On the Existence of Extended 5-Cycle Systems Having a Prescribed Number of Idempotent Elements

Michael E. Raines*
Western Michigan University
Department of Mathematics and Statistics
Kalamazoo, Michigan
USA 49008-5152
raines@math-stat.wmich.edu

Abstract

In this paper, necessary and sufficient conditions are given for the existence of extended 5-cycle systems of order n which have x idempotent elements.

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1 Introduction

Let λK_n^+ denote the complete multigraph of order n in which exactly λ edges join each pair of vertices and exactly λ loops occur at each vertex. Define an *extended* $(2k+1)$*-cycle* to be a *loop*, a k*-tadpole*, or a $(2k+1)$*-cycle* (see Figure 1). An *extended* $(2k+1)$*-cycle system* of order n and index λ is a decomposition of λK_n^+ into extended $(2k+1)$-cycles (in what follows, we will assume that the index is 1, unless otherwise specified). We refer to a loop as an *idempotent element*. Let (S, C) be an extended $(2k+1)$-cycle system, and for each extended triple $c \in C$, let $c(2)$ denote the distance 2 graph of c (so the graphs $c(2)$ and c have precisely the same vertex set, but two vertices are joined in $c(2)$ if and only if they are distance two apart in c). Let $C(2) = \{c(2) \mid c \in C\}$. Then (S, C) is said to be *2-perfect* if $(S, C(2))$ is also an extended $(2k+1)$-cycle system.

Extended $(2k+1)$-cycle systems are generalizations of extended triple systems, which are equivalent to totally symmetric quasigroups when $\lambda = 1$ (see, for example, [4]). Although every extended triple system gives rise to a quasigroup, it is not true that every extended $(2k+1)$-cycle system yields a quasigroup. However, if the extended $(2k+1)$-cycle system is 2-perfect, then it will yield a quasigroup by the standard construction which is described in [7].

It has been shown by Lindner and Rodger [7] that extended m-cycle systems can be equationally defined if and only if $m \in \{3, 5, 7\}$. Furthermore, it was shown that the equations which define the extended cycle systems are the same as those which define cycle systems, except that the idempotent law does not apply.

Since they were first introduced in 1972 by D. Johnson and N. S. Mendelsohn [5], extended triple systems have attracted their share of attention.

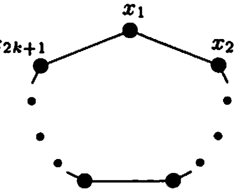
Extended $(2k+1)$-cycle	Graphical representation	$(2k+1)$-tuple representation
loop		$\{x_1, x_1, \ldots, x_1\}$
k-tadpole		$\{x_1, x_1, x_2, \ldots, x_{k+1}, x_k, \ldots, x_2\}$
$(2k+1)$-cycle		$\{x_1, x_2, \ldots, x_{2k+1}\}$

Figure 1:

For example, the spectrum for extended triple systems, i.e., the set of orders for which an extended triple system of order n exists, has been determined [1] (in fact, Bennett and Mendelsohn gave constructions for extended triple systems of order n with all possible numbers of idempotent elements). Furthermore, necessary and sufficient conditions for the existence of extended triple systems of order v and index λ which contain extended triple systems of order n and index λ have been determined when $\lambda = 1$ by Hoffman and Rodger [4] and when $\lambda > 1$ by Raines [8]. However, virtually nothing is known about extended k-cycle systems when $k > 3$. In particular, the spectrum for extended 5-cycle systems has not been determined. The main objective of this paper is to prove the following theorem.

Theorem 1.1 *Let $n = 1$ or $n \geq 4$. Then there exists an extended 5-cycle system of order n which contains x idempotent elements iff $x \equiv 0 \pmod 5$ if $n \equiv 0 \pmod 5$, $x \equiv 1 \pmod 5$ if $n \equiv 1, 4 \pmod 5$, $x \equiv 4 \pmod 5$ if $n \equiv 2, 3 \pmod 5$, and $x \leq n/2$ if n is even.*

It turns out that we obtain a solution for the spectrum problem for extended 5-cycle systems as a corollary.

