Combinatorial properties of the divisibility of mn by $am + bn + c$

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Abstract

In this paper, we investigate the divisibility of mn by $am+bn+c$ for given a, b and c. We give the necessary and sufficient condition for the divisibility, that is, $am+bn+c$ divides mn. We then present the structure of the set of pairs $[m,n]$ that satisfies the divisibility. This structure is represented by a directed graph and we prove the necessary and sufficient condition for the graph to have a binary tree structure. In particular, for $c=-1$, we show double binary tree structures on the set.

1 Introduction

This paper deals with the divisibility of mn by $am + bn + c$ and several combinatorial properties of the set of pairs $[m,n]$ such that $am + bn + c$ divides mn for given a, b, and c. We especially pay attention to binary tree structures of the set.

Shibata and Seki [4] have studied the divisibility of mn by $m + n - 1$. In [4], the necessary and sufficient conditions for the divisibility of mn by $m+n-1$ was proved. They then defined an order relation on the set of pairs $[m,n]$ such that $m + n - 1$ divides mn, and represented the relation with directed graphs to have shown that the graph has a binary tree structure.

One purpose of this paper is to generalize this result. We first give characterizations of the divisibility of mn by $am + bn + c$ for given a,b,c. The characterizations are stated as extensions of the results given in [4]. In order to investigate properties of the set of pairs $[m,n]$ such that $am+bn+c$ divides mn, we classify the set into subsets so that each of the subsets has a linear ordering. The order relations can be represented as a directed graph.

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We then give the necessary and sufficient condition for the graph to have a binary tree structure. In particular, for $c = -1$, we find that infinitely many binary trees form other binary trees. We call this structure *double binary tree*. These investigations give combinatorics of divisibility of mn by $am + bn + c$, which is also considered to be divisibility of a number in product form by a number in additive form.

In Section 2, the necessary and sufficient conditions for the divisibility are stated. Section 3 deals with the classification of the set into ordered sets, then the structures of directed graphs are discussed in Section 4. In Section 5 and Section 6, we prove the set has double binary tree structures.

For integers m, n, if m divides n, then we write $m|n$. $(n_1, n_2, \ldots, n_k)$ stands for the greatest common divisor of $n_1, n_2, \ldots, n_k$. $\{n_1, n_2, \ldots, n_k\}$ stands for the least common multiple of $n_1, n_2, \ldots, n_k$. For other number theoretic terminology and notation, we refer to Shapiro [3].

2 Necessary and sufficient conditions for divisibility

In this section, we give necessary and sufficient conditions for integers m, n, such that $am + bn + c$ divides mn.

Theorem 2.1 *Let a, b and c be any integers. For integers m, n,*

$$(am + bn + c, mn) = \frac{(m, bn + c)(am + c, n)}{\theta}, \tag{1}$$

where

$$\theta = \frac{(d_m, d_n)}{(d_m, d_n, a\alpha + \alpha', b\beta + \beta')},$$

$$d_m = (m, bn + c), \ m = d_m\alpha, \ bn + c = d_m\alpha',$$

$$d_n = (am + c, n), \ n = d_n\beta \ am + c = d_n\beta'.$$

Proof. First we assume that $am + bn + c \neq 0$. Then $(am + bn + c, mn) = (am + bn + c, (am + bn + c, m)(am + bn + c, n)) = (am + bn + c, (bn + c, m)(am + c, n)) = (am + bn + c, d_m d_n)$. Since $am + bn + c$ is a common multiple of d_m and d_n, we have $\{d_m, d_n\}|(am + bn + c)$.

$$\begin{aligned}(am + bn + c, mn) &= \{d_m, d_n\}\left(\frac{am + bn + c}{\{d_m, d_n\}}, \frac{d_m d_n}{\{d_m, d_n\}}\right), \\ &= \frac{d_m d_n}{(d_m, d_n)}\left(\frac{am + bn + c}{\{d_m, d_n\}}, d_m, d_n\right).\end{aligned}$$

For any integer $k \neq 0$ and integers $a_1, a_2, \ldots, a_n$, if $a_i | k$, $i = 1, \ldots, n$, then

$$\left(\frac{k}{a_1}, \ldots, \frac{k}{a_n}\right) = \frac{|k|}{\{a_1, \ldots, a_n\}}. \tag{2}$$

By equation (2), we have

$$\begin{aligned}
\left(\frac{am+bn+c}{\{d_m,d_n\}}, d_m, d_n\right) &= \left(\frac{am+bn+c}{\{d_m,d_n\}}, \frac{am+bn+c}{a\alpha+\alpha'}, \frac{am+bn+c}{b\beta+\beta'}\right) \\
&= \frac{|am+bn+c|}{\{d_m, d_n, a\alpha+\alpha', b\beta+\beta'\}} \\
&= \frac{|am+bn+c|}{\{\{d_m, a\alpha+\alpha'\}, \{d_n, b\beta+\beta'\}\}} \\
&= \left(\frac{am+bn+c}{\{d_m, a\alpha+\alpha'\}}, \frac{am+bn+c}{\{d_n, b\beta+\beta'\}}\right) \\
&= \left(\frac{d_m(a\alpha+\alpha')}{\{d_m, a\alpha+\alpha'\}}, \frac{d_n(b\beta+\beta')}{\{d_n, b\beta+\beta'\}}\right) \\
&= \left((d_m, a\alpha+\alpha'), (d_n, b\beta+\beta')\right) \\
&= (d_m, d_n, a\alpha+\alpha', b\beta+\beta').
\end{aligned}$$

Let θ be

$$\theta = \frac{(d_m, d_n)}{(d_m, d_n, a\alpha+\alpha', b\beta+\beta')},$$

then we obtain $(am+bn+c, mn) = \dfrac{d_m d_n}{\theta}$.

If $am+bn+c=0$, then $a\alpha+\alpha' = b\beta+\beta' = 0$. The left hand side of the equation (1) becomes $(am+bn+c, mn) = (am+bn+c, d_m d_n) = d_m d_n$. On the other hand, the right hand side is $d_m d_n$ because

$$\theta = \frac{(d_m, d_n)}{(d_m, d_n, a\alpha+\alpha', b\beta+\beta')} = 1,$$

and the theorem is proved. ∎

We define the new symbol ϑ by

$$\vartheta = \operatorname{sign}(am+bn+c)\theta, \tag{3}$$

where $\operatorname{sign}(x)$ is a function defined as follows:

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Theorem 2.2 *Let a, b and c be integers. For the integers m, n such that $am + bn + c \neq 0$, $(am + bn + c)|mn$ if and only if*

$$\frac{(m, bn + c)(am + c, n)}{\vartheta} = am + bn + c.$$

From Theorem 2.2, we can obtain several necessary and sufficient conditions for the divisibility.

Theorem 2.3 *For integers m, n such that $am + bn + c \neq 0$, we have*
(i) *$(am + bn + c)|mn$ if and only if $d_m = \vartheta(b\beta + \beta')$.*
(ii) *$(am + bn + c)|mn$ if and only if $d_n = \vartheta(a\alpha + \alpha')$.*

Proof.
(i) First, assume that a pair $[m, n]$ satisfies $(am + bn + c)|mn$. By Theorem 2.2, $d_m d_n = \vartheta(am + bn + c) = \vartheta(bn + (am + c)) = \vartheta d_n(b\beta + \beta')$. Since $d_n > 0$, we obtain $d_m = \vartheta(b\beta + \beta')$.

For the converse, assume that $d_m = \vartheta(b\beta + \beta')$. By multiplying d_n to both sides of the equation, we obtain $d_m d_n = \vartheta d_n(b\beta + \beta') = \vartheta(bn + (am + c)) = \vartheta(am + bn + c)$. By Theorem 2.2, $am + bn + c$ divides mn.

(ii) Similarly proved. ∎

Theorem 2.4 *For integers m, n such that $am + bn + c \neq 0$, we have*
(i) *if $c \neq 0$, then $(am + bn + c)|mn$ if and only if $ab\alpha\beta - \alpha'\beta' = -\frac{c}{\vartheta}$.*
(ii) *if $m \neq n$, then $(am + bn + c)|mn$ if and only if $\alpha\beta' - \alpha'\beta + (b - a)\alpha\beta = \dfrac{m - n}{\vartheta} \neq 0$.*

Proof.
(i) Let us assume that a pair $[m, n]$ satisfies $(am + bn + c)|mn$ and $c \neq 0$. From the definition, we have $(am + c)(bn + c) = d_m d_n \alpha'\beta'$. Then $abmn + c(am + bn + c) = d_m d_n \alpha'\beta'$. Substituting m and n from $m = d_m\alpha$ and $n = d_n\beta$, we obtain $d_m d_n(ab\alpha\beta - \alpha'\beta') = -c(am + bn + c)$. By Theorem 2.2, $ab\alpha\beta - \alpha'\beta' = -c/\vartheta$.

For the converse, assume that $ab\alpha\beta - \alpha'\beta' = -c/\vartheta$ and $c \neq 0$. Multiplying $d_m d_n$ to both sides of the equation, we have $d_m d_n(ab\alpha\beta - \alpha'\beta') = -\frac{c d_m d_n}{\vartheta}$. Then $\vartheta(abmn - (am + c)(bn + c)) = -c d_m d_n$. Since $c \neq 0$, $\vartheta(am + bn + c) = d_m d_n$. By Theorem 2.2, $(am + bn + c)|mn$.

