in N(v)} d(u) \; - \; r \; + \; \frac{1}{2}\,[n - d(v) - 1]\,[n - d(v) - 2].$$

Theorem 2.2 *Any minimal coloring of K_{4n+3} is an extension of a minimal coloring of K_{4n+2} and the number of non-isomorphic minimal colorings of K_{4n+3} is at least $2n$ less than the number of non-isomorphic minimal colorings of K_{4n+2}.*

Proof : Consider K_{4n+3}, where n is a nonnegative integer. Any 2-coloring of the edges of K_{4n+3} will be a minimal coloring if and only if the degree pairs of $4n+2$ vertices are (2n+1,2n+1) and the degree pair of one exceptional vertex, say P is $(2n, 2n+2)$ or $(2n+2, 2n)$ (1.4). The number of monochromatic triangles that lie on the vertex P is $2n^2$ (2.1) and using Goodman's formula (1.2) we get

$$M(K_3, K_{4n+3}) - M(K_3, K_{4n+2}) = 2n^2.$$

So, by removing this vertex P from K_{4n+3} we get a minimal coloring of K_{4n+2}. In other words, any minimal coloring of K_{4n+3} is obtained precisely by extending a minimal coloring of K_{4n+2}.

We consider a minimal coloring of K_{4n+2} and find all possible extensions of this to a minimal coloring of K_{4n+3}. In any minimal coloring of K_{4n+2}, the degree pair of any vertex is $(2n, 2n+1)$ or $(2n+1, 2n)$ (1.4). Hence to extend a minimal coloring of K_{4n+2} to a minimal coloring of K_{4n+3}, we have to add a new vertex P and join this with the vertices of K_{4n+2} in such a way that the degree pair of P is $(2n, 2n+2)$ or $(2n+2, 2n)$ and the degree pairs of all other vertices are $(2n+1, 2n+1)$. It is clear that there is at most one extension possible. In fact, an extension is possible only when exactly $2n$ vertices of K_{4n+2} have degree pair $(2n, 2n+1)$ or exactly $2n$ vertices of K_{4n+2} have degree pair $(2n+1, 2n)$. If $C_1(K_{4n+3})$ and $C_2(K_{4n+3})$ are two minimal colorings of K_{4n+3} which are extensions of two non-isomorphic minimal colorings of K_{4n+2}, then $C_1(K_{4n+3})$ and $C_2(K_{4n+3})$ are also non-isomorphic, for an exceptional vertex must be mapped to an exceptional vertex under any isomorphism.

We claim that there exist $2n$ non-isomorphic minimal colorings of K_{4n+2} which are not extendable to a minimal coloring of K_{4n+3}.

For each integer k such that $0 \leq k \leq 2n+1$, we construct a minimal coloring C_k of K_{4n+2} such that C_i is non-isomorphic to C_j, for $i \neq j$. Suppose $u_1, u_2, \ldots, u_{2n+1}$ and $v_1, v_2, \ldots v_{2n+1}$ are the vertices of K_{4n+2} and the edges are colored with red and blue. The red edges of C_k are precisely u_iu_j, for $1 \leq i \leq 2n+1, 1 \leq j \leq 2n+1, i \neq j$, v_iv_j for $1 \leq i \leq 2n+1, 1 \leq j \leq 2n+1, i \neq j$, and if $k > 0$, u_iv_i, for $1 \leq i \leq k$. It is easy check that C_k is a minimal coloring of K_{4n+2}, for each $0 \leq k \leq 2n+1$, Also C_ks are all mutually non-isomorphic, because in these colorings all the monochromatic triangles are of red color and the number of red edges are distinct.

The minimal colorings

$$C_0, C_1, \ldots, C_{n-1}, C_{n+2}, C_{n+3}, \ldots, C_{2n+1}$$

of K_{4n+2} are not extendable to get minimal colorings of K_{4n+3}. These are $2n$ in number. Hence the theorem. □

Remark: We note that in Theorem 2.2, the colorings C_u and C_{u+1} are extendable to get minimal colorings of K_{4u+3} and hence there exists at least 2 non-isomorphic minimal colorings of K_{4u+3}.

