On the Connectivity of $(4;g)$-Cages

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Abstract

A $(k;g)$-graph is a k-regular graph with girth g. A $(k;g)$-cage is a $(k;g)$-graph with the least possible number of vertices. In this paper, we prove that all $(4;g)$-cages are 4-connected, a special case of the conjecture about $(k;g)$-cages' connectivity made by H.L.Fu et al [1].

1 Introduction

We consider finite simple graphs. Any undefined notation follows Bondy and Murty [2]. The vertex set, edge set of a graph G are denoted, respectively, by $V(G), E(G)$. Suppose that V' (or E') is a nonempty subset of $V(G)$ (or $E(G)$). The induced subgraph (or the edge-induced subgraph) is denoted by $G[V']$ (or $G[E']$). The subgraph obtained from G by deleting the vertices in V' together with their incident edges is denoted by $G - V'$. The graph obtained from G by adding a set of edges E' is denoted by $G + E'$. Let $X, Y \subseteq V(G)$, $\overline{X} = V(G) - X$ and $E(X,Y) = \{xy \in E(G) \mid x \in X, y \in Y\}$. For a vertex v of G and a set of vertices $S \subseteq V(G)$, we use $N_S(v)$ to denote the set of vertices in S that are adjacent to v. In this paper, we used the subindex to denote the neighbours of a vertex. For example, $N(v) = \{v_1, v_2, v_3, v_4\}$. One exception is that we use $X = \{x_1, x_2, x_3\}$ to denote a vertex cut of a $(k;4)$-cage. For two vertices u, v in $S \subseteq G$, let $d_S(u,v)$ denote the distance between u and v in S. The distance between a vertex u and a set of vertices X in $S \subseteq G$, denoted by $d_S(u,X)$, is the minimum distance between u and any vertex in X. For $S = G$, we simple use $N(G), d(u,v)$ and $d(u,X)$. Since all cages are connected, the distances between two vertices are always finite. The following notation is used very often in this paper. For $x \in X, H \subseteq G$, $y \in V(H)$ and a positive integer d_0, $D_{d_0}(x, N_H(y)) = \{v \in N_H(y) \mid d_H(x,y) \leq d_0\}$. The length of a shortest cycle in a graph G is called the girth of G. Clearly, adding edges to a graph G might decrease the girth of G. For convenience, we use smaller cycle to denote

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any cycle of length smaller than g. A k-regular graph with girth g is called a $(k;g)$-graph and a $(k;g)$-cage is a $(k;g)$-graph with the least possible number of vertices. Let $f(k;g)$ denote the number of vertices of the $(k;g)$-cage.

Cages have been studied widely since introduced by Tutte in 1947 [3]. The problem of finding cages has a prominent place in both Extremal and Algebraic Graph Theory. A survey paper by P. K. Wong [4] in 1982 refers to 70 publications. The study of cages has led to interesting applications of algebra to graph theory. Recently, it also attracted some attention from researchers in computer science (see [5], [6]). They use new computer search algorithms to find new cages or provide better bounds of $f(k;g)$.

In 1997, H.L.Fu et al [1] proved a fundamental property of cages. They first proved that all cages are 2-connected, then they showed that all the cubic cages are 3-edge-connected and it follows that all the cubic cages are 3-connected. Furthermore, they conjectured that all $(k;g)$-cages are k-connected. In this paper, we shall prove that all $(4;g)$-cages are 4-connected. Note that we cannot use their approach to prove all $(4;g)$-cages are 4-connected since the edge connectivity may not equal to the vertex connectivity for k regular graphs where $k \geq 4$. We shall try to shed some new light on this conjecture by using a new technic to prove the connectivity of cages. Let X be a 3-cut of a $(4;g)$-cage G. Our new technic involves finding two vertices in different components of $G-X$ with distance furthest away from each other. Then we delete these two vertices and construct a new 4-regular graph with girth at least g. This leads to a contradiction with the fact that the original graph is a cage. In [7] we have successfully used this technic to prove $(k;g)$-cages are k-edge-connected for k is odd.

2 Preliminary results on the structure of cages

From the fact that $(k;3)$-cage is the K_{k+1} and $(k;4)$-cage is the bipartite graph $K_{k,k}$, we assume that $g \geq 5$. We shall often use the following monotonicity theorem (see [1]) with respect of girth and the number of vertices in a cage.
Monotonicity Theorem: For all $k \geq 3$ and $3 \leq g_1 < g_2$, $f(k;g_1) < f(k;g_2)$.

They also showed that all $(k;g)$-cages are 2-connected. In [7], we proved that $(k;g)$-cages are 3-connected for k is even. It follows immediately that all $(4;g)$-cages are 3-connected. In this paper, we shall prove the following theorem:

Theorem 2.1. *All simple $(4;g)$-cages are 4-connected.*

The following lemmas will be used often. To prove Theorem 2.1, we assume the connectivity of a $(4;g)$-cage is 3. Let G be a $(4;g)$-cage with a vertex cut $X = \{x_1, x_2, x_3\}$.

Lemma 2.1. *Let u, v be two vertices of a $(4;g)$-graph G. If $d(u,v) < \lfloor g/2 \rfloor$,* *then there is only one vertex of $N(v)$, say v_1, such that $d(u, v_1) = d(u, v) - 1$ and $d(u, v_j) = d(u, v) + 1$ for $j = 2, 3, 4$.*

Proof. If there are two vertices of $N(v)$, say v_1 and v_2, such that $d(u, v_1) = d(u, v_2) = d(u, v) - 1$, then we have a smaller cycle with length at most $(d(u, v) - 1) \times 2 + 2 < 2 \times \lfloor g/2 \rfloor \leq g$, a contradiction. Hence, there is only one vertex v_1 of $N(v)$ such that $d(u, v_1) = d(u, v) - 1$.

If there is a vertex in $\{v_2, v_3, v_4\}$, say v_2, such that $d(u, v_2) = d(u, v)$, then we have a cycle of length at most $2 \times (\lfloor g/2 \rfloor - 1) + 1 < g$, a contradiction. This completes the proof. □

From Lemma 2.1, we know that the following equation holds if $d(u, v) = \lfloor g/2 \rfloor - 1$.

$$|D_{g/2-1}(u, N(v))| = |\{v_i \in N(v) \mid d(u, v_i) \leq g/2 - 1\}| = 1 \qquad (1)$$

Lemma 2.2. *Let X be a 3-vertex cut of a $(4; g)$-graph G where $g \geq 5$, and let ω be a component of $G - X$ and $H = G[X \cup V(\omega)] - E(G[X])$. Let $x \in X$, $y \in V(\omega)$ and $|N_H(x)| = i$. The following statements hold:*
(i) If g is even and $d_H(x, y) \geq g/2$, $|D_{g/2-1}(x, N_H(y))| \leq i$, and moreover $|D_{g/2}(x, N_H(y))| = 1$ if $i = 1$ and $d_H(x, y) = g/2$.
(ii) If g is odd and $d_H(x, y) \geq \lfloor g/2 \rfloor$, $|D_{\lfloor g/2 \rfloor}(x, N_H(y))| \leq i$ and $|D_{\lfloor g/2 \rfloor - 1}(x, N_H(y))| \leq 1$.