(ii) Let us assume that a pair $[m, n]$ satisfies $(am + bn + c)$ divides mn and $m \neq n$. By the definition of α, α', β and β',

$$m(am + c) - n(bn + c) + (b - a)mn = d_m d_n \alpha\beta' - d_m d_n \alpha'\beta + (b - a) d_m d_n \alpha\beta.$$

Then $(m-n)(am+bn+c) = d_m d_n(\alpha\beta' - \alpha'\beta + (b-a)\alpha\beta)$. By Theorem 2.2, we obtain $\alpha\beta' - \alpha'\beta + (b-a)\alpha\beta = (m-n)/\vartheta$.

For the converse, assume that $\alpha\beta' - \alpha'\beta + (b-a)\alpha\beta = (m-n)/\vartheta$ and $m \neq n$. Multiplying $d_m d_n$ to both sides of the equation, we have

$$d_m d_n(\alpha\beta' - \alpha'\beta + (b-a)\alpha\beta) = \frac{d_m d_n(m-n)}{\vartheta}.$$

Then $m(am+c) - n(bn+c) + (b-a)mn = d_m d_n(m-n)/\vartheta$, or $(m-n)(am+bn+c) = d_m d_n(m-n)/\vartheta$. Since $m \neq n$, we have $am+bn+c = d_m d_n/\vartheta$, this means $(am+bn+c)|mn$. ∎

Lemma 2.5 *If $(a,c) = 1$, then $(a,\alpha') = (a,\beta') = 1$. If $(b,c) = 1$, then $(b,\alpha') = (b,\beta') = 1$.*

Proof. Since $ab\alpha\beta - \alpha'\beta' = -c/\vartheta$ from Theorem 2.4, each of (a,α'), (a,β'), (b,α') and (b,β') divides c. If $(a,c) = 1$, we obtain $(a,\alpha') = ((a,\alpha'),c) = (\alpha',(a,c)) = 1$. The remaining are similarly proved. ∎

3 Classification of pairs

In this section, we will investigate the structure of the set of pairs $[m,n]$ such that $(am+bn+c)|mn$, and we will classify the pairs.

Definition 3.1 *For integers a, b and c $(abc \neq 0)$, the set $S(a,b,c)$ is a collection of pairs $[m,n]$ such that $am+bn+c \neq 0$ and $(am+bn+c)|mn$.*

For $[m_i, n_i] \in S(a,b,c)$, we write

$$\begin{array}{ll} d_{m_i} = (m_i, bn_i + c), & d_{n_i} = (am_i + c, n_i), \\ m_i = d_{m_i}\alpha_i, & n_i = d_{n_i}\beta_i, \\ bn_i + c = d_{m_i}\alpha_i', & am_i + c = d_{n_i}\beta_i'. \end{array}$$

By Definition 3.1, $[m,n] \in S(a,b,c)$ if and only if $[n,m] \in S(b,a,c)$, and parameters α and α' are exchanged for β and β', respectively. In other words, let

$$a_1 = b,\ b_1 = a,\ m_1 = n \text{ and } n_1 = m,$$

then $[m_1,n_1] \in S(a_1,b_1,c)$ and $\alpha_1 = \beta$, $\alpha_1' = \beta'$, $\beta_1 = \alpha$ and $\beta_1' = \alpha'$. Therefore, if a certain proposition P holds in $S(a,b,c)$, the proposition obtained from P by exchanging a for b, m for n, α for β, α' for β' holds in $S(b,a,c)$. We call this proposition the *symmetric proposition* for P.

Lemma 3.2 *If $[m,n] \in S(a,b,c)$, then*
(i) $\vartheta(b\beta + \beta')\alpha' - c \equiv 0 \pmod{b(a\alpha + \alpha')}$,
(ii) $\vartheta(a\alpha + \alpha')\beta' - c \equiv 0 \pmod{a(b\beta + \beta')}$.

Proof. We prove only (i). (ii) is a proposition symmetric for (i).

Let $[m,n] \in S(a,b,c)$. From the definition,

$$\begin{aligned}\vartheta(a\alpha+\alpha')(b\beta+\beta') &= \vartheta ab\alpha\beta+\vartheta(a\alpha\beta'+b\alpha'\beta+a'\beta') \\ &= (\vartheta\alpha'\beta'-c)+\vartheta(a\alpha\beta'+b\alpha'\beta+\alpha'\beta') \\ &= \vartheta(a\alpha+\alpha')\beta'+\vartheta(b\beta+\beta')\alpha'-c,\end{aligned}$$

then we have $\vartheta b(a\alpha+\alpha')\beta = \vartheta(b\beta+\beta')\alpha'-c$. Hence we obtain $\vartheta(b\beta+\beta')\alpha'-c \equiv 0 \pmod{b(a\alpha+\alpha')}$. ▮

Each congruence in Lemma 3.2 contains two parameters in modulus. Conversely, for example, if α and α' are fixed, the values for $\vartheta(b\beta+\beta')$ of pairs that have these two parameters are congruent under $b(a\alpha+\alpha')$. From this fact, if we classify the pairs in $S(a,b,c)$ with respect to these parameters, there might be an order relation on each subset.

Definition 3.3 *Let $S_{a,b,c}(\alpha,\alpha')$ and $S'_{a,b,c}(\beta,\beta')$ be subsets of $S(a,b,c)$ defined as follows:*

$$S_{a,b,c}(\alpha,\alpha') = \left\{[m,n] \;\middle|\; \frac{m}{d_m}=\alpha, \frac{bn+c}{d_m}=\alpha'\right\},$$

$$S'_{a,b,c}(\beta,\beta') = \left\{[m,n] \;\middle|\; \frac{n}{d_n}=\beta, \frac{am+c}{d_n}=\beta'\right\}.$$

Example. Some elements in $S_{2,1,-3}(3,1)$ of $S(2,1,-3)$ are shown in Fig 1.

$$S_{2,1,-3}(3,1) = \left\{[12,7],[33,14],[54,21],\cdots\right\}$$

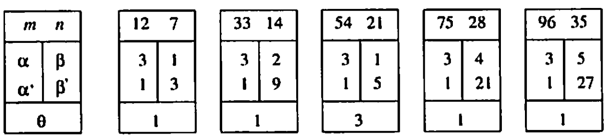

Figure 1: Some elements in $S_{2,1,-3}(3,1)$.

Theorem 3.4
(i) *If $[m_1,n_1],[m_2,n_2] \in S_{a,b,c}(\alpha,\alpha')$, then*

$$\vartheta_1(b\beta_1+\beta_1') \equiv \vartheta_2(b\beta_2+\beta_2') \pmod{\frac{b(a\alpha+\alpha')}{(ab,\alpha')}}.$$

(ii) *If* $[m_1,n_1],[m_2,n_2] \in S'_{a,b,c}(\beta,\beta')$, *then*

$$\vartheta_1(a\alpha_1+\alpha'_1) \equiv \vartheta_2(a\alpha_2+\alpha'_2) \pmod{\frac{a(b\beta+\beta')}{(ab,\beta')}}.$$

Proof. We prove only (i). Assume that $[m_1,n_1]$ and $[m_2,n_2]$ are members of $S_{a,b,c}(\alpha,\alpha')$. By Lemma 3.2,

$$\vartheta_1(b\beta_1+\beta'_1)\alpha'-c \equiv 0 \pmod{b(a\alpha+\alpha')}, \tag{4}$$

$$\vartheta_2(b\beta_2+\beta'_2)\alpha'-c \equiv 0 \pmod{b(a\alpha+\alpha')}. \tag{5}$$

By subtracting (5) from (4), we have $\vartheta_1(b\beta_1+\beta'_1)\alpha' \equiv \vartheta_2(b\beta_2+\beta'_2)\alpha' \pmod{b(a\alpha+\alpha')}$. Since $(\alpha', b(a\alpha+\alpha')) = (\alpha', ab)$, we obtain $\vartheta_1(b\beta_1+\beta'_1) \equiv \vartheta_2(b\beta_2+\beta'_2) \pmod{\frac{b(a\alpha+\alpha')}{(ab,\alpha')}}$. ∎

Lemma 3.5
(i) *If* $[m_1,n_1],[m_2,n_2] \in S_{a,b,c}(\alpha,\alpha')$, *then there exists an integer* k *such that*

$$\vartheta_2 b\beta_2 = \vartheta_1 b\beta_1 + kb\frac{\alpha'}{(ab,\alpha')} \text{ and } \vartheta_2\beta'_2 = \vartheta_1\beta'_1 + k\frac{ab\alpha}{(ab,\alpha')}.$$

(ii) *If* $[m_1,n_1],[m_2,n_2] \in S'_{a,b,c}(\beta,\beta')$, *then there exists an integer* k *such that*

$$\vartheta_2 a\alpha_2 = \vartheta_1 a\alpha_1 + ka\frac{\beta'}{(ab,\beta')} \text{ and } \vartheta_2\alpha'_2 = \vartheta_1\alpha'_1 + k\frac{ab\beta}{(ab,\beta')}.$$

Proof. We prove only (i). By Theorem 3.4,

$$\vartheta_1(b\beta_1+\beta'_1) \equiv \vartheta_2(b\beta_2+\beta'_2) \pmod{\frac{b(a\alpha+\alpha')}{(ab,\alpha')}}.$$

Hence, there exists an integer k such that

$$\vartheta_2(b\beta_2+\beta'_2) = \vartheta_1(b\beta_1+\beta'_1) + k\frac{b(a\alpha+\alpha')}{(ab,\alpha')}. \tag{6}$$

By Theorem 2.4, $\vartheta_1(ab\alpha\beta_1 - \alpha'\beta'_1) = -c = \vartheta_2(ab\alpha\beta_2 - \alpha'\beta'_2)$. Thus we obtain $\vartheta_1(ab\alpha\beta_1 - \alpha\beta'_1) = \vartheta_2 ab\alpha\beta_2 - \alpha'\left(\vartheta_1(b\beta_1+\beta'_1) + k\frac{b(a\alpha+\alpha')}{(ab,\alpha')} - \vartheta_2 b\beta_2\right)$, or $\vartheta_2(a\alpha+\alpha')b\beta_2 = \vartheta_1(a\alpha+\alpha')b\beta_1 + k(a\alpha+\alpha')b\frac{\alpha'}{(ab,\alpha')}$. Since $a\alpha+\alpha' \neq 0$, we have $\vartheta_2 b\beta_2 = \vartheta_1 b\beta_1 + kb\frac{\alpha'}{(ab,\alpha')}$. Substituting this result into (6), we obtain $\vartheta_2\beta'_2 = \vartheta_1\beta'_1 + k\frac{ab\alpha}{(ab,\alpha')}$. ∎

Now we will introduce an order relation on $S_{a,b,c}(\alpha,\alpha')$ and $S'_{a,b,c}(\beta,\beta')$ on the basis of the results of Lemma 3.5.

