Subdirectly Irreducible Squags of Cardinality 3n

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Abstract: *In this paper, we construct a squag SQG(3n) of cardinality 3n that contains three given arbitrary squags SQG(n)s as disjoint subquags. Accordingly, we can construct a subdirectly irreducible squag SQG(3n), for each $n \geq 7$, with $n \equiv 1$ or 3 (mod 6). Also, we want to review the shape of the congruence lattice of non simple squags SQG(n) for some n and to give a classification of the class of all SQG(21)s and the class of all SQG(27)s according to the shape of its congruence lattice. SQG(21)s are classified into three classes and SQG(27)s are classified into four classes. The construction of SQG(3n), which is given in this paper, helps us to construct examples of each class of both SQG(21)s and SQG(27)s.*

Introduction:

A squag is a groupoid S = (S; .) Satisfying the identities:

$$x . x = x \quad , \quad x . y = y . x \quad , \quad x . (x . y) = y .$$

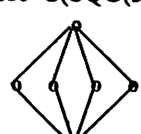
We use the abbreviation SQG(n) for a squag of cardinality n. If a squag satisfies the identity:

$$(x . y) . (z . w) = (x . z) . (y . w),$$

then it is called a medial squag. An extensive study of squags can be found in [5],[7]and [8].

A Steiner triple system is a pair (P; B), where P is a set of points and B is a set of 3-element subsets of P called blocks such that for distinct points p_1, p_2 $\in$ P, there is a unique block $b \in B$ satisfying $\{p_1, p_2\} \subseteq b$. There is a one to one correspondence between the squags and the Steiner triple systems [2] [8]. The Steiner triple system (P; B) is denoted by STS(n), if the cardinality of P is equal to n. It is well known that a necessary and sufficient condition for the existence of an STS(n) is $n \equiv 1$ or 3 (mod 6).

Quackenbush [8] proved that most finite squags are simple. In fact, a squag SQG(n) is simple if n can not be factored into $(6n_1 + i)(6n_2 + j)$ for some n_1, n_2 and some $i, j \in \{1,3\}$. In particular, there are only simple SQG(n)s, if n is prime. Quackenbush [8] also proved that the congruences of squags are permutable, regular, and Lagrangian and he showed also that any finite simple SQG(n) with $n > 3$ is functionally complete. I.e. For any finite simple squag $\mathcal{S}$ = (S; .) with $|S| > 3$, the congruence lattice $C(\mathcal{S}^r) \cong 2^r$ (the Boolean lattice with exactly r atoms). But the congruence lattice $C(SQG(3)^r)$ has skew congruences. In particular, the congruence lattice

$C(SQG(3)^2) \cong$  .

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The basic concepts of universal algebra can be found in [4].
In the following section, we give a construction of an SQG(3n) containing any three given sub-SQG(n)s. By this construction we may construct a subdirectly irreducible squag

of cardinality 3n with congruence lattice ≅  , for each n >9 with

n ≡ 1or 3(mod 6) .

In section 3, we will be concerned with a classification of the classes of SQG(21)s and SQG (27)s according to the shape of its congruence lattice. And by using the construction given in section 2, we can construct examples of each class of the determined classes of SQG(21)s and of SQG(27)s. Specially, examples of subdirectly irreducible squags of SQG(21) and SQG(27) will be constructed.

2-Construction of an SQG(3n)

All SQG(m)s with m ≡ 1 or 3 (mod 6) are simple, if m is not divisible by 3 or if m =3n and n ≢ 1 or 3 (mod 6). Then we may say that the class of SQG(3n) with n ≡ 1 or 3 (mod 6) is the class of squags that may contain non-simple squags.

Moreover, Quackenbush [8] and Armanious [1] proved that a subsquag S_1 of a finite squag S with $|S| = 3|S_1|$ is normal iff there are three disjoint subsquags S_1 , S_2 and S_3 with $|S_1| = |S_2| = |S_3|$. This algebraic properties leads us to construct an SQG(3n) having three disjoint sub-SQG(n)s for each n ≡ 1 or 3 (mod 6).

In order to turn an STS= (P ; B) to a squag =(P ; .) or conversely we have the relation:

$x . y = y . x := z \Leftrightarrow \{x, y, z\} \in B$; for any two distinct elements x and y ∈ P and we also have $x . x = x$ for any $x \in P$ [8].

With the help of this correspondence, both algebraic and combinatorial languages will be used in this article.

At first, we need to write the definitions of some concepts from the graph theory [6]. A complete bipartite graph $K_{n,m}$ is a simple graph, in which its set of vertices $V(K_{n,m})$ can be divided into two disjoint sets A and B such that $|A| = n$ & $|B| = m$ and the set of edges $E(K_{n,m})$ is exactly the set of all edges connecting each vertex of A with each vertex of B. A spanning subgraph F of a graph G is called a 1- factor of G, if deg v = 1 ; ∀ v ∈ V(F). If a graph G is the union of a set of disjoint 1-factors $\{F_1, F_2, \ldots, F_n\}$, then the set $\{F_1, F_2, \ldots, F_n\}$ is called a 1- factorization of G and G itself is called 1-factorable.