Proof. Since the proofs of the part (i) and part (ii) are very similar, we only give the proof for the part (i). It is easy to see that (i) is true for $d_H(x, y) \geq g/2 + 1$ or $i = 4$. Suppose $d_H(x, y) = g/2$ and $i \leq 3$. If (i) is false, there are at least $i + 1$ paths from x to $y_j \in N_H(y)$ of length at most $g/2 - 1$ where $1 \leq j \leq i + 1$. Since $|N_H(x)| = i$, then there are two paths, say P_{x,y_1} and P_{x,y_2}, having a common vertex in $x' \in N_H(x)$. It follows that $x' \to P_{x,y_1} \to y \to P_{x,y_2} \to x'$ is a smaller cycle of length at most $2 \times (g/2 - 2) + 2 = g - 2$, a contradiction. Hence, $|D_{g/2-1}(x, N_H(y))| \leq i$. If $|N_H(x)| = 1$ and $d_H(x, y) = g/2$, there is at most one vertex $y_1 \in N_H(y)$ with distance $g/2 - 1$ to x. Let x' be the vertex in $N_H(x)$ of distance $g/2 - 1$ to y. It follows that the distance between x' and $\{y_2, y_3, y_4\}$ is at least $g/2$. Hence, there is only one vertex in $N_H(y)$ of distance to x less than or equal to $g/2$. □

Lemma 2.3. *If G is a $(4; g)$-cage and has a 3-vertex cut $X = \{x_1, x_2, x_3\}$, it has the following properties:*
(i) For any induced subgraph H of G, $|E(H, \overline{H})| \equiv 0 (mod\ 2)$.
(ii) G is 4-edge-connected.
(iii) $G - X$ has at most three components.
(iv) $|E(G[X])| \leq 1$.
(v) $d = min\{d_H(x_i, x_j) \mid x_i$ and $x_j \in X\} \leq g - 2$ where ω is a component of $G - X$ and $H = G[V(\omega) \bigcup X] - E(G[X])$.

Proof. (i) Since for every induced subgraph H of G, we have $\sum_{v\in V(H)} d_H(v)$ $+|E(H,\overline{H})| = 4 \times |V(H)| \equiv 0(mod\ 2)$ and $\sum_{u\in V(H)} d_H(v) \equiv 0(mod\ 2)$. It follows that $|E(H,\overline{H})| \equiv 0(mod\ 2)$.
(ii) In [7], it has been proved that $(k;g)$-cages are 3-connected for k is even. It follows that G is 3-connected. Since G is 3-edge-connected and $|E(H,\overline{H})|$ is even, $|E(H,\overline{H})| = 4$. That is, G is 4-edge-connected.
(iii) Since G is a 4-regular graph, it follows from (ii) that $G - X$ has at most three components.
(iv) Let $G-X = \bigcup_i \omega_i$, $G_i = G[X \bigcup (V(\omega_i)| - E(G[X]))$ and $d_i = min\{d_{G_i}(x_{j_1}, x_{j_2}) \mid j_1 \neq j_2$ and $x_{j_1}, x_{j_2} \in X\}$ for $i = 1,2,3$.

$|E(G[X]) \leq 2$ since $|X| = 3$ and $g \geq 5$. If $|E(G[X])| = 2$, then $d_i \geq g-2$ otherwise there exists a smaller cycle in G. Furthermore, there are only two components in $G - X$ and there are four vertices in each component that are adjacent to the vertices in X since $g \geq 5$. Let $E(G[X]) = \{x_1x_2, x_2x_3\}$ and $|N_{G_2}(x_1)| = |N_{G_1}(x_3)| = 2$. Let $s_{i,j} \in N_{G_1}(x_i)$ and $t_{i,j} \in N_{G_2}(x_i)$ for $i = 1,2,3$.

$$G' = G - X + \{s_{1,1}t_{2,1}, s_{2,1}t_{3,1}, s_{3,1}t_{1,1}, s_{3,2}t_{1,2}\}$$

is still a 4-regular graph with girth at least g when $g \geq 5$. Clearly, G' has three less vertices than G, a contradiction with G being a $(4;g)$-cage.

(v) By (i) and a simple parity argument, one can easily verify that there is a vertex, say $x_1 \in X$, such that $|N_G(x_1) \bigcap V(\omega)| = 2$. First, we consider the case that $G - X$ has three components or there is a vertex, say x_1, such that $|N_G(x_1) \bigcap V(\omega_1)| = |N_G(x_1) \bigcap V(\omega_2)| = 2$. Suppose, on the contrary, $d \geq g-1$. We may assume that this minimum distance d occurs in a component where x_1 has two neighbours in it. We construct a new graph as follows,

$$G' = G - x_1 + \{s_it_i \mid s_i \in N_G(x_1) \cap V(\omega), t_i \in N_G(x_1) \cap (V(G) - V(\omega) - X), i = 1,2\}.$$

Clearly, it is still a 4-regular graph. If there is a smaller cycle in G', this smaller cycle must contain one of the new edges, say s_1t_1, and a vertex in $X - \{x_1\}$, say x_2. Since $d_H(x_1,x_2) \geq g-1$, $d_H(s_1,x_2) \geq g-2$. It follows that the length of this smaller cycle is at least g, a contradiction. Hence, $d \leq g-2$. Second, we consider the case that there is an edge x_1x_2 in X. Suppose, on the contrary, $d \geq g-1$. Without loss of generality, we may assume that it occurs in ω_1. By (i), we may assume that x_1 has two neighbours, say s_1 and s_2, in ω_1 and x_2 has two neighbours, t_1 and t_2, in ω_2. Let $s_3 = N_G(x_2) \bigcap \omega_1$ and $t_3 = N_G(x_1) \bigcap \omega_2$.

$$G' = G - x_1 - x_2 + \{s_it_i \mid i = 1,2,3\}.$$

Similarly, we can show that G' is a 4-regular graph and has girth at least g, a contradiction. Hence, $d \leq g-2$. This completes the proof of (v). □

Algorithm 2.1. *Let $X = \{x_1, x_2, x_3\}$ be a 3-vertex cut of $(4;g)$-cage G, ω is a component of $G-X$ and $H = G[V(\omega) \bigcup X] - E(G[X])$. Let $d = min\{d_H(x_i,x_j) \mid x_i, x_j \in X\}$. Assume $d_H(x_1,x_2) = d$. Let $m_1 \in V(\omega)$ such that $d_H(m_1,x_1) = \lfloor d/2 \rfloor$ and $d_H(m_1,x_2) = \lceil d/2 \rceil$.*

1. $i = 1$, $M_1 = \{m_1\}$, $X_1 = \{x \in X \mid d_H(x, m_1) = \lfloor d/2 \rfloor\} \bigcup \{x_1, x_2\}$.