Suppose that $[m,n] \in S_{a,b,c}(\alpha,\alpha')$. We define integers m_1, n_1 as follows: let

$$\theta_1 = \left(\beta + \frac{\alpha'}{(ab,\alpha')}, \beta' + \frac{ab\alpha}{(ab,\alpha')}\right), \text{ and } \vartheta_1 = \text{sign}(am+bn+c)\theta_1,$$

$$\alpha_1 = \alpha, \alpha_1' = \alpha', \vartheta_1\beta_1 = \vartheta\beta + \frac{\alpha'}{(ab,\alpha')}, \vartheta_1\beta_1' = \vartheta\beta' + \frac{ab\alpha}{(ab,\alpha')},$$

$$d_{m_1} = \vartheta_1(b\beta_1 + \beta_1'), d_{n_1} = \vartheta_1(a\alpha_1 + \alpha_1'), \text{ and } m_1 = d_{m_1}\alpha_1, n_1 = d_{n_1}\beta_1.$$

Then, we have

$$am_1 + bn_1 + c = (a\alpha + \alpha')\left(\vartheta(b\beta + \beta') + \frac{b(a\alpha + \alpha')}{(ab,\alpha')}\right),$$

hence if $am_1 + bn_1 + c \neq 0$, then $(am_1 + bn_1 + c) | mn$ and $[m_1, n_1] \in S_{a,b,c}(\alpha,\alpha')$.

Similarly, let

$$\theta_2 = \left(\beta - \frac{\alpha'}{(ab,\alpha')}, \beta' - \frac{ab\alpha}{(ab,\alpha')}\right), \vartheta_2 = \text{sign}(am+bn+c)\theta_2,$$

$$\alpha_2 = \alpha, \alpha_2' = \alpha', \vartheta_2\beta_2 = \vartheta\beta - \frac{\alpha'}{(ab,\alpha')}, \vartheta_2\beta_2' = \vartheta\beta' - \frac{ab\alpha}{(ab,\alpha')},$$

$$d_{m_2} = \vartheta_2(b\beta_2 + \beta_2'), d_{n_2} = \vartheta_2(a\alpha_2 + \alpha_2'), \text{ and } m_2 = d_{m_2}\alpha_2, n_2 = d_{n_2}\beta_2.$$

If $am_2 + bn_2 + c \neq 0$, then $(am_2 + bn_2 + c)|mn$ and $[m_2, n_2] \in S_{a,b,c}(\alpha,\alpha')$.

Thus elements in $S_{a,b,c}(\alpha,\alpha')$ are ordered with respect to the order of $\vartheta(b\beta+\beta')$, that is, a pair $[m_1,n_1]$ has precedence over $[m_2,n_2]$ if and only if $\vartheta_1(b\beta_1 + \beta_1') \leq \vartheta_2(b\beta_2 + \beta_2')$. This order relation is a linear ordering. Since $\vartheta(b\beta + \beta') > 0$, each $S_{a,b,c}(\alpha,\alpha')$ has the least element.

Similarly, elements of $S'_{a,b,c}(\beta,\beta')$ are linearly ordered with respect to the order of $\vartheta(a\alpha + \alpha')$.

Example. Linear ordering on $S_{2,1,-3}(3,1)$ is shown in Fig 2.

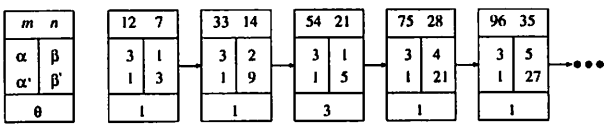

Figure 2: Linear ordering on $S_{2,1,-3}(3,1)$.

Lemma 3.6
(i) *A pair* $[m,n]$ *is the least element of* $S_{a,b,c}(\alpha,\alpha')$ *if and only if* $\vartheta(b\beta+\beta') \le \dfrac{|b(a\alpha+\alpha')|}{(ab,\alpha')}$.
(ii) *A pair* $[m,n]$ *is the least element of* $S'_{a,b,c}(\beta,\beta')$ *if and only if* $\vartheta(a\alpha+\alpha') \le \dfrac{|a(b\beta+\beta')|}{(ab,\beta')}$.

Proof. We prove only (i). Let $[m,n]$ be an element in $S_{a,b,c}(\alpha,\alpha')$ and $[m_1,n_1]$ an adjacent element to $[m,n]$ with respect to the order on $S_{a,b,c}(\alpha,\alpha')$. Then the parameters of $[m_1,n_1]$ are $\vartheta_1\beta_1 = \vartheta\beta + \dfrac{\alpha'}{(ab,\alpha')}$ and $\vartheta_1\beta'_1 = \vartheta\beta' + \dfrac{ab\alpha}{(ab,\alpha')}$, or $\vartheta_1\beta_1 = \vartheta\beta - \dfrac{\alpha'}{(ab,\alpha')}$ and $\vartheta_1\beta'_1 = \vartheta\beta' - \dfrac{ab\alpha}{(ab,\alpha')}$. Thus $\vartheta_1(b\beta_1+\beta'_1) = \vartheta(b\beta+\beta') \pm \dfrac{b(a\alpha+\alpha')}{(ab,\alpha')}$. Hence, if $[m,n]$ is the least element of $S_{a,b,c}(\alpha,\alpha')$, then $\vartheta(b\beta+\beta') - \dfrac{|b(a\alpha+\alpha')|}{(ab,\alpha')} \le 0$.

The converse is easily proved. ∎

Lemma 3.7
(i) *A pair* $[m,n]$ *is the least element of* $S_{a,b,c}(\alpha,\alpha')$ *if and only if* $\vartheta(b\beta+\beta')$ *is the least positive solution of the equation*

$$\frac{\alpha'}{(ab,\alpha')}x - \frac{c}{(ab,\alpha')} \equiv 0 \pmod{\frac{b(a\alpha+\alpha')}{(ab,\alpha')}}.$$

(ii) *A pair* $[m,n]$ *is the least element of* $S'_{a,b,c}(\beta,\beta')$ *if and only if* $\vartheta(a\alpha+\alpha')$ *is the least positive solution of the equation*

$$\frac{\beta'}{(ab,\beta')}y - \frac{c}{(ab,\beta')} \equiv 0 \pmod{\frac{a(b\beta+\beta')}{(ab,\beta')}}.$$

Proof. We prove only (i). Let $[m,n]$ be the least element of $S_{a,b,c}(\alpha,\alpha')$. By Lemma 3.6, we have $\vartheta(b\beta+\beta') \le \dfrac{|b(a\alpha+\alpha')|}{(ab,\alpha')}$. From Lemma 3.2, $[m,n]$ satisfies the congruence

$$\vartheta(b\beta+\beta')\frac{\alpha'}{(ab,\alpha')} - \frac{c}{(ab,\alpha')} \equiv 0 \pmod{\frac{b(a\alpha+\alpha')}{(ab,\alpha')}}.$$

Hence $\vartheta(b\beta+\beta')$ is the least positive solution of the equation

$$\frac{\alpha'}{(ab,\alpha')}x - \frac{c}{(ab,\alpha')} \equiv 0 \pmod{\frac{b(a\alpha+\alpha')}{(ab,\alpha')}}.$$

The converse is easily proved. ∎

Corollary 3.8 *Let $[m,n]$ be a member of $S(1,1,-1)$. Then $[m,n]$ is the least element of both $S_{1,1,-1}(\alpha,\alpha')$ and $S'_{1,1,-1}(\beta,\beta')$ if and only if $m+n-1=\pm 1$.*

Proof. Suppose that $[m,n] \in S(1,1,-1)$ is the least element of both $S_{1,1,-1}(\alpha,\alpha')$ and $S_{1,1,-1}(\alpha,\alpha')$. By Lemma 3.6, we have $d_m \leq d_n$ and $d_n \leq d_m$, and this reduces to $d_m = d_n$. Since $(d_m, d_n) = (m,n,-1) = 1$ and $d_m, d_n \geq 1$, we have $d_m = d_n = 1$. Then the following two cases arise.

Case $\vartheta = 1$. Since $d_m = \vartheta(b\beta + \beta')$ and $d_n = \vartheta(a\alpha + \alpha')$, we have $d_m = \beta + \beta' = 1$ and $d_n = \alpha + \alpha' = 1$. From $\alpha\beta - \alpha'\beta' = 1$, we have $\alpha' = 1 - \alpha$, $\beta = 2 - \alpha$ and $\beta' = \alpha - 1$. Then $m = d_m\alpha = \alpha$ and $n = d_n\beta = 2 - \alpha$, and we obtain $m + n - 1 = \alpha + (2 - \alpha) - 1 = 1$.