Theorem 2.3 *There exists precisely four non-isomorphic minimal colorings of K_7.*

Proof : Suppose we color the edges by red and blue. Harary [2] has constructed all the 6 non-isomorphic minimal colorings of K_6 which are given below in Figures 1 through 6 with only the red edges present.

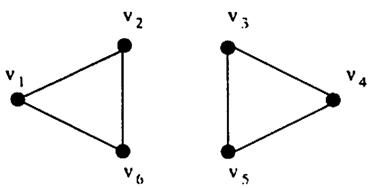

FIGURE 1

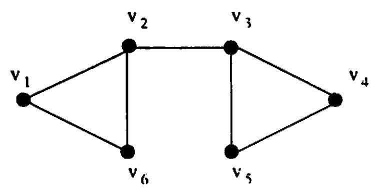

FIGURE 2

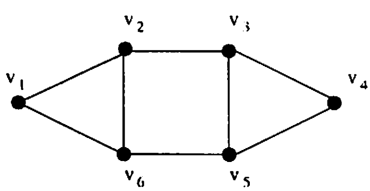

FIGURE 3

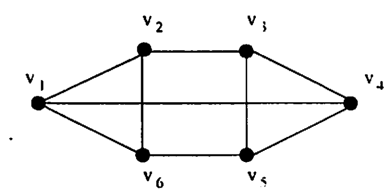

FIGURE 4

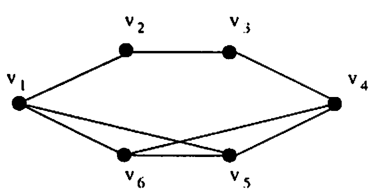

FIGURE 5

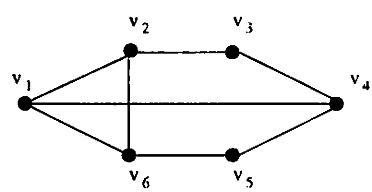

FIGURE 6

By Theorem (2.3) there are at most 4 non-isomorphic minimal colorings of K_7. It is easy to see that the colorings given in figures 2, 3, 5 and 6 are extendable to a minimal coloring of K_7. These are given below in Figures 7 through 10 with only the red edges present.

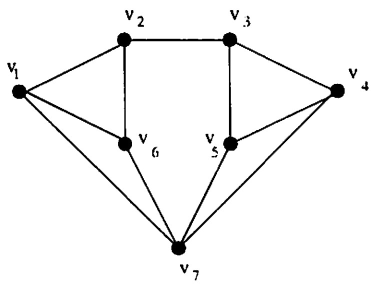

FIGURE 7

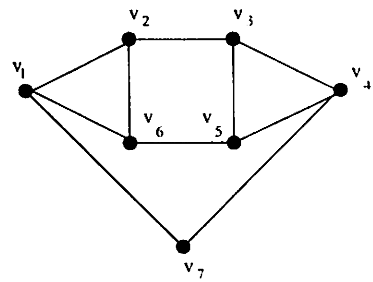

FIGURE 8

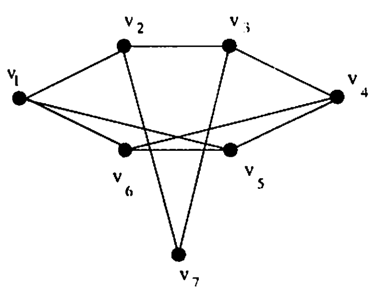

FIGURE 9

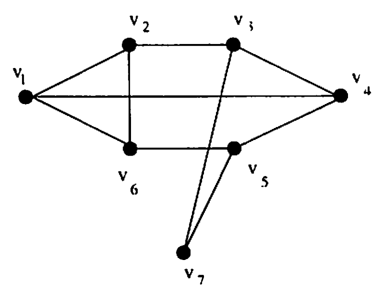

FIGURE 10

Hence the proof. □

Acknowledgement: I thank Prof. S.S. Sane for his help in preparing this paper.

References

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