2. If $d_H(m_i, x_1) \geq \lfloor g/2 \rfloor - 1$, *STOP.*

3. Consider the neighbour of m_i, $N_G(m_i) = \{m_i^1, m_i^2, m_i^3, m_i^4\}$.

Let $m_i^{j_0}$ *be a vertex in* $N_G(m_i)$ *such that* $d_H(m_i^{j_0}, x) = d_H(m_i, x) + 1$ *for all* $x \in X_1$*; and* $d_H(m_i^{j_0}, x) \geq d_H(m_i, x_1) + 1$ *for all* $x \in X \setminus X_i$.

Let $m_{i+1} = m_i^{j_0}$.

4. $i = i + 1$.

Let $X_i = X_i \bigcup \{x \in X \mid d_H(x, m_{i+1}) = d_H(x_1, m_{i+1})\}$, $M_i = M_i \bigcup \{m_{i+1}\}$.
Go to 2.

Obviously, $d \geq 2$ since the induced subgraph $H[X]$ of H contains no edge, which means that $|N_H(m_i)| = 4$ for all i in Algorithm 2.1. Suppose that $d_H(m_1^1, x_1) = d_H(m_1, x_1) - 1$ and $d_H(m_1^1, x_2) = d_H(m_1, x_2) - 1$ If $d_H(m_i, x_1) < g/2 - 2$ (g is even) or $d_H(m_i, x_1) < \lfloor g/2 \rfloor - 1$ (g is odd), $d_H(m_i^j, x) = d_H(m_i, x) + 1$ for all $j \in \{3, 4\}$ and $x \in X_i$. By Lemma 2.1, there is at most one vertex $m_i^j \in \{m_i^3, m_i^3\}$ such that $d_H(x_3, m_i^j) \leq d_H(x_1, m_i)$. In other words, there exists a vertex $m_i^{j_0}$ such that $d_H(m_i^{j_0}, x) = d_H(m_i, x) + 1$ for all $x \in X_i$ and $d_H(m_i^{j_0}, x) \geq d_H(m_i, x_1) + 1$. By (v) of Lemma 2.3, Algorithm 2.1 will finish in finite steps. At the end, Algorithm 2.1 will find a vertex, u, such that $d_H(u, x_1) = \lfloor g/2 \rfloor - 1, d_H(u, x_2) = \lceil g/2 \rceil - 1$ and $d_H(u, x_3) \geq \lfloor g/2 \rfloor - 1$.

Corollary 2.1. *If* g *is odd,* $d = d_H(x_1, x_2) \leq g - 3$ *and* $|N_G(x_2) \bigcap V(\omega)| \leq 2$, *we can find a vertex* $u = m_{l+1} \in N_H(m_l)$ *such that* $d_H(u, x) \geq \lfloor g/2 \rfloor$ *for all* $x \in X$ *and* $d_H(u, x_2) \geq \lceil g/2 \rceil$.

Proof. Suppose that at the end of Algorithm 2.1, find a vertex, m_l, such that $d_H(m_l, x_1) = \lfloor g/2 \rfloor - 1, d_H(m_l, x_2) - \lceil g/2 \rceil - 1$ and $d_H(m_l, x_3) \geq \lfloor g/2 \rfloor - 1$.

Obviously, $m_l \neq m_1$ since $d \leq g - 3$. For each vertex $m_l^j \in \{m_l^2, m_l^3, m_l^4\}$, $d_H(m_l^j, x) \geq \lfloor g/2 \rfloor$ for $x = x_1$ or $x = x_2$ otherwise there is a smaller cycle of length $2\lfloor g/2 \rfloor$. By part (ii) of Lemma 2.2, we have $|D_{\lfloor g/2 \rfloor}(x_2, N_H(u))| \leq 2$ since $|N_G(x_2) \cap V(\omega)| \leq 2$. This plus $d_H(m_{l-1}, x_2) = \lfloor g/2 \rfloor - 1$ implies that there exist two vertices in $\{m_l^2, m_l^3, m_l^4\}$, say m_3 and m_4, such that $d_H(m_l^j, x_2) \geq \lceil g/2 \rceil$ for $j = 3, 4$. For x_3, there is at most one vertex in $\{m_l^3, m_l^4\}$ of distance less that or equal to $\lfloor g/2 \rfloor - 1$ otherwise there is a smaller cycle in G. Hence, there exists one vertex $u \in \{m_l^3, m_l^4\}$ that satisfies this corollary. □

3 The structural properties of the components of $G - X$

From the part (i) of Lemma 2.3, we know that $|E(X, \omega_i)|$ is even and $4 \leq |E(X, \omega_i)| \leq 8$ where ω_i is a component of $G - X$. We will focus on finding a furthest away vertex from X in $\omega = \omega_i$ according to the number of edges in $E(X, \omega)$. In the following, let $H = G[V(\omega) \bigcup X] - E(G[X])$, $d = min\{d_H(x_i, x_j) \mid x_i, x_j \in X\}$.

3.1 $|E(X,\omega)| = 4$

Let $|N_G(x_1) \bigcap V(\omega)| = 2$ and $|N_G(x_2) \bigcap V(\omega)| = |N_G(x_3) \bigcap V(\omega)| = 1$.

Lemma 3.1. *There exists $u \in V(H)$ such that $d_H(u,X) \geq \lfloor g/2 \rfloor$. In particular, $d_H(u,x_2) \geq g/2+1$ or $d_H(u,x_3) \geq g/2+1$ if g is even, and $u \in V(H)$ such that $d_H(u,x) \geq \lceil g/2 \rceil$ for all $x \in X$ if g is odd.*

Proof. We distinguish two cases according to the parity of g.

Case 1 g is even

Subcase 1.1 $d = d_H(x_1,x_2)$ or $d = d_H(x_1,x_3)$

Without loss of generality, we may assume that $d = d_H(x_1,x_2)$. Since $d \leq g-2$ (Lemma 2.3), $x_1x_2 \notin E(G)$ otherwise there is a smaller cycle containing x_1x_2 in G. In this subcase, we consider the case that d is odd. For d is even, the proof is similar.