Theorem 1[3]. *Every regular bipartite graph* $K_{n,n}$ *is 1-factorable.*

Let $(P_i\ ;\ B_i)$; for $i = 1, 2, 3$ be any three STS(n)s with $P_i \cap P_j = \varnothing$; for $i \neq j$ and $K_{n,n}$ be the complete bipartite graph with a set of vertices $V(K_{n,n}) = P_1 \cup P_2$ and the set of edges of $K_{n,n}$ consists exactly of the edges connecting points in P_1 with points in P_2. Also let $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ be a 1-factorization of $K_{n,n}$.

By taking any bijactive map

$\alpha_n : \{1, 2, \ldots, n\} \rightarrow P_3$, we define the set of blocks B_{123} as the following :

$$B_{123} := \{\{x, y, \alpha_n(i)\} : \text{for all } xy \in F_i \text{ and } F_i \in \mathcal{F}\}.$$

For $P := P_1 \cup P_2 \cup P_3$ and $B := B_1 \cup B_2 \cup B_3 \cup B_{123}$, then the system (P ; B) is an STS(3n) . And the proof is in routine manner. We note that $\mathcal{F}$ can be any 1-factorizations and that α_n can be any bijective map. This construction of an STS(3n) = (P; B) will be denoted by:

$[((P_1; B_1) \cup (P_2; B_2)) \cup (P_3; B_3); \mathcal{F}(P_1, P_2), \alpha_n]$.

And the corresponding squag (P ; .) will also be denoted by:

$[((P_1; \cdot_1) \cup (P_2; \cdot_2)) \cup (P_3; \cdot_3); \mathcal{F}(P_1, P_2), \alpha_n]$.

Where the binary operation " . " on P defined by

$$x \cdot y := \begin{cases} x \cdot_i y & \text{if } x, y \in P \\ \alpha_n(i) & \text{if } x \in P_1 \text{ and } y \in P_2 \\ z & \text{if } x \in P_1 \text{ and } y \in P_3 \text{ and } xz \in F_{\alpha_n^{-1}(y)} \\ z & \text{if } x \in P_3 \text{ and } y \in P_2 \text{ and } yz \in F_{\alpha_n^{-1}(x)} \end{cases}$$

and satisfied the identities $x \cdot x = x$ and $x \cdot y = y \cdot x$.

This construction supplies us with examples of SQG(3n)s that can not be constructed by the direct product SQG(n) × SQG(3) such as an SQG(3n) with congruence having non-isomorphic congruence classes each as a sub-SQG(n).

In the next theorem, we give a combinatorial equivalent condition of a subsquag to be normal.

Let $(P_1\ ;\ B_1)$ be a subspace STS(n) of the STS(3n) := (P ; B), where $P_1 := \{a_1, a_2, \ldots, a_n\}$. Let $\mathbf{P} = (P; .)$ be the corresponding squag of (P ; B) and $\mathbf{P}_1 = (P_1 ; .)$ be the corresponding subsquag of $(P_1; B_1)$.

Theorem 2. *The corresponding subsquag* $\mathbf{P}_1$ *of the squag* $\mathbf{P}$ *is normal iff the set* $\mathcal{F} := \{F_1, F_2, \ldots, F_n\}$, *where* $F_i := \{xy : \{x, y\} \subseteq P - P_1 \ \&\ \{x, y, a_i\} \in B\}$

forms a 1-factorization of a complete bipartite graph $K_{n,n}$ *with a set of vertices* $V(K_{n,n})=P-P_1$.

Proof . If P_1 is normal of the squag **P**, then there is a congruence θ on **P** such that the factor algebra $P/\theta = \{P_1=[a_i]\theta, P_2:=[x]\theta, P_3:=[y]\theta\}$. We have the three disjoint subsets P_1 , P_2 and P_3; each forms a subsquag of P and $P_i . P_j = P_k$; for {i, j, k} = { 1 , 2 , 3 }.

It is well known that the number of the blocks in an STS(m) containing a fixed element is equal to (m – 1)/2. Since $a_i . P_2 = P_3$ for $a_i \in P_1$, then for any $a_i \in P_1$ there are ((3n-1)/2) – ((n-1)/2) = n blocks b ∈ B in the form b = { a_i , x , y } such that x ∈ P_2 and y ∈ P_3 .

This means that any factor F_i of ***F*** contains n edges and each edge incident with two vertices one of them x ∈P_2 and the other y ∈ P_3 . Moreover, there is no edges between two vertices belonging to the same set P_2 or P_3 . This implies that the set ***F*** is a 1-factorization of a complete bipartite graph $K_{n,n}$ having the set of vertices $V(K_{n,n}) = P- P_1 = P_2 \cup P_3$ and containing exactly of the edges connecting points in P_1 with points in P_2.