Case $\vartheta = -1$. Similarly we have $d_m = -\beta - \beta' = 1$ and $d_n = -\alpha - \alpha' = 1$. Then $\alpha' = 1 - \alpha$, $\beta = -\alpha$, $\beta' = \alpha - 1$. Then $m = d_m\alpha = \alpha$ and $n = d_n\beta = -\alpha$, and we obtain $m + n - 1 = \alpha - \alpha - 1 = -1$.

Conversely, suppose that $[m,n] \in S(1,1,-1)$ and $[m,n]$ satisfies $m+n-1 = \pm 1$. Then $d_m = (m, n-1) = (m, m+n-1) = 1$ and $d_n = (n, m-1) = (n, m+n-1) = 1$, hence we obtain $d_m = d_n$. By Lemma 3.6, the pair $[m,n]$ is the least element of both $S_{1,1,-1}(\alpha,\alpha')$ and $S_{1,1,-1}(\alpha,\alpha')$. ∎

4 Structure of $S(a,b,c)$

In the previous section, we introduced $S_{a,b,c}(\alpha,\alpha')$ and $S'_{a,b,c}(\beta,\beta')$ and proved that each one was a linearly ordered set. The order relation can be represented by a directed graph. We define a directed graph on $S(a,b,c)$, and give the necessary and sufficient condition for the graph to be a binary tree.

4.1 A graph on $S(a,b,c)$

Definition 4.1 *Let a directed graph $G(a,b,c)$ be defined as follows:*
1. the vertex set of $G(a,b,c)$ is $S(a,b,c)$,
2. there exists a directed arc from v to v' if and only if these vertices satisfy either of the following two conditions:
(a) $v, v' \in S_{a,b,c}(\alpha,\alpha')$ and v' is the next element of v with respect to the order on $S_{a,b,c}(\alpha,\alpha')$;
(b) $v, v' \in S'_{a,b,c}(\beta,\beta')$ and v' is the next element of v with respect to the order on $S'_{a,b,c}(\beta,\beta')$.

Fig. 3 and Fig. 4 show $G(3,2,-6)$ and $G(3,2,2)$, respectively.

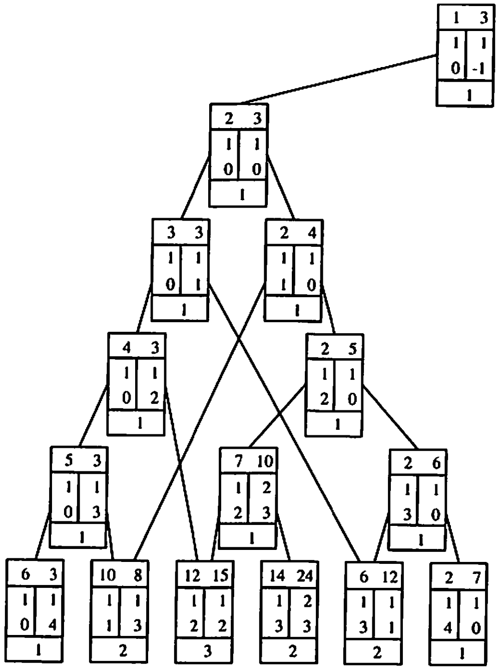

Figure 3: A part of the graph $G(3, 2, -6)$.

A graph $G(a, b, c)$ can be regarded as a representation of the partial order relation on $S(a, b, c)$. Shibata and Seki [4] have shown that the graph $G(1, 1, -1)$ has a binary tree structure. $G(a, b, c)$ however does not have binary tree structures in general, for example such as $G(3, 2, -6)$. On the other hand, we see that a connected component of graph $G(3, 2, 2)$ is a binary tree.

In order to investigate the structure of $G(a, b, c)$, we first have an insight into the order relation on $S(a, b, c)$. Each vertex $[m, n]$ of $G(a, b, c)$ satisfies one of the following conditions:

Type 1 the least element of both $S_{a,b,c}(\alpha, \alpha')$ and $S'_{a,b,c}(\beta, \beta')$;

Type 2 the least element of either $S_{a,b,c}(\alpha, \alpha')$ or $S'_{a,b,c}(\beta, \beta')$;

Type 3 Otherwise, that is, $[m, n]$ is not the least element of $S_{a,b,c}(\alpha, \alpha')$, and is not of $S'_{a,b,c}(\beta, \beta')$.

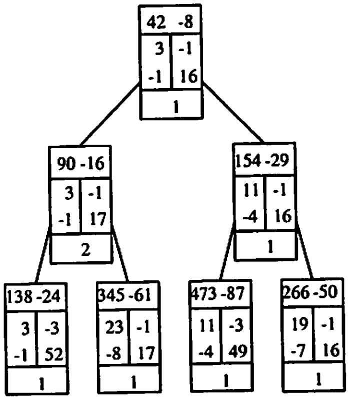

Figure 4: A part of the graph $G(3,2,2)$.

From Lemma 3.6, a vertex $[m,n]$ of $G(a,b,c)$ is of Type 3 if and only if

$$\vartheta(b\beta+\beta') > \frac{|b(a\alpha+\alpha')|}{(ab,\alpha')} \text{ and } \vartheta(a\alpha+\alpha') > \frac{|a(b\beta+\beta')|}{(ab,\beta')}. \tag{7}$$

The following theorem gives a necessary and sufficient condition for the existence of a pair of Type 3 in a graph $G(a,b,c)$.

Theorem 4.2 *$G(a,b,c)$ has elements of Type 3 if and only if $|ab| < c^2$.*

Proof. Let us assume that $|ab| < c^2$. We will prove that there is a pair such that $\vartheta = c$ and the condition holds. If $\vartheta = c$, then $(ab,\alpha') = (ab,\beta') = 1$ since $ab\alpha\beta - \alpha'\beta' = -c/\vartheta = -1$. Thus, $[m,n]$ with $\vartheta = c$ is a pair or Type 3 if and only if

$$c(b\beta+\beta') > |b(a\alpha+\alpha')| \text{ and } c(a\alpha+\alpha') > |a(b\beta+\beta')|. \tag{8}$$

Let A, B be any integers such that

$$\begin{cases} cB > |bA|, \\ cA > |aB|, \end{cases} \tag{9}$$

$$(A,B) = (a,A) = (b,B) = 1 \text{ and } (a,b)|(AB-1). \tag{10}$$

Let α, β be any solution of a diophantine equation

$$ab\alpha\beta - (A - a\alpha)(B - b\beta) = -1,$$

that is,

$$aB\alpha + bA\beta = AB - 1,$$

and $\alpha' = A - a\alpha$, $\beta' = B - b\beta$. Then, we have $(\alpha, \alpha') = (\alpha, A - a\alpha) = (\alpha, A) = (aB\alpha, A) = (-bA\beta + AB - 1, A) = 1$, $(\beta, \beta') = (\beta, B - b\beta) = (\beta, B) = (bA\beta, B) = (-aB\alpha + AB - 1, B) = 1$. Hence, if we define m, n by $d_m = c(b\beta + \beta') = cB$, $m = d_m\alpha$, $d_n = c(a\alpha + \alpha') = cA$ and $n = d_n\beta$, then $[m, n]$ satisfies $(am + bn + c)|mn$ and the condition (8), that is, $[m, n]$ is a pair of Type 3. From this reason, if we prove the existence of any integers satisfying (9) and (10), the proof completes.

Without loss of generality, we assume that $|a| \geq |b|$. Then the possible orderings are $|b| \leq |a| < |c|$, $|b| < |a| = |c|$ and $|b| < |c| < |a|$. Through this proof, we assume that a and b are positive, since we consider only absolute values of a, b. Three cases arise.

Case 1. $b \leq a < |c|$. Putting $A = B = \mathrm{sign}(c)$, we find that inequalities (9) and (10) hold.

Case 2. $b < a = |c|$. Putting $A = \mathrm{sign}(c)(a + ab + 1)$ and $B = \mathrm{sign}(c)(ab + 1)$, we have $(A, B) = (a, A) = (b, B) = 1$ and $(a, b)|(AB - 1)$, and the inequalities (9) hold because

$$\begin{aligned}
cB - |bA| &= (ab|c| + |c|) - (ab + ab^2 + b), \\
&= ab(|c| - b) + (|c| - b) - ab \\
&\geq ab - ab + (|c| - b) \\
&= |c| - b \\
&> 0, \\
cA - |aB| &= (a|c| + ab|c| + |c|) - (a^2b + a) \\
&= ab(|c| - a) + (|c| - a) + a|c| \\
&= a^2 \\
&> 0.
\end{aligned}$$

Case 3. $b < |c| < a$. Let $A = \mathrm{sign}(c)(ak + abl + 1)$ and $B = \mathrm{sign}(c)(abl + 1)$, where k and l are any integers such that

$$k > \frac{b(a - |c|)(|c| - b)}{c^2 - ab},$$

$$\frac{k}{|c| - b} - \frac{1}{ab} < l < \frac{k|c|}{b(a - |c|)} - \frac{1}{ab}. \tag{11}$$

Then A and B satisfy $(A, B) = (a, A) = (b, B) = 1$ and $(a, b)|(AB - 1)$, and the inequalities (9) follow, since

$$\begin{aligned}
cB - |bA| &= (ab|c|l + |c|) - b(ak + abl + 1) \\
&= ab(|c| - b)l - (abk - |c| + b) \\
&> (abk - |c| + b) - (abk - |c| + b) \\
&= 0, \\
cA - |aB| &= (a|c|k + ab|c|l + |c|) - a(abl + 1) \\
&= -ab(a - |c|)l + a|c|k - (a - |c|) \\
&> -a|c|k + (a - |c|) + a|c|k - (a - |c|) \\
&= 0.
\end{aligned}$$

Finally, we show that the existence of the positive integer l satisfying (11). The difference between the left and right hand side of (11) is

$$\begin{aligned}
\left(\frac{k|c|}{b(a-|c|)} - \frac{1}{ab}\right) - \left(\frac{k}{|c|-b} - \frac{1}{ab}\right) &= \frac{k|c|}{b(a-|c|)} - \frac{k}{|c|-b} \\
&= \frac{k(c^2 - ab)}{b(a-|c|)(|c|-b)} \\
&> 1,
\end{aligned}$$

hence there is a positive integer in the interval (11).