From Algorithm 2.1, we can find a vertex $m_l \in H$ such that $d_H(m_l,x_1) = g/2-1, d_H(m_l,x_2) = g/2$, $d_H(m_l,x_3) \geq g/2-1$. Let $N_H(m_l) = \{m_l^1 = m_{l-1}, m_l^2, m_l^3, m_l^4\}$. Obviously $m_l \neq m_1$. From Lemma 2.1, we know that $d_H(m_l^j,x_1) = g/2$ for $j = 2,3,4$. Since $d_H(m_l,x_2) = g/2$ and $|N_H(x_2)| = 1$, we have $|D_{g/2}(x_2,N_H(m_l))| = 1$ from the part (i) of Lemma 2.2. This implies that $d_H(m_l^j,x_2) = g/2+1$ for $j = 2,3,4$. Since $d_H(m_l,x_3) \geq g/2-1$, we know that $|D_{g/2-1}(x_3,N_H(m_l))| = 1$ from (1) and part (ii) of Lemma 2.2. Thus, there exists a vertex $m_l^{jo} \in \{m_l^2,m_l^3,m_l^4\}$ such that $d_H(m_l^{jo},x_3) \geq g/2$. Clearly, $u = m_l^{jo}$ satisfies Lemma 3.1 if g is even. If m_l^{jo} also satisfies $d_H(m_l^{jo},x_3) \geq g/2+1$, then $u = m_l^{jo}$ satisfies the distance requirement of the lemma for x_2. Otherwise, we have $d_H(m_l^{jo},x_3) = g/2$. Let $m_{l+1} = m_l^{jo}$, consider $N_H(m_{l+1}) = \{m_{l+1}^1 = m_l, m_{l+1}^2, m_{l+1}^3, m_{l+1}^4\}$. From $d_H(m_{l+1},x_2) = g/2+1$, we have $d_H(m_{l+1}^i,x_2) \geq g/2$, for $i = 2,3,4$. From the part (i) of Lemma 2.2, we have $|D_{g/2}(x_3,N_H(m_{l+1}))| = 1$, and $|D_{g/2-1}(x_1,N_H(m_{l+1}))| \leq 2$. Since $d_H(m_l,x_1) = g/2-1$, thus one vertex $u \in \{m_{l+1}^2, m_{l+1}^3, m_{l+1}^4\}$ satisfies $d_H(u,X) \geq g/2$ and $d_H(u,x_3) \geq g/2+1$.

Subcase 1.2 $d = d_H(x_2,x_3)$

Suppose d is odd and $d \leq g-3$. By Algorithm 2.1, we can find a vertex $m_l \in H$ such that $d_H(m_l,x_2) = g/2-1, d_H(m_l,x_3) = g/2$. $N_G(m_l) = \{m_l^1 = m_{l-1}, m_l^2, m_l^3, m_l^4\}$. Obviously, $m_l \neq m_1$. Since $|N_H(x_2)| = |N_H(x_3)| = 1$, we have $d_H(m_l^j,x_2) = g/2$, $d_H(m_l^j,x_3) = g/2+1$, for $j = 2,3,4$ by (1) and the part (i) of Lemma 2.2. Note that $d_H(m_l,x_1) \geq g/2-1$. From (1) and the part (i) of Lemma 2.2, we have $|D_{g/2-1}(x_1,N_H(m_l))| \leq 2$. Thus, we can find a vertex m_l^{jo} such that $d_H(m_l^{jo},x_1) = g/2, d_H(m_l^{jo},x_2) = g/2$, and $d_H(m_l^{jo},x_3) \geq g/2+1$. Let $u = m_l^{jo}$. If we take $m_l \in H$ such that $d_H(m_l,x_2) = g/2, d_H(m_l,x_3) = g/2-1$, similarly as above, we get a vertex u satisfying $d_H(u,X) \geq g/2$ and $d_H(u,x_2) \geq g/2+1$. Similarly, we can prove the result when d is even and $d \leq g-3$.

By Lemma 2.3, we know that $d \leq g-2$. Hence, it suffices to show the lemma for $d = g-2$. Clearly, d is even since g is even. Let $m_1 \in V(\omega)$ such that $d_H(m_1,x_2) = g/2-1$ and $d_H(m_1,x_3) = g/2-1$. It follows that

$d_H(m_1, x_1) \geq g/2$ otherwise $d_H(x_1, x_2) \leq g-2$ and one can use the argument in **Subcase 1.1** to prove the result. Now consider $N_H(m_1) = \{m_l^1, \cdots, m_l^4\}$. Assume that $d_H(m_1^1, x_2) = g/2-2$ and $d_H(m_1^2, x_3) = g/2-2$. Let $m_2 = m_1^3$. From Lemma 2.1, $d_H(m_2, x_j) = g/2$ for $j = 2,3$ and $d_H(m_2, x_1) \geq g/2-1$. Again, we consider the neighbours of m_2, that is, $N_H(m_2) = \{m_2^1 = m_1, \cdots, m_2^4\}$. Note that $|N_H(x_2)| = |N_H(x_3)| = 1$. From the part (i) of Lemma 2.2, $d_H(m_2^j. x_2) = d_H(m_2^j, x_3) = g/2+1$ for $j = 2,3,4$. From (1) and the part (i) of Lemma 2.2, we have $|D_{g/2-1}(x_1, N_H(m_2))| \leq 2$. This implies that there exists a vertex $u = m_2^{j_0}$ such that $d_H(x_1, u) \geq g/2$.

Case 2 g is odd

Again, we only consider the case that d is odd. A similar argument can be used to prove the case that d is even.

Subcase 2.1 $d = d_H(x_1, x_2)$ or $d = d_H(x_1, x_3)$

Without loss of generality, we may assume that $d = d_H(x_1, x_2)$. Let $M_{l-1} = \{m_1, \cdots, m_{l-1}\}$ be the set of vertices as defined in Algorithm 2.1 such that $d_H(m_{l-1}, x_1) = \lfloor g/2 \rfloor - 1, d_H(m_{l-1}, x_2) = \lfloor g/2 \rfloor$ and $d_H(m_{l-1}, x_3) \geq \lfloor g/2 \rfloor - 1$. Since $|N_H(x_1)| = 2$ and $|N_H(x_i)| = 1 (i = 2,3)$, we have $|D_{\lfloor g/2 \rfloor -1}(x_1, N_H(m_{l-1}))| \leq 1$, $|D_{\lfloor g/2 \rfloor}(x_2, N_H(m_{l-1}))| \leq 1$ and $|D_{\lfloor g/2 \rfloor -1}(x_3, N_H(m_{l-1}))| \leq 1$ by (1) and part (ii) of Lemma 2.2. Thus, there exists a vertex, say m_l, in $N_H(m_{l-1})$, such that $d_H(m_l, x_1) = \lfloor g/2 \rfloor. d_H(m_l, x_2) = \lceil g/2 \rceil$ and $d_H(m_l, x_3) \geq \lfloor g/2 \rfloor$.