Corollary 1.2 *There exists an extended 5-cycle system of order n if and only if $n = 1$ or $n \geq 4$.*

2 Preliminary Results

Before presenting the constructions for extended 5-cycle systems, we introduce some useful definitions and theorems.

Let $Q = \{1, 2, \ldots, 2k\}$, and let $H = \{\{1, 2\}, \{3, 4\}, \ldots, \{2k-1, 2k\}\}$. We refer to the two-element subsets of H as *holes*. If $(Q, \circ)$ is a commutative quasigroup with the property that, for each hole $h \in H$, $(h, \circ)$ is a subquasigroup, then $(Q, \circ)$ is a *commutative quasigroup with holes H*. Figure 2 gives an example of a commutative quasigroup of order 6 with holes.

We have the following theorem regarding the existence of commutative quasigroups with holes.

Theorem 2.1 ([6]) *For every $2k \geq 6$, there exists a commutative quasigroup with holes of order $2k$.*

A quasigroup $(Q, \circ)$ of order n is said to be *idempotent* if $i \circ i = i$ for each $i \in Q$. We also have the following well-known theorem which will enable us to complete our construction.

$\circ$	1	2	3	4	5	6
1	1	2	5	6	3	4
2	2	1	6	5	4	3
3	5	6	3	4	1	2
4	6	5	4	3	2	1
5	3	4	1	2	5	6
6	4	3	2	1	6	5

Figure 2: A commutative quasigroup or order 6 with holes.

Theorem 2.2 **([6])** *For every positive integer t, there exists an idempotent commutative quasigroup of order $2t+1$.*

Let $K_n \setminus K_v$ denote the complete graph of order n with the edges of a complete graph of order v removed (also referred to as a complete graph of order n with a hole of size v). To prove our main result, we will employ a useful theorem on the spectrum for 5-cycle systems of $K_n \setminus K_v$.

Lemma 2.3 **([3])** *There exists a 5-cycle system of $K_n \setminus K_v$ whenever $n \geq \frac{3v}{2} + 1$ and*

(i) $n, v \equiv 1$ or 5 (mod 10),

(ii) $n, v \equiv 7$ or 9 (mod 10), or

(iii) $n, v \equiv 3$ (mod 10).

3 The Construction

We begin by proving the necessity of Theorem 1.1.

Proposition 3.1 *If there exists an extended 5-cycle system of order n which contains x idempotent elements then*

(i) $n = 1$ *or* $n \geq 4$*;*

(ii) $x \equiv 0$ *(mod 5) if* $n \equiv 0$ *(mod 5),* $x \equiv 1$ *(mod 5) if* $n \equiv 1, 4$ *(mod 5),* $x \equiv 4$ *(mod 5) if* $n \equiv 2, 3$ *(mod 5); and*

(iii) $x \leq n/2$ *if* n *is even.*

Proof: Suppose (V, C) is an extended 5-cycle system of order n. We will show that $n \notin \{2, 3\}$. If $n < 5$, then C cannot contain any 5-cycles, so C must consist entirely of loops and 2-tadpoles. This means that the number of edges in K_n^+ must be even, so $n \notin \{2, 3\}$.

Now suppose that (V, C) is a 5ECS(n) which contains x idempotent elements. Suppose n is even. Then each vertex in K_n^+ has odd degree, so each vertex must be contained as a vertex of degree 1 in at least one 2-tadpole. This means that there must be at least $n/2$ 2-tadpoles, so at least $n/2$ loops will be contained in some 2-tadpole. Therefore, if n is even, then $x \leq n/2$.