Conversely, assume that a pair $[m,n]$ is of Type 3 in $G(a,b,c)$. From Lemma 3.6, $[m,n]$ satisfies

$$\vartheta(b\beta + \beta') > \frac{|b(a\alpha + \alpha')|}{(ab, \alpha')} \text{ and } \vartheta(a\alpha + \alpha') > \frac{|a(b\beta + \beta')|}{(ab, \beta')}.$$

Multiplying each side of the inequalities, we have $\vartheta^2(ab,\alpha')(ab,\beta') > |ab|$. Since $\vartheta(ab\alpha\beta - \alpha'\beta') = -c$, both $\vartheta(ab,\alpha')$ and $\vartheta(ab,\beta')$ divides c. Hence $\vartheta(ab,\alpha') \leq c$ and $\vartheta(ab,\beta') \leq c$. So, we obtain $c^2 > |ab|$. ∎

Corollary 4.3 *If $c^2 \leq |ab|$, each element in $S(a,b,c)$ is the least element of $S_{a,b,c}(\alpha,\alpha')$, or of $S'_{a,b,c}(\beta,\beta')$, or of both.*

Proof. By Lemma 4.2, if $c^2 \leq |ab|$, $G(a,b,c)$ has no pairs of Type 3. Hence each pair in $G(a,b,c)$ is either Type 1 or Type 2. ∎

4.2 Binary tree structures

Lemma 4.4 *$S_{a,b,c}(\alpha,\alpha')$ and $S'_{a,b,c}(\beta,\beta')$ have at most one element in common.*

Proof. Let us assume that $[m_1, n_1], [m_2, n_2] \in S_{a,b,c}(\alpha, \alpha') \cap S'_{a,b,c}(\beta, \beta')$. Since $[m_1, n_1], [m_2, n_2] \in S_{a,b,c}(\alpha, \alpha')$, we have $\alpha_2 = \alpha_1$, $\alpha'_2 = \alpha'_1$ and there exists an integer k such that

$$\vartheta_2\beta_2 = \vartheta_1\beta_1 + k\frac{\alpha'_1}{(ab, \alpha'_1)}, \quad \vartheta_2\beta'_2 = \vartheta_1\beta'_1 + k\frac{ab\alpha_1}{(ab, \alpha'_1)}.$$

Furthermore, since $[m_1, n_1], [m_2, n_2] \in S'_{a,b,c}(\beta, \beta')$, then $\beta_1 = \beta_2$, $\beta'_2 = \beta'_1$. So we have

$$\begin{aligned}\vartheta_2 - \vartheta_1 &= -c\left(\frac{1}{ab\alpha_1\beta_1 - \alpha'_1\beta'_1} - \frac{1}{ab\alpha_2\beta_2 - \alpha'_1\beta'_2}\right) \\ &= 0.\end{aligned}$$

Thus $[m_1, n_1]$ corresponds to $[m_2, n_2]$. ∎

From Corollary 4.3 and Lemma 4.4, we obtain a necessary and sufficient condition so that $G(a, b, c)$ has a binary tree structure.

Theorem 4.5 *Each connected component of a graph $G(a, b, c)$ has a binary tree structure if and only if $c^2 \leq |ab|$. This binary tree has the root $[m, n]$ such that*

$$\left\{\begin{aligned} d_m &\leq \frac{|b(a\alpha + \alpha')|}{(ab, \alpha')}, \\ d_n &\leq \frac{|a(b\beta + \beta')|}{(ab, \beta')}. \end{aligned}\right. \tag{12}$$

Proof. Every vertex of $G(a, b, c)$ is of outdegree 2. From Lemma 4.3, every vertex, other than the pairs satisfying the condition (12), is of indegree 1.

Let $[m_1, n_1]$ be any vertex of $G(a, b, c)$. If $[m_1, n_1]$ does not satisfy the inequalities (12), either it is the least element of $S_{a,b,c}(\alpha_1, \alpha'_1)$ or the least element of $S'_{a,b,c}(\beta_1, \beta'_1)$. Without loss of generality, assume that $[m, n]$ is the least element of $S_{a,b,c}(\alpha_1, \alpha'_1)$. Then we can trace it to the least element of $S'_{a,b,c}(\beta_1, \beta'_1)$. It is an element of $S_{a,b,c}(\alpha_2, \alpha'_2)$ for some α_2, α'_2, β_2 and β'_2. If it is the least element in $S_{a,b,c}(\alpha_2, \alpha'_2)$, it satisfies (12). If it is not, we can trace it to the least element of $S_{a,b,c}(\alpha_2, \alpha'_2)$.

When the step is repeated, the value of $\vartheta(a\alpha + \alpha')$ and $\vartheta(b\beta + \beta')$ decreases keeping positivity. We can arrive, consequently, at the element $[m, n]$ which is the least element of both $S_{a,b,c}(\alpha, \alpha')$ and $S'_{a,b,c}(\beta, \beta')$ for some α, α', β and β'. It must satisfy (12). Any element is therefore reached from the element $[m, n]$ by a unique directed path. Hence, each component of $G(a, b, c)$ is an infinite binary tree with the root $[m, n]$. ∎

Corollary 4.6 *If $c = \pm 1$, each connected component of $G(a,b,c)$ is a binary tree with the root $[m,n]$ such that*

$$\begin{cases} d_m \leq |bd_n|, \\ d_n \leq |ad_m|. \end{cases} \tag{13}$$

Proof. From the assumption, $c^2 = 1 \leq |ab|$ always holds. Hence each connected component of $G(a,b,-1)$ is a binary tree from Theorem 4.5. Since $(a,c) = (b,c) = 1$ from Lemma 2.5, we have $(ab,\alpha') = (ab,\beta') = 1$. Since ϑ is a divisor of c, we have $|\vartheta| = 1$. Hence $d_m = \vartheta(b\beta + \beta') = |b\beta + \beta'|$ and $d_n = \vartheta(a\alpha + \alpha') = |a\alpha + \alpha'|$, and the condition (12) is rewritten by the condition (13). ∎

Corollary 4.7 ([4]) *Each connected component of $G(1,1,-1)$ is an infinite binary tree with the root $[m,n]$ such that $m + n - 1 = \pm 1$.*

Proof. By Corollary 4.6, each connected component of $G(1,1,-1)$ is a binary tree. By Corollary 3.8, each root must satisfy $m + n - 1 = \pm 1$. ∎

There are infinite components in $G(a,b,c)$ in general. In particular, Shibata and Seki [4] have shown that the structure of the set of pairs such that $(m+n-1)|mn$ and m,n are positive is a binary tree with the root $[1,1]$.

5 Duality of the set $S(a,b,-1)$

We will investigate the structure of binary trees in the case of $c = -1$ through Section 5, 6. First, we show that the set $S(a,b,-1)$ includes other linear orders, and another binary tree can be constructed in $S(a,b,-1)$.

5.1 Transformation of a subset of $S(a,b,c)$

We define a set $S(a,b)$ which is a subset of $S(a,b,-1)$.

Definition 5.1 *For the positive integers a,b, a set $S(a,b)$ is a collection of pairs in $S(a,b,-1)$ such that m,n are positive.*

We define the following subsets of $S(a,b)$,

$$\begin{aligned} S_{a,b}(\alpha,\alpha') &= S_{a,b,-1}(\alpha,\alpha') \cap S(a,b), \\ S'_{a,b}(\beta,\beta') &= S'_{a,b,-1}(\beta,\beta') \cap S(a,b). \end{aligned}$$

For any pair $[m,n]$ in $S(a,b)$, we have $\theta = 1$ and $\vartheta = 1$, since the value of θ is a divisor of 1 and a, b, m, n are positive.

Since $S_{a,b}(\alpha,\alpha')$ and $S'_{a,b}(\beta,\beta')$ are subsets of $S_{a,b,-1}(\alpha,\alpha')$ and $S'_{a,b,-1}(\beta,\beta')$, respectively, Lemma 3.2, Theorem 3.4 and Theorem 3.5 also hold for $S_{a,b}(\alpha,\alpha')$ and $S'_{a,b}(\beta,\beta')$. Hence these sets are linearly ordered sets. Furthermore, the necessary and sufficient condition for the least element in $S(a,b)$ also holds.

Lemma 5.2
(i) *A pair* $[m,n]$ *is the least element of* $S_{a,b}(\alpha,\alpha')$ *if and only if* $(b\beta+\beta') \leq b(a\alpha+\alpha')$.
(ii) *A pair* $[m,n]$ *is the least element of* $S'_{a,b}(\beta,\beta')$ *if and only if* $(a\alpha+\alpha') \leq a(b\beta+\beta')$.