Now consider $N_H(m_l) = \{m_l^1 = m_l - 1, m_l^2, \cdots, m_l^4\}$. From part (ii) of lemma 2.2, we have $|D_{\lfloor g/2 \rfloor}(x_1, N_H(m_l))| \leq 2$ and $|D_{\lfloor g/2 \rfloor}(x_2, N_H(m_l))| \leq 1$. Since $d_H(m_{l-1}, x_1) = \lfloor g/2 \rfloor - 1$ and $d_H(m_{l-1}, x_2) = \lfloor g/2 \rfloor$, there must be two vertices in $\{m_l^2, m_l^3, m_l^4\}$, say m_l^3 and m_l^4, that have distance at least $\lceil g/2 \rceil$ to both x_1 and x_2. Since $|D_{\lfloor g/2 \rfloor}(x_3, N_H(m_l))| \leq 1$, there exists a vertex $u \in \{m_l^3, m_l^4\}$ such that $d_H(u, x_3) \geq \lceil g/2 \rceil$. Hence, $d_H(u, x) \geq \lceil g/2 \rceil$ for all $x \in X$.

Subcase 2.2 $d = d_H(x_2, x_3)$

First, we prove this lemma for $d \leq g-3$. Similar to **Subcase 2.1** we can find $M_l = \{m_1, \cdots, m_l\} (m_l \neq m_1)$ such that $d_H(m_l, x_2) = \lfloor g/2 \rfloor, d_H(m_l, x_3) = \lceil g/2 \rceil$ and $d_H(m_l, x_1) \geq \lfloor g/2 \rfloor$ by Corollary 2.1. Since $|N_H(x_2)| = |N_H(x_3)| = 1$, $d_H(m_l^j, x_2) = \lceil g/2 \rceil$ and $d_H(m_l^j, x_3) = \lceil g/2 \rceil$ for $j = 2,3,4$. By part (ii) of Lemma 2.3, we have $|D_{\lfloor g/2 \rfloor}(x_1, N_H(m_l))| \leq 2$. There exists a vertex of $m_l^{j_0}$ such that $d_H(m_l^{j_0}, x) \geq \lceil g/2 \rceil$ for all $x \in X$ and $j = 2,3,4$.

Now we consider the case that $d = g-2$. Clearly, d is odd since g is odd. Let $m_1 \in V(\omega)$ such that $d_H(m_1, x_2) = \lfloor g/2 \rfloor - 1$ and $d_H(m_1, x_3) = \lfloor g/2 \rfloor$. It follows that $d_H(m_1, x_1) \geq \lfloor g/2 \rfloor$ otherwise $d_H(x_1, x_2) \leq g-3 < d$. From part (ii) of Lemma 2.2, we know that $|D_{\lfloor g/2 \rfloor -1}(x_1, N_H(m_1))| \leq 1$. Let $d_H(m_1^1. x_2) = \lfloor g/2 \rfloor - 2$ and $d_H(m_1^2, x_3) = \lfloor g/2 \rfloor - 1$. Consider $\{m_1^3, m_l^4\}$. From Lemma 2.1 and part (ii) of Lemma 2.2, $d_H(m_1^j, x_2) = \lfloor g/2 \rfloor$ and $d_H(m_1^j, x_3) = \lceil g/2 \rceil$ for $j = 3,4$. Hence, there exists a vertex, say $m_2 \in \{m_1^3, m_l^4\}$, such that $d_H(m_2, x_1) \geq \lfloor g/2 \rfloor, d_H(m_2, x_2) = \lfloor g/2 \rfloor$ and $d_H(m_2, x_3) = \lceil g/2 \rceil$. Now we consider $N_H(m_2) = \{m_2^1 = m_1, m_2^2, m_2^3, m_2^4\}$. From part (ii) of Lemma 2.2, $d_H(m_2^j, x_i) \geq \lceil g/2 \rceil$ for $j = 2,3,4$ and $i = 2,3$. From part (ii) of Lemma 2.2, we know that $|D_{\lfloor g/2 \rfloor}(x_1, N_H(m_2))| \leq 2$ since $|N_H(x_1)| = 2$. Hence, there exists a vertex $m_2^{j_0} \in \{m_2^2, m_2^3, m_2^4\}$ such that $d_H(x_1, m_2^{j_0}) = \lceil g/2 \rceil$. This completes the proof. □

3.2 $|E(X,\omega)| = 8$

Since G is 4-edge-connected, $|E(G[X])| = 0$. Without loss of generality, we assume that $|N_G(x_1) \bigcap V(\omega)| = 2$ and $|N_G(x_i) \cap V(\omega)| = 3$ for $i = 2,3$.

Lemma 3.2. *There exists $u \in V(H)$ such that $d_H(u,x) \geq \lfloor g/2 \rfloor$ for all $x \in X$ and $d_H(u,x_1) \geq \lceil g/2 \rceil$ if g is odd, and $d_H(u,x_1) \geq g/2, d_H(u,x_2) \geq g/2$ and $d_H(y_1,x_3) \geq g/2 - 1$ or $d_H(u,x_1) \geq g/2, d_H(u,x_2) \geq g/2 - 1$ and $d_H(y_1,x_3) \geq g/2$.*

Proof. Suppose $N_G(x_i) \bigcap V(\overline{\omega}) = x_i'(i = 2,3)$. It follows that $x_2' \neq x_3'$ otherwise the connectivity of G will be 2, a contradiction G being 3 connected. Let $X' = X - x_2 - x_3 + x_2' + x_3'$ and $\omega' = G[\{x_2,x_3\} \bigcup V(\omega)]$. It follows that $|E(X', V(\omega'))| = 4$. Note that conclusions on the furthest away vertex in Lemma 3.1 implies this lemma. □

3.3 $|E(X,\omega)| = 6$

Case 1 $|N_G(x) \bigcap V(\omega)| = 2$ for all $x \in X$

The proofs of the next three lemmas are similar with the proof of Lemma 3.1. We omit the proofs from this paper and will put them on the second author's home page .