Next consider the number x of idempotent elements which occur in a 5ECS (V, C). Each loop in K_n^+ is contained in exactly one loop or in exactly one 2-tadpole in C, so the number of loops x plus the number of 2-tadpoles is n. Furthermore, each edge in K_n^+ is contained in exactly one 5-cycle or in exactly one 2-tadpole. Therefore, the number of edges in K_n^+ which do not occur in any 2-tadpole must be divisible by 5. That is, $n(n-1)/2 - 2(n-x) \equiv 0 \pmod 5$. Now if $n \equiv 0 \pmod 5$, then $x \equiv 0 \pmod 5$; if $n \equiv 1, 4 \pmod 5$, then $x \equiv 1 \pmod 5$; and if $n \equiv 2, 3 \pmod 5$, then $x \equiv 4 \pmod 5$. □

For $4 \leq n \leq 13$, the appendix contains a 5ECS(n) with x idempotent elements for all admissible x. In the next two propositions we will construct some 5ECS(n) for other small values of n which are not handled by the general construction that will appear later.

Proposition 3.2 *If $n \in \{14, 16, 17, 18, 19\}$, there exists an extended 5-cycle system of order n with x idempotent elements for all admissible x.*

Proof: In all cases, we will start with an extended 5-cycle decomposition of $K^+_{10+v} \setminus K^+_v$ with x' idempotents (note that for $v = 4, 6$ and 8, the appendix contains such decompositions for $x' = 0$ and $x' = 5$, and for $v = 7$ and 9, the appendix contains such decompositions for $x' = 0, 5$, and 10). Then we will "fill in" the hole of size v with the appropriate extended 5-cycle decomposition of K^+_v with x^* idempotents.

First suppose $n = 14$. By Proposition 3.1, $x = 1$ or $x = 6$. The appendix contains an extended 5-cycle decomposition of K^+_4 with 1 idempotent element. Using this decomposition to fill in a decomposition of $K^+_{14} \setminus K^+_4$ with 0 and 5 idempotents, respectively, gives the desired result.

Next suppose $n = 16$. By Proposition 3.1, $x = 1$ or $x = 6$. The appendix contains an extended 5-cycle decomposition of K^+_6 with 1 idempotent element. Using this decomposition to fill in a decomposition of $K^+_{16} \setminus K^+_6$ with 0 and 5 idempotents, respectively, gives the desired result.

Now suppose $n = 17$. By Proposition 3.1, $x = 4, 9$, or 14. The appendix contains an extended 5-cycle decomposition of K^+_7 with 4 idempotent elements. Using this decomposition to fill in a decomposition of $K^+_{17} \setminus K^+_7$ with 0, 5, and 10 idempotents, respectively, gives the desired result.

Next suppose $n = 18$. By Proposition 3.1, $x = 4$ or $x = 9$. The appendix contains an extended 5-cycle decomposition of K^+_8 with 4 idempotent elements. Using this decomposition to fill in a decomposition of $K^+_{18} \setminus K^+_8$ with 0 and 5 idempotents, respectively, gives the desired result.

Finally, suppose $n = 19$. By Proposition 3.1, $x = 1, 6, 11$, or 16. The appendix contains extended 5-cycle decompositions of K_9^+ with 1 and 6 idempotent elements. Using the decomposition with 1 idempotent to fill in a decomposition of $K_{19}^+ \setminus K_9^+$ with 0, 5, and 10 idempotents, respectively, handles the cases when $x = 1, 6$, and 11. Also, using the decomposition with 6 idempotents to fill in a decomposition of $K_{19}^+ \setminus K_9^+$ with 10 idempotents handles the case when $x = 16$. □

Proposition 3.3 *Let $n \in \{21, 23, 27, 29\}$. Then there exists an extended 5-cycle systems of order n with x idempotent elements for all admissible $x > n/2$.*