The other direction, let ***F*** be a 1- factorization of a complete bipartite graph $K_{n,n}$ with a set of vertices P- P_1 divided into two disjoint sets $P_2 = \{x_1, x_2, \dots, x_n\}$ and $P_3 = \{y_1, y_2, \dots, y_n\}$; i.e. $V(K_{n,n}) = P_2 \cup P_3$ and the set of edges of $K_{n,n}$ consists exactly of the edges connecting points in P_2 with points in P_3. From the definition of the 1-factor F_i , we have n blocks in the form { a_i, x_j , y_k } which $x_j\ y_k$ is an edge in F_i ; for any i = 1, 2 , ... , n . This implies that for any 2-element set { x_i, x_j } ⊆ P_2 , there is no 1-factor F_k with k ∈ { 1, 2 , ... , n } satisfying the edge $x_i x_j \in F_k$ .Therefore, the block b in B which contains the 2-element set {x_i , x_j } is one of the following two cases:

(1) b = {x_i , x_j , y_k} or (2) b = {x_i , x_j , x_k}.

For case (1), we have $x_i\ y_k$ is an edge in a 1-factor F_j ; for some j , then there is a block in B in the form {a_j, x_i, y_k}, contradicting the definition of Steiner triple systems that there is exactly one block containing {x_i , y_k}. Hence b must be in the form of the second case namely b = {x_i , x_j , x_k}. This means for any 2-element subset {x_i , x_j } ⊆ P_2 that there is a block b = {x_i , x_j , x_k}⊆ P_2 . Which implies that the set of elements of P_2 forms a subspace of the system (P , B) . Similarly, the set of elements of P_3 forms also a subspace. This completes the proof of the theorem.

In view of the above Theorem, we may say that any STS(3n) = (P ; B) with three disjoint subspaces (P_1; B_1), (P_2; B_2) and (P_3; B_3) of cardinality n can be formulated by the construction given in this section. I.e. (P ; B) = [((P_1; B_1) ∪ (P_2; B_2)) ∪ (P_3; B_3) ; ***F***(P_1,P_2) , α_n] for a specified 1-factorization ***F*** of $K_{n,n}$ with a set of vertices $V(K_{n,n}) = P_1 \cup P_2$ and a set of edges $E(K_{n,n})$ consists exactly of the edges connecting points in P_1 with points in P_2 and a certain bijective map α_n .

Theorem 3. *Let* $(S;\ B_s)$ *be not a sub-STS of any of* $(P_i\ ;\ B_i)$; $i = 1,2,3$. *Then* $(S;\ B_s)$ *is a sub-STS*(3r) *of* $[((P_1;\ B_1) \cup (P_2;\ B_2)) \cup (P_3;\ B_3)\ ;\ F(P_1,P_2)\ ,\ \alpha_n\ )\ ]$ *iff there are three sub-STS*(r)= $(\ S_i\ ;\ B_{S_i}\ ) \leq (P_i\ ;\ B_i)$; $i = 1,2,3$ *and a sub-1-factorization* $f = \{\ f_{i_1}\ ,\ f_{i_2}, \dots, f_{i_r}\}$ *of F on the set of vertices* $V(K_{r,r}) = S_1 \cup S_2$ *and a bijective map* $\alpha_{n_{|r}} := \alpha_n$ *restricted on the subset* $\{i_1\ ,\ i_2\ , \dots ,\ i_r\}$ *such that* $(S\ ;\ B_S) = \ [((S_1\ ;\ B_{S_1}) \cup (\ S_2\ ;\ B_{S_2}\ )\ ) \cup (\ S_3\ ;\ B_{S_3}\ )\ ;\ f\,(S_1\ ,\ S_2)\ ,\ \alpha_{n_{|r}}\ ]$.

Proof: Let $(S\ ;B_S)$ be a sub-STS(m) of $[\ ((P_1;\ B_1) \cup (P_2;\ B_2)) \cup (P_3;\ B_3)\ ;\ F(P_1,P_2)\ ,\ \alpha_n\ ]$, then $S \cap P_i := S_i \neq \emptyset$; for $i = 1,2,3$. Otherwise, if $S_1 = \emptyset$ & $S_i \neq \emptyset$; for $i = 2, 3$, then for $a \in S_2 \Rightarrow aS_3 \subseteq S_1 \Rightarrow$ a contradiction. Also, for $a \in S_2 \Rightarrow aS_3 \subseteq S_1$ & $aS_1 \subseteq S_3 \Rightarrow |S_1| = |S_3|$. Therefore, $|S_1| = |S_2| = |S_3| = r$, for some r.