Proof. We prove only (i). (ii) is a proposition symmetric to (i). Assume that $[m,n]$ is the least element of $S_{a,b}(\alpha,\alpha')$. Then we have $b\beta - b\alpha' \leq 0$ or $\beta' - ba\alpha < 0$.
The case for $b\beta - b\alpha' \leq 0$. Since $[m,n] \in S(a,b)$, we have $ab\alpha\beta - \alpha'\beta' = 1$ by Theorem 2.4. Then we have $ab\alpha(\beta-\alpha') - \alpha'(\beta' - ba\alpha) = 1$, this yields $\alpha'(\beta' - ab\alpha) = ab\alpha(\beta - \alpha') - 1 < 0$. Since $\alpha' > 0$, we obtain $\beta' - ba\alpha < 0$. By adding both sides of the inequalities $b\beta - b\alpha' < 0$ and $\beta' - ba\alpha < 0$, we have $(b\beta+\beta') - b(a\alpha+\alpha') < 0$.

The case for $\beta' - ab\alpha < 0$. In this case, two cases arise.
1. the case of $\alpha' > 0$. In a similar way, we obtain the same result.
2. the case of $\alpha' = 0$. Since $ab\alpha(b-\alpha') - 1 = \alpha'(\beta' - ba\alpha) = 0$, we have $ab\alpha(\beta - \alpha') = 1$. Hence $a = b = \alpha = 1$, $\alpha' = 0$ and $\beta = 1$ and $\beta' - ab\alpha = \beta' - 1 < 0$. Because of $\beta' \geq 0$, we have $\beta' = 0$. Hence $b\beta + \beta' = 1$ and $b(a\alpha+\alpha') = 1$, we obtain $(b\beta+\beta') \leq b(a\alpha+\alpha')$.
The converse is easily proved. ∎

We define a transformation D of $S(a,b)$ as follows.

Definition 5.3 *Let* D *be a transformation of a set* $S(a,b)$ *defined as follows:*

$$D([m,n]) = [m_1, n_1],$$

where

$$\begin{cases} \alpha_1 = \alpha, & \alpha'_1 = \beta' \\ \beta_1 = \beta, & \beta'_1 = \alpha' \\ d_{m_1} = b\beta_1 + \beta'_1, & d_{n_1} = a\alpha_1 + \alpha'_1 \\ m_1 = d_{m_1}\alpha_1, & n_1 = d_{n_1}\beta_1. \end{cases}$$

By this definition, we have $m_1 n_1 = (\alpha' + b\beta)(a\alpha+\beta')\alpha\beta$ and

$$\begin{aligned} am_1 + bn_1 - 1 &= a(b\beta+\alpha')\alpha + b(a\alpha+\beta')\beta - 1 \\ &= ab\alpha\beta + a\alpha\alpha' + b\beta\beta' + (ab\alpha\beta - 1) \end{aligned}$$

$$\begin{aligned} &= ab\alpha\beta + a\alpha\alpha' + b\beta\beta' + \alpha'\beta' \\ &= (a\alpha + \beta')(b\beta + \alpha'), \end{aligned}$$

so that we have $(am_1 + bn_1 - 1)|m_1n_1$. Therefore D is a transformation of $S(a,b)$. It is easily proved that D is a bijection and $D^{-1} = D$.

Example. [22,39] and [26,33] are members of $S(2,1)$. We have $D([22,39]) = [26,33]$ and $D([26,33]) = [22,39]$.

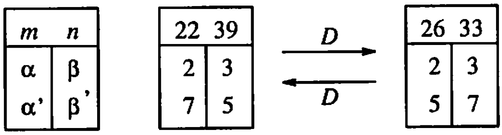

Figure 5: An example of transformation D.

5.2 Duality of $S(a,b)$

Definition 5.4 *Let $R_{a,b}(\alpha,\beta')$ and $R'_{a,b}(\alpha',\beta)$ be subsets of $S(a,b)$ defined as follows:*

$$\begin{aligned} R_{a,b}(\alpha,\beta') &= \left\{[m,n] \in S(a,b) \mid \frac{m}{d_m} = \alpha,\ \frac{am-1}{d_n} = \beta'\right\}, \\ R'_{a,b}(\alpha',\beta) &= \left\{[m,n] \in S(a,b) \mid \frac{bn-1}{d_m} = \alpha',\ \frac{n}{d_n} = \beta\right\}. \end{aligned}$$

Let $D([m,n]) = [m_1,n_1]$. Since D is a bijection of $S(a,b)$, $[m,n]$ corresponds to $[m_1,n_1]$ in one-to-one manner by the transformation D. If $[m,n]$ is a member of $S_{a,b}(\alpha,\alpha')$, then it is immediately proved that $[m_1,n_1]$ is a member of $R_{a,b}(\alpha',\alpha)$. Conversely, if $[m_1,n_1]$ is a member of $R_{a,b}(\alpha,\beta')$, then $[m,n]$ is a member of $S_{a,b}(\alpha,\beta')$. Let P be a proposition on $S_{a,b}(\alpha,\alpha')$, and P_1 be a proposition obtained by exchanging $S_{a,b}(\alpha,\alpha')$ for $R_{a,b}(\alpha,\beta')$ and α' for β'. Since D is a bijection of $S(a,b)$, the proof obtained by exchanging $S_{a,b}(\alpha,\alpha')$ for $R_{a,b}(\alpha,\beta')$ and α' for β' in the proof for P is valid. Hence P_1 is a proposition on $R_{a,b}(\alpha,\beta')$. Similarly, we can rewrite the proposition Q on $S'_{a,b}(\beta,\beta')$ to a proposition Q_1 on $R'_{a,b}(\alpha',\beta)$ by exchanging $S'_{a,b}(\beta,\beta')$ for $R'_{a,b}(\alpha',\beta)$ and α' for β'. We call P_1 and Q_1 *dual propositions* of P and Q, respectively. The symmetry between $S_{a,b}(\alpha,\alpha')$ and $S'_{a,b}(\beta,\beta')$ also holds between $R_{a,b}(\alpha,\beta')$ and $R'_{a,b}(\alpha',\beta)$. Lemma 3.2, Theorem 3.4 and Lemma 3.5 have dual propositions.

Theorem 5.5 *Let $[m,n]$ be any element of $S(a,b)$.*
(i) $(\alpha' + b\beta)\beta' + 1 \equiv 0 \pmod{b(a\alpha + \beta')}$.
(ii) $(a\alpha + \beta')\alpha' + 1 \equiv 0 \pmod{a(\alpha' + b\beta)}$.

Lemma 5.6
(i) *If $[m_1,n_1],[m_2,n_2] \in R_{a,b}(\alpha,\beta')$, then*

$$(\alpha_1' + b\beta_1) \equiv (\alpha_2' + b\beta_2) \pmod{b(a\alpha + \beta')}.$$

(ii) *If $[m_1,n_1],[m_2,n_2] \in R'_{a,b}(\alpha',\beta)$, then*

$$(a\alpha_1 + \beta_1') \equiv (a\alpha_2 + \beta_2') \pmod{a(\alpha' + b\beta)}.$$

Lemma 5.7
(i) *For $[m_1,n_1],[m_2,n_2] \in R_{a,b}(\alpha,\beta')$, there is an integer k, such that $b\beta_2 = b\beta_1 + kb\alpha'$ and $\beta_2' = \beta_1' + kab\alpha$.*
(ii) *For $[m_1,n_1],[m_2,n_2] \in R'_{a,b}(\alpha',\beta)$, there is an integer k, such that $a\alpha_2 = a\alpha_1 + ka\beta'$ and $\alpha_2' = \alpha_1' + kab\beta$.*

We introduce order relations on $R_{a,b}(\alpha,\beta')$ and $R'_{a,b}(\alpha',\beta)$ in similar way to Section 3.

Theorem 5.8
(i) *$R_{a,b}(\alpha,\beta')$ is a linearly ordered set with respect to the order of $(\alpha' + b\beta)$. The difference of the value $(\alpha' + b\beta)$ of an adjacent pair is $(a\alpha + \beta')$.*
(ii) *$R'_{a,b}(\alpha',\beta)$ is a linearly ordered set with respect to the order of $(a\alpha + \beta')$. The difference of the value $(a\alpha + \beta')$ of an adjacent pair is $(\alpha' + b\beta)$.*

Example. Fig. 6 shows linear ordering on $R_{3,2}(2,1)$.

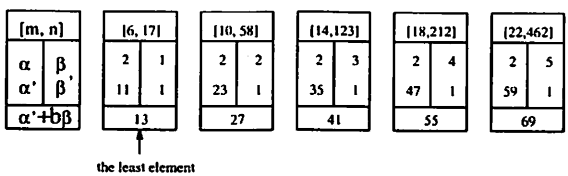

Figure 6: Linear ordering on $R_{3,2}(2,1)$.

By the dual proposition of Lemma 5.2, the necessary and sufficient condition for the least element of $R_{a,b}(\alpha,\beta')$ and $R'_{a,b}(\alpha',\beta)$ is given as follows.

Lemma 5.9 *For $[m,n] \in S(a,b)$,*
(i) *A pair $[m,n]$ is the least element of $R_{a,b}(\alpha,\beta')$ if and only if $\alpha' + b\beta \leq b(a\alpha + \beta')$.*
(ii) *A pair $[m,n]$ is the least element of $R'_{a,b}(\alpha',\beta)$ if and only if $a\alpha + \beta' \leq a(\alpha' + b\beta)$.*

5.3 Binary tree structure

We will represent the structures of $S(a, b)$ by $R_{a,b}(\alpha, \beta')$ and $R'_{a,b}(\alpha', \beta)$ in similar way to Section 4.