Proof: We need only consider odd values of n since $x \leq n/2$ when n is even. By Lemma 2.3, there exists a 5-cycle system of $K_n \setminus K_v$ (and, thus, a 5-cycle system of $K_n^+ \setminus K_v^+$) whenever $n \geq \frac{3v}{2} + 1$ and $n, v \equiv 1$ or 5 (mod 10), $n, v \equiv 7$ or 9 (mod 10), or $n, v \equiv 7$ (mod 10). So if $n = 21$, then $v \in \{1, 6, 11\}$; if $n = 23$, then $v \in \{3, 13\}$; if $n = 27$, then $v \in \{7, 9, 17\}$; and if $n = 29$, then $v \in \{7, 9, 17\}$. In particular, there exists a 5-cycle system of $K_n^+ \setminus K_{n-10}^+$ if $n \in \{21, 23, 27\}$, and there exists a 5-cycle system of $K_{29}^+ \setminus K_{17}^+$. Consider $n = 21$; then there exists a 5-cycle system of $K_{21}^+ \setminus K_{11}^+$. Replace the hole of size 11 by a 5ECS(11) with 1, 6, or 11 idempotent elements so that our resulting 5ECS of K_{21}^+ contains 11, 16, or 21 idempotent elements. The cases when $n = 21$ and $x = 1$ or 6 are handled in the appendix.

Next consider $n = 23$. Then there exists a 5-cycle system of $K_{23}^+ \setminus K_{13}^+$. Replace the hole of size 13 with a 5ECS(13) with 4 or 9 idempotent elements to produce a 5ECS(23) with 14 or 19 idempotent elements. The cases when $x = 4$ or 9 are handled in the appendix.

Now consider $n = 27$. Then there exists a 5-cycle system of $K_{27}^+ \setminus K_{17}^+$. Replace the hole of size 17 with a 5ECS(17) with 4, 9, or 14 idempotent elements, so that the resulting 5ECS(27) has 14, 19, or 24 idempotent elements. The cases when $x = 4$ or 9 are handled in the appendix.

Finally consider $n = 29$. Then there exists a 5-cycle system of $K_{29}^+ \setminus K_{17}^+$. Replace the hole of size 17 with a 5ECS(17) with 4, 9, or 14 idempotent elements, so that the resulting 5ECS(29) contains 16, 21, or 26 idempotents. The cases when $x = 1, 6$ or 11 are handled in the appendix.

We are now ready to prove the main theorem.

Theorem 3.4 *Let $n = 1$ or $n \geq 4$. Then there exists an extended 5-cycle system of order n which contains x idempotent elements iff $x \equiv 0 \pmod 5$ if $n \equiv 0 \pmod 5$, $x \equiv 1 \pmod 5$ if $n \equiv 1, 4 \pmod 5$, $x \equiv 4 \pmod 5$ if $n \equiv 2, 3 \pmod 5$, and $x \leq n/2$ if n is even.*

Proof: The necessity is handled by Proposition 3.1, so we only prove the sufficiency here. We first note that for $n = 1$ and for $4 \leq n \leq 29$, the appendix contains 5ECS(n) with all admissible numbers of idempotent elements for those values which have not been considered previously and which are not addressed by the proof of this theorem. Furthermore, the appendix contains extended 5-cycle systems with all admissible numbers of idempotent elements for the graph $K_{10+j}^+ \setminus K_j^+$ (that is, K_{10+j}^+ with the edges and loops of K_j^+ removed), where $0 \leq j \leq 9$, $j \neq 5$.

We will show by direct construction that there exists a 5ECS(n) (V, C) with all possible admissible numbers of idempotent elements for the remaining values of n which are not handled in the appendix. First, suppose $n = 10k + j$, where $n \geq 30$, $0 \leq j \leq 9$ and $j \neq 5$. We define (V, C) with $x = 5y + z$ idempotent elements. Let $V = (\{1, 2, \ldots, 2k\} \times \{0, 1, 2, 3, 4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$. We break up the construction according to the different values of j.

Case 1: j is even.