Now, we have for $x \in S_1$, $y \in S_2$ and $z_{\alpha(i)} \in S_3 \Leftrightarrow \{x,\ y,\ z_{\alpha(i)}\} \in B_S$. This implies that for all i_j with $z_{\alpha(i_j)} \in S_3$, there is a sub-1-factor $f_{i_j} \subseteq F_{i_j}$ on $K_{r,r}$ with a set of vertices $V(K_{r,r}) = S_1 \cup S_2$ and a set of edges $E(K_{r,r})$ consists exactly of the edges connecting points in S_1 with points in S_2.

Let $R = \{\ i_j\ :\ z_{\alpha(i_j)} \in S_3\ \}$, then $|R| = r$ and $f = \{\ f_{i_j} : i_j \in R\ \}$ is a sub 1- factorization of F on $K_{r,r}$ with $V(K_{r,r}) = S_1 \cup S_2$. Therefore, $(S\ ;\ B_S) = [((S_1\ ;\ B_{S_1}) \cup (\ S_2\ ;\ B_{S_2}\ )\ ) \cup (\ S_3\ ;\ B_{S_3}\ )\ ; f\,(S_1\ ,\ S_2)\ ,\ \alpha_{n_{|r}}]$, where $\alpha_{n_{|r}}$ is equal to α_n restricted on the subset R of the set $\{1, 2, \dots, n\}$.

The other direction, if $(\ S_i\ ;\ B_{S_i}\ )$ are sub-STS(r)s of $(\ P_i\ ;\ B_i)$; $i = 1,2,3$, and $f = \{\ f_{i_1}\ , f_{i_2}, \dots, f_{i_r}\}$ is a sub 1-factorization of F on the set of vertices $V(K_{r,r}) = S_1 \cup S_2$ and a bijective map $\alpha_{n_{|r}} : \{i_1\ ,\ i_2, \dots ,\ i_r\} \to S_3$ given by $\alpha_{n_{|r}}(i_j) := \alpha_n(i_j)$ for $j = 1, 2, \dots , r$.

Then $[((S_1\ ;\ B_{S_1}\ ) \cup (\ S_2\ ;\ B_{S_2}\ )\ ) \cup (\ S_3\ ;\ B_{S_3}\ )\ ;\ f\,(S_1\ ,\ S_2)\ ,\ \alpha_{n_{|r}}]$ is directly a sub-STS(3r) of $[\ ((P_1;\ B_1) \cup (P_2;\ B_2)) \cup (P_3;\ B_3)\ ;\ F(P_1,P_2)\ ,\ \alpha_n\ ]$. This completes the proof of the theorem.

According to the above Theorem, examples of SQG(3n)s with certain congruence lattice will be constructed in the following theorem and in the next section.

An STS is planar if it is generated by every triangle and contains a triangle. A planar STS(n) exits for each $n \geq 7$ and $n \equiv 1$ or 3 (mod 6) [3]. Quackenbush [8] proved in the next theorem that almost all planar SQG(n)s are simple.

Theorem 4[8]. *Let* (P ; B) *be a planar STS(n) and* (P ; .) *be the corresponding squag, then either* (P ; .) *is simple or* n = 9 .

Now, we are ready to construct squags having exactly one proper congruence.

Theorem 5. *For each* $n > 9$ *with* $n \equiv 1$ or 3 (mod 6), *there is a subdirectly irreducible squag of cardinality* $3n$.

Proof . For each $n > 3$ with $n \equiv 1$ or 3 (mod 6), there is a finite planar STS(n) := $(P_0 \; ; \; B_0)$ [3] and according to Theorem4, we may say that there is a simple squag SQG(n) :=$(P_0 \, ; .)$ for all $n > 9$ with $n \equiv 1$ or 3 (mod 6).

By taking $P_0 := \{1, 2, \ldots, n\}$, $P_1 := \{x_1, x_2, \ldots, x_n\}$, $P_2 = \{y_1, y_2, \ldots, y_n\}$, $P_3 = \{z_1, z_2, \ldots, z_n\}$ and $(P_i \, ; B_i) \cong (P_0 \, ; B_0)$ for $i = 1,2,3$, where $\{x_i, x_j, x_k\}$, $\{y_i, y_j, y_k\}$ and $\{z_i, z_j, z_k\}$ are blocks of B_1, B_2 and B_3 respectively iff $\{i, j, k\}$ is a block in B_0.

Consider the 1-factorization $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ with $F_i = \{x_j y_k : \{j, k, i\} \in B_0$ or $i = j = k\}$ on $K_{n,n}$ with a set of vertices $V(K_{n,n}) = P_1 \cup P_2$ and a set of edges $E(K_{n,n})$ consists exactly of the edges connecting points in P_1 with points in P_2.