Lemma 3.3. *If $d = g - 2$, then $E(G[X]) \neq \emptyset$.*

Lemma 3.4. *If $d = g-2$ and $|E(G[X])| = 1$ (assume that $x_ix_j \in E(G)$), there exists a vertex $u \in V(\omega)$ such that $d_H(u,x_i) \geq g/2, d_H(u,x_j) \geq g/2 - 1$ and $d_H(u,x_k) \geq g/2$, or $d_H(u,x_i) \geq g/2 - 1, d_H(u,x_j) \geq g/2$ and $d_H(u,x_k) \geq g/2$ for $x_k \in X - \{x_i,x_j\}$ and g even. And $d_H(u,x_i) \geq \lfloor g/2 \rfloor$ for $i = 1,2,3$ and g is odd.*

Lemma 3.5. *If $d \leq g - 3$, there is a vertex $u \in \omega$ such that $d_H(u,X) \geq g/2$ for g is even, and there is a vertex $u \in \omega$ such that $d_H(u,X) \geq \lfloor g/2 \rfloor$ and $|\{x \in X \mid d_H(u,x) \geq \lceil g/2 \rceil\}| \geq 2$ for g is odd. Moreover, there is $u \in V(H)$ for each given $x \in X$ such that $d_H(u,X) \geq \lfloor g/2 \rfloor$ and $d_H(u,x) \geq \lceil g/2 \rceil$ if g is odd.*

Case 2: $|N_G(x_i) \cap V(\omega)| = 3$ for some vertex $x_i \in X$

Without loss of generality, we may assume that $N_{\omega_1}(x_1) = 3$. It follows that $N_{\omega_2}(x_1) = 1$. Let $N_{\omega_2}(x_1) = x_1'$. This implies that $X' = \{x_1', x_2, x_3\}$ is also a 3-cut of G and $|E(X',\omega')| = 4$, where $\omega' = G[V(\omega) \bigcup \{x_1\}]$. By Lemma 3.1, we know that there is a vertex $u \in V(\omega')$ such that $d_H(u,x) \geq g/2$ for all $x \in X'$ for g is even and $d_H(u,x_1) \geq \lceil g/2 \rceil, d_H(u,x_2) \geq \lceil g/2 \rceil$ and $d_H(u,x_3) \geq \lfloor g/2 \rfloor$ for g is odd.

Lemma 3.6. *If $|N_G(x) \cap V(\omega)| = 3$ for some vertex $x' \in X$, there is a vertex $u \in V(\omega)$ such that $d_H(u, x) \geq g/2$ for all $x \in X$ if g is even, and $d_H(y_1, x') \geq \lfloor g/2 \rfloor$ and $d_H(u, x) \geq \lceil g/2 \rceil$ if g is odd and $x \in X \setminus \{x'\}$.*

4 Proof of Theorem 2.1

Now we give an operation that can be used on a $(4; g)$-cage G to construct a new 4-regular graph with girth at least g. In the other words, G cannot be a cage if one can apply the Half Split Operation on it.

Half Split Operation
Let G be a $(4; g)$-cage, $g \geq 5$, and u, v be two vertices in G of distance $d(u, v) \geq g$. Let $u_i \in N_G(u)$ and $v_i \in N_G(v)$ for $i = 1, 2, 3, 4$. If $d_{G-u-v}(u_i, v_i) \geq g-1$ for $i = 1, 2$, we construct a new graph $G' = G - u - v + w + \{wu_3, wu_4, wv_3, wv_4\} + \{u_1v_1, u_2v_2\}$, where w is a new vertex. Then G' is a 4-regular graph and has girth at least g.

Clearly, G' is a 4-regular graph. In the following, we show that G' also has girth at least g. Let C be a smallest cycle of G'. Suppose $|C| \leq g - 1$. Since G has girth g, C must contain w or one of edges in $\{u_1v_1, u_2v_2\}$. If C contains w and none of $\{u_1v_1, u_2v_2\}$, then it contains exactly two edges of $\{wu_3, wu_4, wv_3, wv_4\}$. Without loss of generality, we assume that C contains $\{wu_i, wv_j\}$ $(i, j = 3, 4)$. $|C| \geq g$ since $d_G(u_i, v_j) \geq g - 2$. If C contains exactly one of the edges from $\{u_1v_1, u_2v_2\}$, say u_1v_1, and contains no w, then $|C| \geq g$ since $d_G(u_1, v_1) \geq g - 1$. If C contains two edges $\{u_1v_1, u_2v_2\}$ and contains no w, $|C| \geq g$ since $d_{G-u-v}(u_1, u_2) \geq g - 1$, $d_{G-u-v}(v_1, v_2) \geq g - 1$ and $d_G(u_i, v_j) \geq g - 2 (i \neq j)$. If C contains w and at least one edge of $\{u_1v_1, u_2v_2\}$, then $|C| \geq g$ since $d_G(x, y) \geq g-2$ for $x \in \{u_3, u_4, v_3, v_4\}$ and $y \in \{u_1, u_2, v_1, v_2\}$ and $d_{G-u-v}(u_1, u_2) \geq g - 1$. Hence, $|C| \geq g$.

Let $X = \{x_1, x_2, x_3\}$ be a minimum 3-cuts of $(4; g)$-graph of G, let $\cup_i \omega_i$ be the components of $G - X$. $G_i = G[X \cup V(\omega_i)] - E(G[X])$, $d_i = min\{d_{G_i}(x_i, x_j) \mid i \neq j, x_i, x_j \in X\}$. From Lemma 2.3, we know that $|E(X, \omega_i)|$ is even and $4 \leq |E(X, \omega_i)| \leq 8$, $|E(G[X])| \leq 1$ and $d_i \leq g - 2$.

Proof of Theorem 2.1 We distinguish three cases according to $|E(X, \omega)|$.

Case 1: $|E(G[X])| = 1$, $|E(X, V(\omega_1))| = 4$ and $|E(X, V(\omega_2))| = 6$

Without loss of generality, we may assume that $|N_G(x_1) \cap V(\omega_1)| = 2$, $|N_G(x_2) \cap V(\omega_1)| = 1$ and $|N_G(x_3) \cap V(\omega_1)| = 1$.

We first consider the case that g is even. Since $|E(X, V(\omega_2))| = 6$, suppose that there exists a vertex $u_2 \in V(\omega_2)$ such that $d_{G_2}(u_2, x_1) \geq g/2$, $d_{G_2}(u_2, x_2) \geq g/2$,and $d_{G_2}(u_2, x_3) \geq g/2 - 1$ by Lemma 3.4, Lemma 3.5 and Lemma 3.6. By Lemma 3.1, there exists a vertex $u_1 \in V(\omega_1)$ such that $d_{G_1}(u_1, x) \geq g/2$ for all $x \in X$ and $d_{G_1}(u_1, x_3) \geq g/2 + 1$. Now consider x_3. $d_G(u_1^i, x_3) \geq g/2$ for all $u_1^i \in N_{G_1}(u_1)$. Since $|N_G(x_1) \cap V(\omega_1)| = 2$ and $|N_G(x_2) \cap V(\omega_1)| = 1$,