Then $x = 5y + z \le (10k + j)/2 = n/2$, so $y \le k$. We begin by assuming that z is the smallest admissible number of idempotent elements that a 5ECS$(10 + j)$ may contain (so $z = 0$ if $j = 0$; $z = 1$ if $j = 4$ or 6; and $z = 4$ if $j = 2$ or 8). Furthermore, we assume that $0 \le y \le k - 1$. We place a 5ECS$(10 + j)$ with z idempotents on the vertices $(\{1, 2\} \times \{0, 1, 2, 3, 4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$ and, for $1 \le i \le y$, we place a 5ECS of $K_{10+j}^+ \setminus K_j^+$ with 5 idempotents on $(\{2i+1, 2i+2\} \times \{0, 1, 2, 3, 4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$. Furthermore, for $y + 1 \le i \le k - 1$, we place a 5ECS of $K_{10+j}^+ \setminus K_j^+$ with 0 idempotents on $(\{2i + 1, 2i + 2\} \times \{0, 1, 2, 3, 4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$.

Up to this point, we are able to obtain up to $5k - 5$ idempotents if $j = 0$, up to $5k - 4$ idempotents if $j = 4$ or 6, and up to $5k - 1$ idempotents if $j = 2$ or 8. It remains to show that we can obtain the following: $5k$ idempotents if $j = 0$; $5k + 1$ idempotents if $j = 4$ or 6; and $5k + 4$ idempotents if $j = 8$ (notice that $5k - 1$ is the maximum number of idempotents possible if $j = 2$). In all cases, we place a 5ECS of $K_{10+j}^+ \setminus K_j^+$ with 5 idempotent elements on $(\{2i+1, 2i+2\} \times \{0, 1, 2, 3, 4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$, for $1 \le i \le k-1$. If $j = 0$, we place a 5ECS(10) with 5 idempotents on $\{1, 2\} \times \{0, 1, 2, 3, 4\}$. Furthermore, if $j = 2$, we have already produced an ECS for every admissible number of idempotent elements. Next, if $j = 4$ or 6, we place a 5ECS$(10+j)$ with 6 idempotents on $(\{1, 2\} \times \{0, 1, 2, 3, 4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$. Finally, if $j = 8$, we place a 5ECS(18) with 9 idempotent elements on $(\{1, 2\} \times \{0, 1, 2, 3, 4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_8\}$.

At this point, each loop and edge contained entirely in the sets $\{\infty_1, \infty_2, \ldots, \infty_j\}$ and $\{2i + 1, 2i + 2\} \times \{0, 1, 2, 3, 4\}$, for $0 \le i \le k - 1$, has been placed in exactly one extended 5-cycle. Furthermore, for $0 \le i \le k - 1$, each edge which joins a vertex in $\{2i + 1, 2i + 2\} \times \{0, 1, 2, 3, 4\}$ to a vertex in $\{\infty_1, \infty_2, \ldots, \infty_j\}$ has been placed in exactly one extended 5-cycle. We partition the remaining edges into 5-cycles by using a commutative quasigroup with holes $(Q, \circ)$. For each pair $a, b, \in Q$, where a and b are contained in different holes, we form the 5-cycles $\{(a, \ell), (b, \ell), (a, \ell+1), (a \circ b, \ell+3), (b, \ell+1)\}$, where $0 \leq \ell \leq 4$ and all sums are reduced modulo 5.

Case 2: j is odd, $j \neq 5$.

We let $x = 5y + z \leq 10k + j = n$, so we assume that $y \leq 2k$. We begin, as in the previous case, by assuming that z is the smallest admissible number of idempotents that a 5ECS$(10+j)$ may contain (so $z = 1$ if $j = 1$ or 9 and $z = 4$ if $j = 3$ or 7). Furthermore, we consider the values of y such that $0 \leq y \leq k-1$. We place a 5ECS$(10+j)$ with z idempotents on the vertices $\{1,2\} \times \{0,1,2,3,4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$ and, for $1 \leq i \leq y$, we place a 5ECS of $K_{10+j}^+ \setminus K_j^+$ with 5 idempotents on $(\{2i+1, 2i+2\} \times \{0,1,2,3,4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$. Also, for $y + 1 \leq i \leq k-1$, we place a 5ECS of $K_{10+j}^+ \setminus K_j^+$ with 0 idempotents on $(\{2i+1, 2i+2\} \times \{0,1,2,3,4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$.