Without loss of generality we may assume that $\{1, i, i+1\} \in B_0$; for $i = 2, 4, \ldots, n-1$, and α be a bijective map from $\{1, 2, .., n\}$ on to P_3 by $i \to z_{i+1}$ for $i = 2, 3, \ldots, n-1$, $\alpha(n) = z_2$ and $\alpha(1) = z_1$. Then the corresponding squag SQG(3n) of the STS(3n) = $[((P_1; B_1) \cup (P_2; B_2)) \cup (P_3; B_3) \, ; \mathcal{F}(P_1,P_2), \alpha]$ has only one proper congruence Φ determined by its congruence classes $[x_1]\Phi = P_1$, $[y_1]\Phi = P_2$ and $[z_1]\Phi = P_3$.

To prove that there is no other proper congruences on SQG(3n), suppose θ is a proper congruence on SQG(3n) differ than Φ, then $\theta|_{P_i}$ restricted on P_i for $i = 1, 2, 3$ is congruence on the corresponding squag $(P_i \, ; .)$ of $(P_i \, ; B_i)$. But $(P_i \, ; .)$ is simple, this implies that the only possible case of the cardinality of the congruence class of θ is $|[a]\theta| = 3$ with $|[a]\theta \cap P_i| = 1$.

We have $b_i = \{x_1, x_i, x_{i+1}\} \in B_1$, then $[x_1]\theta \cup [x_i]\theta \cup [x_{i+1}]\theta$ = sub-STS(9), for each $i = 2, 4, \ldots, n-1$. Each of these sub-STS(9)s contains x_1, a set of points of a block $b_2 \in B_2$ and a set of points of a block $b_3 \in B_3$. Therefore, there is one of these sub-STS(9) containing the element z_1.

By using the definition of F_i, we may verify that $x_1y_1, x_iy_{i+1}, x_{i+1}y_i \in F_1$, for some even number i. Hence the sub-STS(9) containing the element z_1 contains also the blocks $\{x_1, x_i, x_{i+1}\}$ and $\{y_1, y_i, y_{i+1}\}$ for some even number i.

By Theorem 3 and from the definition of F_i the sub-STS(9) containing z_1 may be constructed by the sub 1-factorization $f = \{f_1, f_i, f_{i+1}\}$, where $f_1 = \{x_1y_1, x_iy_{i+1}, x_{i+1}y_i\} \subseteq F_1$, $f_i = \{x_iy_i, x_1y_{i+1}, x_{i+1}y_1\} \subseteq F_i$ and $f_{i+1} = \{x_{i+1}y_{i+1}, x_1y_i, x_iy_1\} \subseteq F_{i+1}$ for some even number i and the bijective map α restricted on the set $\{1, i, i+1\}$. I.e.the sub-STS(9) containing z_1 must contain the triple $\{\alpha(1), \alpha(i), \alpha(i+1)\} = \{z_1, z_{i+1}, z_{i+2}\}$.

But according to the set of blocks of B_0, the block containing the two points z_1 and z_{i+1} is $\{z_1, z_i, z_{i+1}\}$, which is impossible. Then the proof is complete.

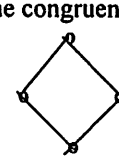
The above Theorem is true for n = 7, we use the same idea to construct an example of a subdirectly irreducible SQG(21) in the next section. But for the case of n= 9, the method used in the proof of the above theorem is not enough to construct a subdirectly irreducible SQG(27), since the corresponding squag of the STS(9) is not simple.

To complete our discussion, we will construct examples of subdirectly irreducible SQG(27)s in the next section.

3-Congruence lattices of SQG(21)s and of SQG(27)s

First of all, a simple SQG(21) and a simple SQG(27) exist according to [3] and Theorem4.

Now we start to discuss the different classes of SQG(21)s.

If the congruence lattice of SQG(21) has more than one atom , then SQG(21) must be isomorphic to the direct product SQG(7) × SQG(3). And according to the next theorem given by Quackenbush [8], we can deduce that the congruence lattice of an SQG(21) having more than one atom is isomorphic to 

Theorem 6[8]. If SQG(n) *is a simple planar squag, then the direct product* SQG(n) × SQG(3) *has no skew congruences.*

To complete the discussion of the congruence lattices of SQG(21)s, we turn our attention to the class of SQG(21)s having exactly one atom. As a consequence, we have

SQG(21)s with congruence lattice isomorphic to  Φ , and the cardinality of the congruence classes of Φ is 7 or 3 .

To give an example of an SQG(21) having a unique atom Φ with congruence classes of cardinality 7, we use the construction given in section 2 and apply the idea of the proof of Theorem5 as follows:
Consider the STS(7) = (P_0 ; B_0) with P_0 = {0, 1, 2, 3, 4, 5, 6} and the blocks of B_0 are the lines of the projective plane PG(2,2); i.e. the set of blocks B_0 are the 3-element set {i, i+1, i+3 } (mod 7). By taking three disjoint STS(7) = (P_i ; B_i) ; i = 1, 2, 3 given by the disjoint sets $P_1 = \{a_0, a_1, \ldots, a_6\}$, $P_2 = \{b_0, b_1, \ldots, b_6\}$, $P_3 = \{c_0, c_1, \ldots, c_6\}$ and the sets of blocks B_j ; j=1, 2, 3 defined by: a block $\{x_i, x_{i+1}, x_{i+3}\}$ belongs to B_j for x = a, b, c iff {i, i+1, i+3} (mod 7).

