Improved Inclusion-Exclusion for Valuations on Distributive Lattices

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Abstract. We restate a recent improvement of the inclusion-exclusion principle in terms of valuations on distributive lattices and present a completely new proof of the result. Moreover, we establish set-theoretic identities and logical equivalences of inclusion-exclusion type, which have not been considered before.

1 Introduction

A *valuation* on a distributive lattice $(L, \wedge, \vee)$ is a mapping v from L into an abelian group such that $v(a \wedge b) + v(a \vee b) = v(a) + v(b)$ for all $a, b \in L$. This concept was introduced by Rota [5] and further investigated by Geissinger [3]. For valuations on distributive lattices, Rota [5] established the following variant of the well-known inclusion-exclusion principle:

Proposition 1 ([5]) *Let v be a valuation on a distributive lattice $(L, \wedge, \vee)$. Then, for any $a_1, \ldots, a_n \in L$,*

$$v\left(\bigvee_{i=1}^{n} a_i\right) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} v\left(\bigwedge_{i \in I} a_i\right).$$

In many applications of the inclusion-exclusion principle such as network reliability computations (see e.g., [2]) a lot of cancellations can be observed. [1] provides an improvement of the principle, formulated as a theorem of measure theory, which covers some (but not necessarily all) of the cancellations. In this paper, we generalize this improvement from measures to

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valuations and present a completely new proof, which is based on a graph-theoretic argument that replaces the difficult partition in the former proof. The paper ends with set-theoretic identities and logical equivalences of inclusion-exclusion type, which seemingly have not been considered before.

2 Improved inclusion-exclusion for valuations

The following theorem generalizes the main result of [1].

Theorem 1 *Let v be a valuation on a distributive lattice $(L, \wedge, \vee)$, $a_1, \ldots, a_n \in L$, and $\mathfrak{X}$ be a set of non-empty subsets of $\{1, \ldots, n\}$ such that for any $X \in \mathfrak{X}$,*

$$\bigwedge_{x \in X} a_x \leq_L \bigvee_{i > \max X} a_i \,. \tag{1}$$

Then,

$$v\left(\bigvee_{i=1}^{n} a_i\right) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \not\supseteq X \forall X \in \mathfrak{X} \\ I \neq \emptyset}} (-1)^{|I|-1} v\left(\bigwedge_{i \in I} a_i\right) . \tag{2}$$

Proof: For $A, B \subseteq \{1, \ldots, n\}$ define $A \sqsubseteq B$ if $A \subseteq B$ and $\max A < \min B \setminus A$, where $\max \emptyset := 0$ and $\min \emptyset := n + 1$. Then, it is easy to verify that $\sqsubseteq$ is a partial order on the power set of $\{1, \ldots, n\}$, whose minimum is $\emptyset$. In particular, the Hasse diagram of this partial order is connected. (Figure 1 shows the Hasse diagram for $n = 4$.) We claim that the Hasse diagram is also cycle-free: It is a well-known fact from graph theory that a connected simple graph is cycle-free if and only if the number of edges is one less than the number of vertices. Thus, since the number of vertices in the Hasse diagram is 2^n, we have to check that the number of edges is $2^n - 1$. Since any $I \subseteq \{1, \ldots, n\}$ has $n - \max I$ immediate successors, the number of edges is

$$\sum_{I \subseteq \{1,\ldots,n\}} (n - \max I) = n\,2^n - \sum_{I \subseteq \{1,\ldots,n\}} \max I = n\,2^n - \sum_{k=1}^{n} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ \max I = k}} k$$
$$= n\,2^n - \sum_{k=1}^{n} k\,2^{k-1} = n\,2^n - ((n-1)2^n + 1) = 2^n - 1 \,.$$

Therefore, the Hasse diagram of $\sqsubseteq$ is a tree, which is rooted at the minimum $\emptyset$. By the proposition and the preceding observation, the theorem is proved if

$$\sum_{J \sqsupseteq I} (-1)^{|J|-1} v\left(\bigwedge_{j \in J} a_j\right) = 0$$

for each I which is $\sqsubseteq$-minimal in $\{I \subseteq \{1, \ldots, n\} \mid I \supseteq X \text{ for some } X \in \mathfrak{X}\}$. It is easy to see that

$$\sum_{J \sqsupseteq I} (-1)^{|J|-1} v\left(\bigwedge_{j \in J} a_j\right) \equiv v\left(\bigwedge_{i \in I} a_i\right) - \sum_{\substack{J \neq \emptyset \\ \min J > \max I}} (-1)^{|J|-1} v\left(\bigwedge_{i \in I} a_i \wedge \bigwedge_{j \in J} a_j\right)$$

where $\equiv$ means equality up to sign. By applying Proposition 1 we obtain

$$\sum_{J \sqsupseteq I} (-1)^{|J|-1} v\left(\bigwedge_{j \in J} a_j\right) \equiv v\left(\bigwedge_{i \in I} a_i\right) - v\left(\bigwedge_{i \in I} a_i \wedge \bigvee_{j > \max I} a_j\right).$$