$|D_{g/2-1}(x_1, N_{G_1}(u_1))| \leq 2$ and $|D_{g/2-1}(x_2, N_{G_1}(u_1))| \leq 1$. This implies that we can find two vertices, u_1^1 and u_1^2, from $N_{G_1}(u_1)$ such that $d_{G_1}(u_1^1, x) \geq g/2$ for all $x \in X$, and $d_{G_1}(u_1^2, x_1) \geq g/2 - 1, d_{G_1}(u_1^2, x_2) \geq g/2$ and $d_{G_1}(u_1^2, x_3) \geq g/2$. Let $U_2 = \{u \in N_G(u_2) \mid d_{G_2}(u, x_3) \geq g/2 - 1\}$. Since $d_H(u_1, x_3) \geq g/2 - 1$, $|D_{g/2-2}(x_3, N_{G_2}(u_2))| \leq 1$ by Lemma 2.1. Hence, $|U_2| \geq 3$. Since $|N_G(x_1) \bigcap V(\omega_2)| \leq 2$, $|D_{g/2-1}(x_1, N_{G_2}(u_2))| \leq 2$ by Lemma 2.2. It follows that one of the vertex in U_2, say u_2^2, satisfies $d_{G_2}(u_2^2, x_1) \geq g/2, d_{G_1}(u_1^2, x_2) \geq g/2 - 1$ and $d_{G_1}(u_2^2, x_3) \geq g/2 - 1$. Select one vertex from $U_2 \setminus \{u_2^2\}$, say u_2^1. It follows that $d_{G_2}(u_2^1, x) \geq g/2 - 1$ for all $x \in X$. Hence, $d_G(u_1, u_2) \geq g$ and $d_G(u_1^i, u_2^i) \geq g - 1$ for $i = 1, 2$. Clearly, we can use the Half Split Operation on G, a contradiction.

Now we consider the case that g is odd. By Lemma 3.1, there exists a vertex u_1 in ω_1 such that $d_{G_1}(u_1, x) \geq \lceil g/2 \rceil$ for all $x \in X$. By Lemma 3.4, Lemma 3.5 and Lemma 3.6, there exists a vertex u_2 in ω_2 such that $d_{G_2}(u_2, x) \geq \lfloor g/2 \rfloor$ for all $x \in X$. Since $d_{G_1}(u_1, x) \geq \lceil g/2 \rceil$, $d_{G_1}(u_1^i, x) \geq \lfloor g/2 \rfloor$ for all $x \in X$ and $u_1^i \in N_{G_1}(u_1)$. Since $|N_G(x_i) \bigcap V(\omega_1)| = 1$ for $i = 2, 3$, $|D_{\lfloor g/2 \rfloor}(x_i, N_{G_1}(u_1))| \leq 1$ for $i = 2, 3$ by Lemma 2.2. Thus, we can find two vertices from $N_{G_1}(u_1)$, say u_1^1 and u_1^2, such that $d_H(x_i, u_1^j) \geq \lceil g/2 \rceil$ for $i = 2, 3$ and $d_H(x_1, u_1^j) \geq \lfloor g/2 \rfloor$ for $j = 1, 2$. Since $|E(X, V(\omega_2))| = 6$, and $|N_G(x_1) \bigcap V(\omega_2)| \leq 2$ and $d_{G_2}(u_2, x) \geq \lfloor g/2 \rfloor$ for all $x \in X$, $|D_{\lfloor g/2 \rfloor - 1}(x_1, N_{G_2}(u_2))| \leq 1$. This implies that we can find two vertices from $N_{\omega_2}(u_2)$, say u_2^1 and u_2^2, such that $d_{G_2}(x_i, u_1^j) \geq \lfloor g/2 \rfloor - 1$ for $i = 2, 3$ and $d_{G_2}(x_1, u_1^j) \geq \lfloor g/2 \rfloor$ for $j = 1, 2$. Hence, $d_G(u_1, u_2) \geq g$ and $d_G(u_1^i, u_2^i) \geq g - 1$ for $i = 1, 2$. Clearly, we can apply the Half Split Operation on G, a contradiction.

Case 2: $|E(G[X])| = 0$, $|E(X, V(\omega_1))| = 4$ and $|E(X, V(\omega_2))| = 8$

Without loss of generality, we may assume that $|N_G(x_1) \bigcap V(\omega_1)| = 2$, $|N_G(x_2) \bigcap V(\omega_1)| = 1$ and $|N_G(x_3) \bigcap V(\omega_1)| = 1$. It follows that $|N_G(x_1) \bigcap V(\omega_2)| = 2$, $|N_G(x_2) \bigcap V(\omega_2)| = 3$ and $|N_G(x_3) \bigcap V(\omega_2)| = 3$.

If g is even, from Lemma 3.2, suppose that there exists a vertex $u_2 \in \omega_2$ such that $d_{G_2}(u_2, x_1) \geq g/2$, $d_{G_2}(u_2, x_2) \geq g/2$, and $d_{G_2}(u_2, x_3) \geq g/2 - 1$. By Lemma 3.1, there exists a vertex $u_1 \in \omega_1$ such that $d_{G_1}(u_1, x_1) = g/2$, $d_{G_1}(u_1, x_2) \geq g/2$, and $d_{G_1}(y_1, x_3) \geq g/2 + 1$.

If g is odd, there exists a vertex $u_1 \in \omega_1$ such that $d_{G_1}(u_1, x) \geq \lceil g/2 \rceil$ for all $x \in X$ by Lemma 3.1. By Lemma 3.2, there exists a vertex $u_2 \in \omega_2$ such that $d_{G_2}(u_1, x) \geq \lfloor g/2 \rfloor$ for all $x \in X$.

Similarly to case 1, we can apply the Half Split Operation on G, a contradiction.

Case 3: $|E(G[X])| = 0$, $|E(X, V(\omega_1))| = 6$ and $|E(X, V(\omega_2))| = 6$

If $|N_G(x_i) \cap V(\omega_j)| = 3$, for some $x_i \in X$ and ω_j, which can be reduced to Case 1 by the discussion of Lemma 3.6. Thus, we may assume that $|N_G(x) \bigcap V(\omega_i)| = 2$ for all $x \in X$. By Lemma 3.3, we have $d_i \leq g - 3$ where $i = 1, 2$.