Let F_i be the 1-factor defined by:

$F_i = \{a_i b_i\} \cup \{a_j b_k : \{i, j, k\} \in B_0\}$, then $\mathcal{F} = \{F_0, F_1, \ldots, F_6\}$ is a 1-factorization of the complete bipartite graph $K_{7,7}$ with a set of vertices $V(K_{7,7}) = P_1 \cup P_2$ and a set of edges consists exactly of edges connecting each poins in P_1 with each points of P_2. And by choosing the bijective map $\alpha : \{0, 1, , \ldots, 6\} \to P_3$ defined by $\alpha(i) = c_{\alpha_7(i)}$, where α_7 is the permutation (134526) on P_0. Hence the corresponding squag SQG(21) = (P ; .) of the STS(21) = (P ; B) = $[((P_1; B_1) \cup (P_2; B_2)) \cup (P_3; B_3) ; \mathcal{F}(P_1,P_2) , \alpha]$ has only one proper congruence Φ determined by its congruence classes $[a_0]\Phi = P_1$, $[b_0]\Phi = P_2$ and $[c_0]\Phi = P_3$.

To prove that Φ is the only proper congruence, suppose θ is a proper congruence of (P ; .) differ than Φ, then $|[a]\theta| = 3$ and $|[a]\theta \cap P_i| = 1$.

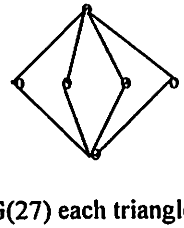
Moreover, for any block $\{a_i, a_{i+1}, a_{i+3}\} \in B_1$ the union $[a_i]\theta \cup [a_{i+1}]\theta \cup [a_{i+3}]\theta$ is a sub-STS(9), for any $i = 0,1,\ldots,6$, then there is a sub-STS(9) containing c_0. From the definition of α_7, we have c_0 related with F_0 and any sub 1-factor in F_0 must contain the edge a_0b_0. This implies that $\{a_0, b_0, c_0\}$ is a block in this sub-STS(9) and the sub 1-factor f_0 of F_0 must be equal (i) $\{a_0b_0, a_1b_3, a_3b_1\}$, (ii) $\{a_0b_0, a_2b_6, a_6b_2\}$ or (iii) $\{a_0b_0, a_4b_5, a_5b_4\}$.
As a consequence, when one of the conditions (i), (ii) or (iii) holds, then the sub-STS(9) must contain one of the triples $\{c_0, c_3, c_4\}$, $\{c_0, c_6, c_1\}$or $\{c_0, c_5, c_2\}$ as a block respectively. But none of these triples is a block in B_3. This contradicts the result of Theorem 3 that this sub-STS(9) must consist of a block of B_1 , a block of B_2 and a block of B_3. Therefore, there is no other proper congruence differ than Φ .

In fact, we are faced with the question; is there an irreducible SQG(21) having a proper congruence Φ with congruence classes of cardinality 3 ?

Secondly, we turn our attention to the class of SQG(27)s.

If the congruence lattice of an SQG(27) has more than one atom then such SQG(27) is a subdirect product of medial squags, hence SQG(27) must be a medial squag. Since each triangle in a medial squag generates a sub- SQG(9), then a medial squag has 39 distinct sub-SQG(9). Moreover, each sub-SQG of a medial squag is normal, then the congruence lattice of a medial SQG(27) has 13 atoms and 13 maximum congruences.

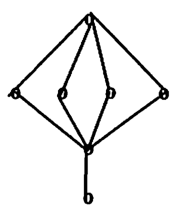
For any atom θ of a medial SQG(27), the interval $[\theta, 1]$ is isomorphic to the

congruence lattice $C(SQG(3)^2) \cong$  .

We observe that if in an SQG(27) each triangle generates an SQG(9), then it satisfies the distributive law $x.(y.z) = (x.y).(x.z)$ [5][7] and contains 39 distinct sub-SQG(9)s. This implies for any sub-SQG(9) = (P_1; .) and any x $\notin P_1$ that (x . P_1 ; .) and (P_1; .) are disjoint sub-SQG(9)s. Therefore, any sub-SQG(9) is normal [8]. Consequently, the congruence lattice of SQG(27) contains 13 maximum congruences. Moreover, the intersection of any two distinct maximal congruences of such SQG(27) is a congruence with congruence classes of cardinality 3. Hence for any maximal congruence of such SQG(27) there is a non-comparable minimal congruence (i.e. an atom). This means that if in an SQG(27) each triangle generates an SQG(9), then it is isomorphic to the direct product SQG(9) × SQG(3), which implies that it must be a medial squag.