Next suppose that $k \leq y \leq 2k-2$, say $y = k + \alpha$. Then place a 5ECS$(10+j)$ with z idempotents on the vertices $(\{1,2\} \times \{0,1,2,3,4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$. Furthemore, for $1 \leq i \leq \alpha + 1$, place a 5ECS of $K_{10+j}^+ \setminus K_j^+$ with 10 idempotents on $(\{2i+1, 2i+2\} \times \{0,1,2,3,4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$, and for $\alpha + 2 \leq i \leq k-1$, place a 5ECS of $K_{10+j}^+ \setminus K_j^+$ with 5 idempotents on $(\{2i+1, 2i+2\} \times \{0,1,2,3,4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$.

To this point, we are able to obtain up to $10k-9$ idempotents if $j = 1$ or 9 and up to $10k-6$ idempotents if $j = 3$ or 7. It remains to show that we can obtain the following: $10k-4$ or $10k+1$ idempotents if $j = 1$; $10k-1$ idempotents if $j = 3$; $10k-1$ or $10k+4$ idempotents if $j = 7$; and $10k-4, 10k+1$, or $10k+6$ idempotents if $j = 9$. In all cases, we place a 5ECS of $K_{10+j}^+ \setminus K_j^+$ with 10 idempotents on $(\{2i+1, 2i+2\} \times \{0,1,2,3,4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_j\}$, for $1 \leq i \leq k-1$. If $j = 1$, we place a 5ECS(11) with either 6 or 11 idempotents on $(\{1,2\} \times \{0,1,2,3,4\}) \cup$

$\{\infty_1\}$. If $j = 3$, we place a 5ECS(13) with 9 idempotents on $(\{1,2\} \times \{0,1,2,3,4\}) \cup \{\infty_1, \infty_2, \infty_3\}$. Next, if $j = 7$, we place a 5ECS(17) with either 9 or 14 idempotents on $(\{1,2\} \times \{0,1,2,3,4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_7\}$. Finally, if $j = 9$, we place a 5ECS(19) with 6, 11, or 16 idempotents on $(\{1,2\} \times \{0,1,2,3,4\}) \cup \{\infty_1, \infty_2, \ldots, \infty_9\}$.

To finish the construction, we partition the remaining edges into 5-cycles using a commutative quasigroup with holes $(Q, \circ)$. For each pair $a, b, \in Q$, where a and b are contained in different holes, we form the 5-cycles $\{(a,\ell),(b,\ell),(a,\ell+1),(a \circ b, \ell+3),(b,\ell+1)\}$, where $0 \le \ell \le 4$ and all sums are reduced modulo 5.

Case 3: $j = 5$.

Let $V = \{1,2,\ldots,2k+1\} \times \{0,1,2,3,4\}$, and define a 5ECS($n$) (V, C) having $x = 5y$ idempotent elements as follows. For $1 \le i \le y$, place in C the extended 5-cycles of an ECS(5) having 5 idempotent elements on each of the sets of vertices $\{i\} \times \{0,1,2,3,4\}$, and for $y+1 \le i \le 2k+1$, place in C the extended 5-cycles of a 5ECS(5) having no idempotent elements on each of the sets of vertices $\{i\} \times \{0,1,2,3,4\}$. Next, let $(V, \circ)$ be an idempotent commutative quasigroup. For each pair $a, b \in V$ and for $0 \le \ell \le 4$, place in C the 5-cycles of the form $\{(a,\ell),(b,\ell),(a,\ell+1),(a \circ b, \ell+3),(b,\ell+1)\}$ where all sums are reduced modulo 5. This construction gives ECS(n)s having all possible admissible numbers of idempotent elements, for $n \equiv 5 \pmod{10}$, $n \ge 15$. □

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