Definition 5.10 *Let $G_R(a, b)$ be a directed graph defined as follows:*
1. *A vertex set of $G_R(a, b)$ is $S(a, b)$;*
2. *There is a directed arc from v to v' if and only if these vertices satisfy either of the following conditions:*
(a) *$v, v' \in R_{a,b}(\alpha, \beta')$ and v' is the next element of v with respect to the order on $R_{a,b}(\alpha, \beta')$;*
(b) *$v, v' \in R'_{a,b}(\alpha', \beta)$ and v' is the next element of v with respect to the order on $R'_{a,b}(\alpha', \beta)$.*

By the dual proposition of Corollary 4.6, the structure of $G_R(a, b)$ is a binary tree.

Theorem 5.11 *Each connected component of a graph $G_R(a, b)$ is an infinite binary tree with the root $[m, n]$ such that $\alpha' + b\beta \leq b(a\alpha + \beta')$ and $a\alpha + \beta' \leq a(\alpha' + b\beta)$.*

Let a graph $G_S(a, b)$ be an induced subgraph of $G(a, b, -1)$ by $S(a, b)$. $G_S(a, b)$ has the binary tree structure presented in Corollary 4.6. In particular, for $a = b = 1$, both $G_S(1, 1)$ and $G_R(1, 1)$ are binary trees with the root $[1, 1]$.

Theorem 5.12 *Graph $G_S(1, 1)$ is a binary tree with the root* $[1, 1]$.

Proof. From Corollary 4.7, any root of $G_S(1, 1)$ satisfies $m + n - 1 = \pm 1$. Since m, n are positive, $m = n = 1$ is the unique solution. Therefore $G_S(1, 1)$ has only one connected component with the root $[1, 1]$. ∎

Theorem 5.13 *Graph $G_R(1, 1)$ is a binary tree with the root* $[1, 1]$.

Proof. The necessary and sufficient condition for the root of the binary tree is $(\alpha' + \beta) \leq (\alpha + \beta')$ and $(\alpha + \beta') \leq (\alpha' + \beta)$ from Theorem 5.11. Hence we have $(\alpha' + \beta) = (\alpha + \beta')$ or $\alpha - \alpha' = \beta - \beta'$. Since $\alpha\beta - \alpha'\beta' = \beta(\alpha - \alpha') + \alpha'(\beta - \beta') = 1$, we have $(\beta + \alpha')(\alpha - \alpha') = 1$. Since $\alpha, \beta \geq 1$ and $\alpha', \beta' \geq 0$, we obtain $\beta + \alpha = \alpha - \alpha' = 1$. Hence $\alpha = \beta = 1$ and $\alpha' = \beta' = 0$. This pair is $[1, 1]$. ∎

6 Structure of double binary trees

In this section, we show that $S(a,b)$ has a double binary tree structure, that is, a graph $G_S(a,b)$ is connected by arcs of graph $G_R(a,b)$. For $v, v' \in S(a,b)$, if v' is the next element of v in $S_{a,b}(\alpha,\alpha')$ (resp. $S'_{a,b}(\beta,\beta')$), then we call v' the left son (resp. right son) of v in $G_R(a,b)$, and v the father of v' in $G_R(a,b)$.

6.1 Preservation of father-son relation

Let $[m_{R1}, n_{R1}]$ be elements in $S_{a,b}(\alpha_{R1}, \alpha'_{R1})$, and $[m_{R2}, n_{R2}]$ be the next element of $[m_{R1}, n_{R1}]$ in $S_{a,b}(\alpha_{R1}, \alpha'_{R1})$. Their parameters are

$$\alpha_{R2} = \alpha_{R1},\ \alpha'_{R2} = \alpha'_{R1},\ \beta_{R2} = \beta_{R1} + \alpha'_{R1},\ \beta'_{R2} = \beta'_{R1} + ab\alpha_{R1}.$$

Let $[m_{S1}, n_{S1}]$ be the next element of $[m_{R1}, n_{R1}]$ with respect to the order on $R_{a,b}(\alpha_{R1}, \beta'_{R1})$. Then

$$\alpha_{S1} = \alpha_{R1},\ \alpha'_{S1} = \alpha'_{R1} + ab\alpha_{R1},\ \beta_{S1} = \beta_{R1} + \beta'_{R1},\ \beta'_{S1} = \beta'_{R1}.$$

Let $[m_{S2}, n_{S2}]$ be the next element of $[m_{R2}, n_{R2}]$ with respect to the order on $R_{a,b}(\alpha_{R2}, \beta'_{R2})$. Then

$$\alpha_{S2} = \alpha_{R2},\ \alpha'_{S2} = \alpha'_{R2} + ab\alpha_{R2},\ \beta_{S2} = \beta_{R2} + \beta'_{R2},\ \beta'_{S2} = \beta'_{R2}.$$

Hence we have

$$\begin{aligned}
\alpha_{S2} &= \alpha_{R2} = \alpha_{R1} = \alpha_{S1},\\
\beta_{S2} &= \beta_{R2} + \beta'_{R2} = (\beta_{R1} + \alpha'_{R1}) + (\beta'_{R1} + ab\alpha_{R1})\\
&= (\beta_{R1} + \beta'_{R1}) + (\alpha_{R1} + ab\alpha_{R1})\\
&= \beta_{S1} + \alpha'_{S1},\\
\alpha'_{S2} &= \alpha'_{R2} + ab\alpha_{R2} = \alpha'_{R1} + ab\alpha_{R1} = \alpha'_{S1},\\
\beta'_{S2} &= \beta'_{R2} = \beta'_{R1} + ab\alpha_{R1} = \beta'_{S1} + ab\alpha_{S1}.
\end{aligned}$$

From above equations, both $[m_{S1}, n_{S1}]$ and $[m_{S2}, n_{S2}]$ are elements of $S_{a,b}(\alpha_{S1}, \alpha'_{S1})$, and $[m_{S2}, n_{S2}]$ is the next element of $[m_{S1}, n_{S1}]$ with respect to the order on $S_{a,b}(\alpha_{S1}, \alpha'_{S1})$. This relation is shown in Figure 7. By symmetry between $S_{a,b}(\alpha,\alpha')$ and $S'_{a,b}(\beta,\beta')$, any father-son relation in $G_S(a,b)$ is preserved with respect to the tracing of directed arcs by $R_{a,b}(\alpha,\beta')$. Furthermore, by symmetry between $R_{a,b}(\alpha,\beta')$ and $R'_{a,b}(\alpha',\beta)$, any father-son relations in $G_S(a,b)$ are preserved with respect to the tracing of directed arcs by $R'_{a,b}(\alpha',\beta)$ of $G_R(a,b)$. Now the following theorem is obtained.

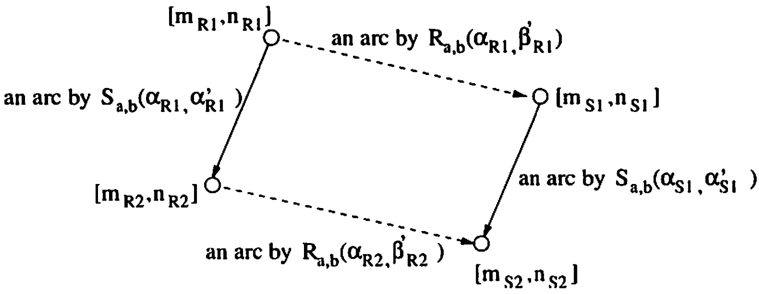

Figure 7: Preservation of the father-son relation in $G_S(a,b)$.

Theorem 6.1 *Any father-son relation in $G_S(a,b)$ is preserved with respect to the tracing directed arcs of $G_R(a,b)$.*

This theorem implies $G_S(a,b)$ and $G_R(a,b)$ construct double binary tree structures in $S(a,b)$.

6.2 Preservation of the least element

Let $[m,n]$ be the least element of $S_{a,b}(\alpha,\alpha')$, and let $[m,n]$ and $[m_1,n_1]$ belong to $R_{a,b}(\alpha,\beta')$. Then $\alpha_1 = \alpha$, $\beta'_1 = \beta'$, and there exists a positive integer k such that $\alpha'_1 = \alpha' + kab\alpha$, $\beta_1 = \beta + k\beta'$. Thus we obtain

$$d_{m_1} = b\beta_1 + \beta'_1 = (b\beta + \beta') + kb\beta' = d_m + kb\beta',$$

$$d_{n_1} = a\alpha_1 + \alpha'_1 = (a\alpha + \alpha') + kab\alpha = d_n + kab\alpha.$$

Since $[m,n]$ is the least element of $S_{a,b}(\alpha,\alpha')$, we have $bd_n - d_m \geq 0$ and

$$bd_{n_1} - d_{m_1} = (bd_n - d_m) + kb(ab\alpha - \beta'). \tag{14}$$

By the definition of the parameters, we have $d_n abm = d_m d_n ab\alpha$ and $d_m(am-1) = d_m d_n \beta'$. By subtracting the second equation from the first, we obtain $d_m d_n(ab\alpha - \beta') = am(bd_n - d_m) + d_m$. The right hand side of this equation is positive, this yields $ab\alpha - \beta' > 0$. We have $bd_{n_1} > d_{m_1}$ from the equation (14), that is, $[m_1,n_1]$ is the least element of $S_{a,b}(\alpha_1,\alpha'_1)$.