From the foregoing discussion, we have only two interesting classes of SQG(27)s each of them is subdirectly irreducible as follows:

(i) The class of SQG(27)s in which the congruence lattice

C(SQG(27) ≅  .

(ii) The class of SQG(27)s in which the congruence lattice

C(SQG(27) ≅  .

To give examples for these two classes, we use a description of three STS(9)s similar to the previous construction of the SQG(21). We consider the STS(9) = (P_0 ; B_0) as the affine plane on the set of points P_0 = { 1, 2, 3, 4, 5, 6, 7, 8, 9 } and its set of lines given by the set of blocks:

B_0 = {{1, 2, 3}, {4, 5, 6}, {7, 8, 9}, {1, 4, 7}, {2, 5, 8}, {3, 6, 9}, {1, 6, 8}, {2, 4, 9}, {3, 5, 7}, {1, 5, 9}, {2, 6, 7}, {3, 4, 8}}.

By taking three disjoint STS(9)s = (P_r ; B_r) for r = 1, 2, 3 given by the disjoint sets $P_1 = \{ a_1, a_2, \ldots, a_9 \}$, $P_2 = \{ b_1, b_2, \ldots, b_9\}$, $P_3 = \{c_1, c_2, \ldots, c_9 \}$and the sets of blocks B_r for r = 1, 2, 3 defined by:

$\{x_i, x_j, x_k\} \in B_r$; for x = a, b, c iff $\{i, j, k\} \in B_0$.

Let F_i be the 1-factor defined similarly as the construction of SQG(21) by $F_i = \{a_ib_i \} \cup \{ a_j b_k : \{i, j, k \} \in B_0 \}$, then $\mathcal{F} = \{F_1, F_2, \ldots, F_9\}$ is a 1-factorization of the complete bipartite graph $K_{9,9}$ with a set of vertices $V(K_{9,9}) = P_1 \cup P_2$, and a set of edges consists exactly of edges connecting each poins in P_1 with each points in P_2.

Hence the corresponding squag SQG(27) = (P ; .) of the STS(27) = (P ; B) = [((P_1; B_1) ∪ (P_2; B_2)) ∪ (P_3; B_3) ; $F(P_1,P_2)$, α] gives examples of the above two cases as follows:

Example of case (i) if the bijective map α : {1, 2, ... , 9} → P_3 defined by α (i) = $c_{\alpha_9(i)}$, where α_9 is the permutation (46) on P_0 .

Example of case (ii) if the bijective map α : {1 , 2 , ... , 9} → P_3 defined by α (i) = $c_{\alpha_9(i)}$, where α_9 is the permutation (12)(34)(68) on P_0.

To prove that the constructed squag SQG(27) = (P ; .) with the permutation α_9 = (46) having a congruence lattice as in the case (i), it is enough to show that the relation θ = { (x, y) : (x . a) . y ∈ {a_1, a_2, a_3} and a ∈ {a_1, a_2, a_3} } is the unique atom of the congruence lattice C(SQG(27).

First, we have to prove that {a_1, a_2, a_3} is a normal subsquag.

We have three subsloop S_1 = {a_1 , a_2 , a_3} of P_1 , S_2 = {b_1 , b_2 , b_3} of P_2 and S_3 = {c_1 , c_2 , c_3} of P_3 and the sub 1-factorization f ={f_1 , f_2 , f_3} given by f_1 = {a_1 b_1 , a_2 b_3 , a_3 b_2 }⊆ F_1 , f_2 = {a_2 b_2 , a_1 b_3 , a_3 b_1 }⊆ F_2 and f_3 = {a_3 b_3 , a_2 b_1 , a_1 b_2 }⊆ F_3 of F . Then by Theorem 3, the structure (S ; B_S) = [((S_1 ; B_{S_1}) ∪ (S_2 ; B_{S_2})) ∪ (S_3 ; B_{S_3}) ; f(S_1 , S_2) , $\alpha_{n_{13}}$] is a subsloop of [((P_1; B_1) ∪ (P_2; B_2)) ∪ (P_3; B_3) ; F (P_1,P_2) , α], where $\alpha_{n_{13}}$(i) = α(i) = i for i = 1, 2, 3.