If g is even, there exists a vertex $u_1 \in \omega_1$ such that $d_{G_1}(u_1, x) \geq g/2$ for all $x \in X$ and exists a vertex $u_2 \in \omega_2$ such that $d_{G_2}(u_2, x) \geq g/2$ for all $x \in X$. Let $N_{\omega_1}(u_1) = \{u_1^1, u_1^2, u_1^3, u_1^4\}$ and $N_{\omega_2}(u_2) = \{u_2^1, u_2^2, u_2^3, u_2^4\}$. Since $d_{G_1}(u_1, x) \geq g/2$ for all $x \in X$, $|D_{g/2-1}(x_i, N_{G_1}(u_1))| \leq 2$ for $i = 1, 2, 3$ by Lemma 2.2. If there is a vertex $u_0 \in V(G_1)$ such that $d_{G_1}(u_0, x) \geq g/2 + 1$ for all $x \in X$, then $d_G(u_0, u_2) = g + 1$. Clearly, the following newly constructed graph G' is a 4-regular graph with girth at least g, a contradiction.

$$G' = G - u_0 - u_2 + \{s_i t_i | s_i \in N_G(u_0), t_i \in N_G(u_1), i = 1, \cdots, 4\}$$

This implies that there exist two vertices in $N_{G_1}(u_1)$, say u_1^1 and u_1^2, such that $|\{x \in X | d_{G_1}(u_1^i, x) = g/2 - 1\}| \leq 1$ for $i = 1, 2$ since $|X| = 3$ and $N_G(u_1) = 4$. We consider two subcases here. First, there is one vertex, say x_1, of X such that $d_{G_1}(u_1^1, x_1) = g/2 - 1$ and $d_{G_1}(u_1^2, x_1) = g/2 - 1$. It follows that $d_{G_1}(y_1^i, x_j) \geq g/2$ for $i = 1, 2$ and $j = 2, 3$. In G_2, there exist two vertices in $N_{G_2}(u_2)$, say u_2^1 and u_2^2, such that $d_{G_2}(u_2^i, x_1) \geq g/2$ for $i = 1, 2$ and $d_{G_2}(u_2^i, x_j) \geq g/2 - 1$ for $i = 1, 2$ and $j = 2, 3$ by Lemma 2.2. Hence, $d_G(u_1^i, u_2^i) = g - 1$ for $i = 1, 2$ and $d_G(u_1, u_2) = g$. We can apply the Half Split Operation on G, a contradiction. Second, there are two distinct vertices, say x_1 and x_2, such that $d_H(u_1^1, x_1) = g/2 - 1$ and $d_H(u_1^2, x_2) = g/2 - 1$. If follows that $d_H(u_1^i, x_j) \geq g/2$ for $i = 1, 2$, $i \neq j$ and $x_j \in X$. In G_2, we can find two sets of vertices from $N_{G_2}(u_2)$, say $U_1 = \{u_2^{i_1}, u_2^{i_2}\}$ such that $d_{G_2}(u, x_1) \geq g/2$ ($u \in U_1$) and $U_2 = \{u_2^{j_1}, u_2^{j_2}\}$ such that $d_{G_2}(u, x_2) \geq g/2(u \in U_2)$ by Lemma 2.2. If $U_1 = U_2$, $d_G(u', u'') \geq g - 1$ for $u' \in \{u_1^1, u_1^2\}$ and $u'' \in U_1$. Otherwise, let $u_2^{i_1} \in U_1 \setminus U_2$ and $u_2^{j_1} \in U_2 \setminus U_1$. It follows that $d_G(u_1^1, u_2^{i_1}) \geq g - 1$ and $d_G(u_1^2, u_2^{j_1}) \geq g - 1$. We can apply the Half Split Operation on G, a contradiction.

If g is odd, by Lemma 3.5, suppose we can find a vertex $u_2 \in \omega_2$ such that $d_{G_2}(u_2, x_1) \geq \lceil g/2 \rceil$, $d_{G_2}(u_2, x_2) \geq \lceil g/2 \rceil$ and $d_{G_2}(u_2, x_3) \geq \lfloor g/2 \rfloor$. Again by Lemma 3.5, we can find a vertex $u_1 \in \omega_1$ such that $d_{G_1}(u_1, x_1) \geq \lfloor g/2 \rfloor$, $d_{G_1}(u_1, x_2) \geq \lfloor g/2 \rfloor$ and $d_{G_1}(u_1, x_3) \geq \lceil g/2 \rceil$. Let $N_{\omega_1}(u_1) = \{u_1^1, u_1^2, u_1^3, u_1^4\}$ and $N_{\omega_2}(u_2) = \{u_2^1, u_2^2, u_2^3, u_2^4\}$. In G_1, $d_{G_1}(u, x_3) \geq \lfloor g/2 \rfloor$ for all $u \in N_{G_1}(u_1)$ since $d_{G_1}(u_1, x_3) \geq \lceil g/2 \rceil$. Since $d_{G_1}(u_1, x_i) \geq \lfloor g/2 \rfloor$ for $i = 1, 2$, $|D_{\lfloor g/2 \rfloor - 1}(x_1, N_{G_1}(u_1))| \leq 1$ and $|D_{\lfloor g/2 \rfloor - 1}(x_2, N_{G_1}(u_1))| \leq 1$ by Lemma 2.1. Hence, there exists two vertices in $N_{G_1}(u_1)$, say u_1^1 and u_1^2, such that $d_{G_1}(u_1^i, x) \geq \lfloor g/2 \rfloor$ for $i = 1, 2$ and for all $x \in X$. In G_2, since $d_{G_2}(u_2, x_i) \geq \lfloor g/2 \rfloor$ for $i = 1, 2$, so $d_{G_2}(u_2^j, x_i) \geq \lfloor g/2 \rfloor$ for all $j = 1, \cdots, 4$ and $i = 1, 2$. Since $d_{G_2}(u_2, x_3) \geq \lfloor g/2 \rfloor$, $|D_{\lfloor g/2 \rfloor - 1}(x_3, N_{G_2}(u_2))| \leq 1$ by Lemma 2.2. Hence, there exist two vertices in $N_{G_2}(u_2)$, say u_2^1 and u_2^2, such that $d_{G_2}(u_2^i, x) \geq \lfloor g/2 \rfloor$ for all $x \in X$ and $i = 1, 2$. It follows that $d_G(u_1^1, u_2^1) \geq g - 1$ and $d_G(u_1^2, u_2^2) \geq g - 1$. We now can apply the Half Split Operation on G, a contradiction. This completes the proof of Theorem 2.1. □

References

[1] H.L. Fu, K.C. Huang and C.A. Rodger, Connectivity of Cages, *J. of Graph Theory*, Vol. 24(1997), No. 2, 187-191.

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York, 1976.

[3] W.T. Tutte, A Family of Cubical Graphs, *Proc. Cambridge Phil. Soc.*, d(1947), 459-474.

[4] P-K, Wong, Cages-a survey, *J. Graph Theory*, 6(1982), 1-22.

[5] G. Brinkmann, B.D. McKay, and C. Saager, The smallest cubic graphs of girth 9, *Combinatorics, Probability, and Computing*, 4(1995), 317-330.

[6] B.D. McKay, W. Myrvold and J. Nadon, Fast Backtracking Principles Applied to Find New Cages, *Ninth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 1998, 188-191.

[7] P. Wang, B. Xu and J.F. Wang, On the Connectivity of Cages, submitted to J. of Graph Theory.