Lemma 6.2 *Let $[m,n]$ be the least element of $S_{a,b}(\alpha,\alpha')$. If $[m_1,n_1]$ is a succeeding element to $[m,n]$ in $R_{a,b}(\alpha,\beta')$, then $[m_1,n_1]$ is the least element of $S_{a,b}(\alpha_1,\alpha'_1)$.*

Next, let $[m,n]$ be the least element of $S'_{a,b}(\beta,\beta')$ and $[m,n],[m_2,n_2] \in R_{a,b}(\alpha,\beta')$. Then, we have $d_n \le ad_m$ and

$$ad_{m_1} - d_{n_1} = (ad_m - d_n) + kab(\beta' - \alpha). \tag{15}$$

From the definition, we have $d_m(am-1) = d_m d_n \beta'$ and $d_n m = d_m d_n \alpha$. By subtracting the second equation from the first one, we have

$$d_m d_n(\beta' - \alpha) = m(ad_m - d_n) - d_m.$$

The case of $d_n < ad_m$. Since $m \ge d_m$ and $d_m, d_n \ge 1$, we have $\beta' - \alpha \ge 0$. Therefore, (15) yields $ad_{m_1} > d_{n_1}$, so that $[m_1,n_1]$ is the least element of $S'_{a,b}(\beta_1,\beta'_1)$.
The case of $ad_m = d_n$. In this case, we obtain $\beta' - \alpha < 0$ and $ad_{m_1} < d_{n_1}$. Therefore $[m_1,n_1]$ is not the least element of $S'_{a,b}(\beta_1,\beta'_1)$. Since $d_n|(am-1)$ and $(a, am-1) = 1$, we have $(d_n, a) = 1$. Moreover, since $(d_m, d_n) = 1$ and $(ad_m, d_n) = 1$, we have $a = d_m = d_n = 1$. We obtain $m = n = 1$ because $m = (b\beta + \beta')\alpha$ and $n = (a\alpha + \alpha')\beta$.

Lemma 6.3 *Let $[m,n]$ be the least element of $S'_{a,b}(\beta,\beta')$ and let $[m_1,n_1]$ be an element succeeding to $[m,n]$ in $R_{a,b}(\alpha,\beta')$. If $a = b = 1$ and $[m,n] = [1,1]$, then $[m_1,n_1]$ is not the least element of $S'_{a,b}(\alpha_1,\alpha'_1)$. Otherwise, $[m_1,n_1]$ is the least element of $S'_{a,b}(\alpha_1,\alpha'_1)$.*

By symmetry between $R_{a,b}(\alpha,\beta')$ and $R'_{a,b}(\alpha',\beta)$, the following results are obtained.

Lemma 6.4 *Let $[m,n]$ be the least element of $S_{a,b}(\alpha,\alpha')$. If $[m_1,n_1]$ is a succeeding element to $[m,n]$ in $R'_{a,b}(\alpha',\beta)$, $[m_1,n_1]$ is the least element of $S_{a,b}(\alpha_1,\alpha'_1)$.*

Lemma 6.5 *Let $[m,n]$ be the least element of $S'_{a,b}(\beta,\beta')$ and $[m_1,n_1]$ be a succeeding element to $[m,n]$ in $R'_{a,b}(\alpha',\beta)$. If $a = b = 1$ and $[m,n] = [1,1]$, then $[m_1,n_1]$ is not the least element of $S'_{a,b}(\alpha_1,\alpha'_1)$. Otherwise, $[m_1,n_1]$ is the least element of $S'_{a,b}(\alpha_1,\alpha'_1)$.*

From the above lemmas, we obtain the following theorem.

Theorem 6.6 *Let $[m,n]$ be any root of $G_S(a,b)$. For $ab \ge 2$, every pair in the component which contains $[m,n]$ in $G_R(a,b)$ is also a root of some component of $G_S(a,b)$.*

6.3 Double binary tree in the case of $ab \geq 2$

$G_S(a,b)$ and $G_R(a,b)$ have a double binary tree structure in the case of $ab \geq 2$. The following lemmas lead to this fact.

Lemma 6.7 *Let a, b be positive integers.*
(i) *$S_{a,b}(\alpha, \alpha')$ and $R_{a,b}(\alpha, \beta')$ have at most one element in common.*
(ii) *$S'_{a,b}(\beta, \beta')$ and $R'_{a,b}(\alpha', \beta)$ have at most one element in common.*

Lemma 6.8 *Let $ab \geq 2$.*
(i) *$S_{a,b}(\alpha, \alpha')$ and $R'_{a,b}(\alpha', \beta)$ have at most one element in common.*
(ii) *$S'_{a,b}(\beta, \beta')$ and $R_{a,b}(\alpha, \beta')$ have at most one element in common.*

Proof. Easily proved. ∎

Both $G_S(a,b)$ and $G_R(a,b)$ consist of infinite components, and by Theorem 6.1, 6.6, Lemma 6.7 and Lemma 6.8, $S(a,b)$ has the double binary tree structure presented in Fig 8. We draw only arcs in $G_R(a,b)$ which connect roots of $G_S(a,b)$ in Fig. 8. The fact is that there are arcs of $G_R(a,b)$ connecting vertices in the corresponding position in each of binary trees of $G_S(a,b)$.

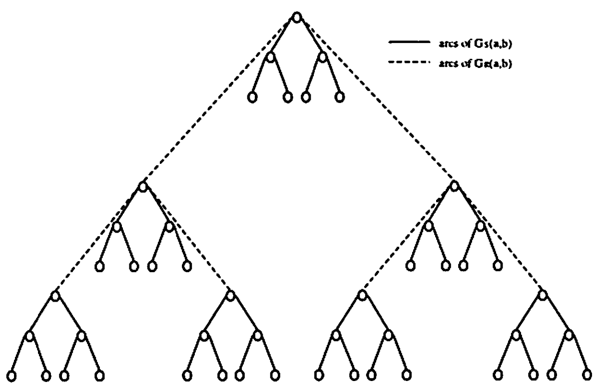

Figure 8: The double binary tree structure for the case of $ab \geq 2$.

6.4 Binary tree structure in $S(1,1)$

In the case of $a = b = 1$, the relations between $G_S(1,1)$ and $G_R(1,1)$ are described in the following lemma.

Lemma 6.9 *Let $a = b = 1$.*
(i) $S_{1,1}(1,0)$ and $R'_{1,1}(0,1)$ are identical. Otherwise, $S_{a,b}(\alpha,\alpha')$ and $R'_{a,b}(\alpha',\beta)$ have at most one element in common.
(ii) $S'_{1,1}(1,0)$ and $R_{1,1}(0,1)$ are identical. Otherwise, $S'_{1,1}(\beta,\beta')$ and $R_{1,1}(\alpha,\beta')$ have at most one element in common.

Proof. We prove only (i). If $a = b = 1$, then $[1,1]$ is the least element of both of $S_{1,1}(1,0)$ and $R'_{1,1}(0,1)$. The parameters of $[1,1]$ are $\alpha = \beta = 1$ and $\alpha' = \beta' = 0$. For any element $[m_1,n_1]$ in $S_{1,1}(1,0)$, there exists an integer k, such that $\alpha_1 = 1$, $\alpha'_1 = 0$ and $\beta_1 = 1$, $\beta'_1 = k$. For any element $[m_2,n_2]$ in $R'_{1,1}(0,1)$, there exists an integer k', such that $\alpha'_2 = 0$, $\beta_2 = 1$ and $\alpha_2 = 1$, $\beta'_2 = k'$. If $[m_1,n_1]$ is a member of $S_{1,1}(1,0)$, then $[m_1,n_1]$ is a member of $R'_{1,1}(0,1)$ since $\alpha' = 0$ and $\beta = 1$. Conversely, if $[m_1,n_1]$ is a member of $R'_{1,1}(0,1)$, then $[m_1,n_1]$ is in $S_{1,1}(1,0)$. Hence these two sets are identical.

Otherwise, we must have $\alpha' \neq 0$. In this case, it is easily proved that $S_{a,b}(\alpha,\alpha')$ and $R'_{a,b}(\alpha',\beta)$ have at most one element in common. ∎

Both $G_S(1,1)$ and $G_R(1,1)$ consist of a single binary tree with the root $[1,1]$. Hence, by Theorem 6.1, Lemma 6.7 and Lemma 6.9, $G_S(1,1)$ and $G_R(1,1)$ are binary trees in dual relation with each other. Fig. 9 shows $G_S(1,1)$ and $G_R(1,1)$.

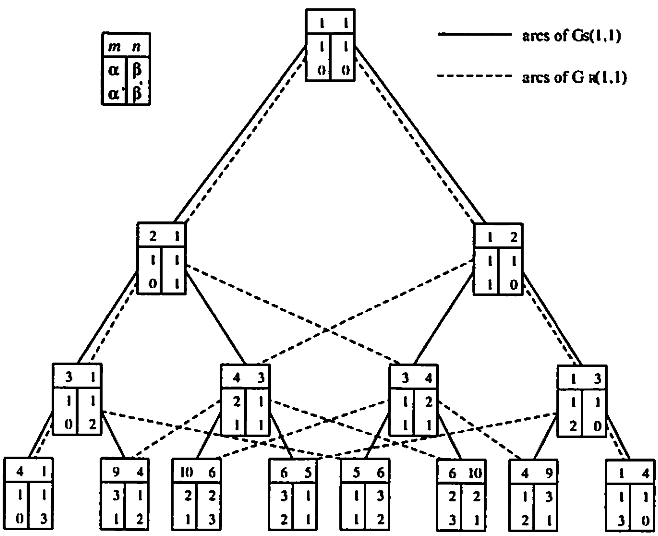

Figure 9: Binary tree structure of $G_R(1,1)$ and $G_S(1,1)$

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