Similarly, if we choose the subsloops S_1' = {a_7 , a_8 , a_9} of P_1 , S_2' = {b_7 , b_8 , b_9} of P_2 and S_3' = {c_7 , c_8 , c_8} of P_3 and the sub 1-factorization f' = { f_7 , f_8 , f_9 } given by f_7 = {a_7 b_7 , a_8 b_9 , a_9 b_8 }⊆ F_7 , f_8 = {a_8 b_8 , a_7 b_9 , a_9 b_7 } ⊆ F_8 and f_9 = {a_9 b_9 , a_7 b_8 , a_8 b_7 }⊆ F_9 of F. Then by Theorem 3, the structure (S' ; $B_{S'}$) = [((S_1' ; $B_{S_1'}$) ∪ (S_2' ; $B_{S_2'}$)) ∪ (S_3' ; $B_{S_3'}$) ; f(S_1' , S_2') , $\alpha'_{n_{13}}$] is a subsloop of [((P_1; B_1) ∪ (P_2; B_2)) ∪ (P_3; B_3) ; F (P_1,P_2) , α], where $\alpha'_{n_{13}}$ (i) = α(i) = i for i = 7, 8, 9.

This means that S and S' are two disjoint subsloop of cardinality 9, then the subset S = { a_1 , a_2 , a_3 , b_1 , b_2 , b_3 , c_1 , c_2 , c_3} forms a normal subsloop of (P ; .) [8]. This implies that the intersection P_1 ∩ S = { a_1 , a_2 , a_3} is a normal subsloop.

To prove that θ is the unique atom; it is enough to prove that the constructed squag SQG (27) is not medial. We have:

(c_1 . c_6) . (a_1 . a_2) = c_8 . a_3 = b_4 and (c_1 . a_1) . (c_6 . a_2) = b_1 . b_9 = b_5 , which implies that (c_1 . c_6) . (a_1 . a_2) ≠ (c_1 . a_1) . (c_6 . a_2) (i.e. the constructed squag SQG(27) is not medial). Then the constructed squag SQG (27) = (P ; .) is an example of case (i).

Now we come to the case (ii), to prove that the corresponding SQG (27) with the permutation α_9 = (12)(34)(68) has a congruence lattice as in the case (ii). We have a maximum congruence Φ determined by its congruence classes

P_i ; $i = 1, 2, 3$, then we have to prove that Φ is the unique proper congruence of the SQG(27).

First, we have the constructed squag SQG(27) is not medial because of: $(c_1 . c_5).(a_1 . a_2) = c_9 . a_3 = b_6$ and $(c_1 . a_1).(c_5 . a_2) = b_3 . b_8 = b_4$, this means that $(c_1 . c_6).(a_1 . a_2) \neq (c_1 . a_1).(c_6 . a_2)$ (i. e. the medial law is not satisfied).

Moreover, if a squag SQG(27) has two distinct congruences with congruence classes of cardinality 9, then the intersection of them is a congruence of SQG(27) with a congruence classes of cardinality 3.
Consequently, it is enough to show that the constructed squag SQG(27) in case (ii) has no congruence θ with congruence classes of cardinality 3.

Suppose that the constructed SQG(27) =(P ; .) has a congruence θ with $|\theta| = 3 \Rightarrow \theta \subseteq \Phi$ or $\theta \cap \Phi = 0$ (the diagonal congruence). Suppose $\theta \cap \Phi = 0$, then (P ; .) is a subdirect product of the direct product of two medial squags. Hence the case $\theta \cap \Phi = 0$ is refused because of the constructed squag (P ; .) is not medial. Consequently, θ is properly contained in Φ.
Therefore, θ restricted on P_i i. e. θ_{P_i} is a congruence on P_i ; $i = 1, 2, 3$ .As a consequence, there is a sub-SQG(9) having three parallel blocks $S_1 = \{a_1, a_2, a_3\} \in B_1$, $S_2 = \{b_i, b_j, b_k\} \in B_2$ and $S_3 = \{c_l, c_m, c_n\} \in B_3$. Furthermore, there is a sub-1-factorization $f = \{ f_1 \subseteq F_1, f_2 \subseteq F_2, f_3 \subseteq F_3 \}$ forming with the triple $\{c_{\alpha(1)}, c_{\alpha(2)}, c_{\alpha(3)}\}$ the same sub-SQG(9), but $\{c_{\alpha(1)}, c_{\alpha(2)}, c_{\alpha(3)}\} = \{c_2, c_1, c_4\}$ is not block in B_3 . This contradicts the result of Theorem3, that this sub-SQG(9) must be equal $[((S_1 ; B_{S_1}) \cup (S_2 ; B_{S_2})) \cup (S_3 ; B_{S_3}) ; f(S_1, S_2), \alpha_{n_{|3}}]$, where where $\alpha_{n_{|3}}(i) = \alpha(i)$

According to the previous example (ii) of a subdirectly irreducible SQG(27), we can improve the result of thoerem5 as follows:

There is a subdirectly irreducible squag of cardinality 3n ; for each n > 3 with n ≡ 1 or 3 (mod 6), having a congruence lattice isomorphic to  *θ .*

Moreover, if n > 9 , then the congruence classes of θ are isomorphic to any simple squag of cardinality n .

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