By this, it remains to show that

$$\bigwedge_{i \in I} a_i \leq_L \bigvee_{i > \max I} a_i .$$

By the choice of I there is some $X \in \mathfrak{X}$ such that $I \supseteq X$. Then, $\max I \leq \max X$, since $\max I > \max X$ would imply $\{i \in I \mid i \leq \max X\} \sqsubset I$, contradicting the minimality of I. Hence,

$$\bigwedge_{i \in I} a_i \leq_L \bigwedge_{i \in X} a_i \leq_L \bigvee_{i > \max X} a_i \leq_L \bigvee_{i > \max I} a_i ,$$

thus finishing the proof. □

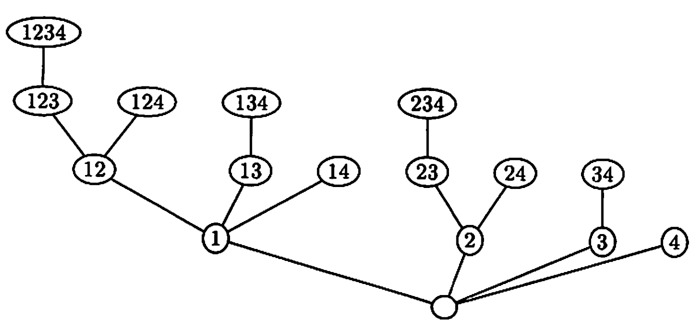

Figure 1: Hasse diagram for $n = 4$

As a special case we deduce the following result of Narushima and Era [4]:

Corollary 1 *Let v be a valuation on a distributive lattice $(L, \wedge, \vee)$, $(S, \curlyvee)$ a finite semilattice, $\{a_s\}_{s\in S} \subseteq L$ and $a_x \wedge a_y \leq_L a_{x\curlyvee y}$ for any $x, y \in S$. Then,*

$$v\left(\bigvee_{s\in S} a_s\right) = \sum_{\substack{I\subseteq S, I\neq\emptyset \\ I \text{ is a chain}}} (-1)^{|I|-1} v\left(\bigwedge_{i\in I} a_i\right).$$

Proof: The corollary follows from Theorem 1 by defining $\mathfrak{X}$ as the set of all incomparable pairs in S and then sorting S topologically. □

3 Set-theoretic identities and logical equivalences of inclusion-exclusion type

We finally deduce some set-theoretic identities and logical equivalences of inclusion-exclusion type. They seem to be new even in the unimproved case, that is, when $\mathfrak{X}$ is chosen as the empty set. (Note that in this case the sum is extended over all non-empty subsets of the index set.)

Theorem 2 *Let $A_1, \ldots, A_n$ be sets, and let $\mathfrak{X}$ be a set of non-empty subsets of $\{1, \ldots, n\}$ such that for any $X \in \mathfrak{X}$, $\bigcap_{x\in X} A_x \subseteq \bigcup_{i>\max X} A_i$. Then,*

$$\bigcup_{i=1}^{n} A_i = \bigoplus_{\substack{I\subseteq\{1,\ldots,n\} \\ I\not\supseteq X\forall X\in\mathfrak{X} \\ I\neq\emptyset}} \bigcap_{i\in I} A_i,$$

where $\oplus$ denotes symmetric difference, defined by $A\oplus B := (A\backslash B)\cup(B\backslash A)$.

Proof: Let L be the Boolean algebra generated by $A_1, \ldots, A_n$. Then, $(L, \oplus)$ is an abelian group, in which every element is its own inverse. Hence, with v being the identity on L, Theorem 2 follows from Theorem 1. □

By choosing $\mathfrak{X} = \emptyset$ (or applying Proposition 1) we obtain the subsequent

Corollary 2 *For any sets $A_1, \ldots, A_n$ the following identity holds:*

$$\bigcup_{i=1}^{n} A_i = \bigoplus_{\substack{I\subseteq\{1,\ldots,n\} \\ I\neq\emptyset}} \bigcap_{i\in I} A_i,$$

In the sequel, let $\leftrightarrow$, $\neg$, $\wedge$, $\vee$, $\boxplus$ stand for logical equivalence, negation, conjunction, disjunction and exclusive disjunction, respectively.

Theorem 3 *Let $F_1, \dots, F_n$ be propositional formulae, and let $\mathfrak{X}$ be a set of non-empty subsets of $\{1, \dots, n\}$ such that for any $X \in \mathfrak{X}$, $\bigvee_{i > \max X} F_i$ is a logical consequence of $\bigwedge_{x \in X} F_x$. Then the following formula is a tautology:*

$$\bigvee_{i=1}^{n} F_i \leftrightarrow \bigboxplus_{\substack{I \subseteq \{1,\dots,n\} \\ I \not\supseteq X \forall X \in \mathfrak{X} \\ I \neq \emptyset}} \bigwedge_{i \in I} F_i .$$

Proof: Let L be the Lindenbaum algebra of propositional logic, and for any classes $[F], [G] \in L$ define $[F] + [G] := [F \boxplus G] = [(\neg F \wedge G) \vee (F \wedge \neg G)]$. Then, as above, $(L, +)$ is an abelian group, in which every element is inverse to itself. Therefore, with $v = \mathrm{id}_L$, Theorem 3 follows from Theorem 1. $\square$

Remark. Theorem 3 can also be deduced from Theorem 2, since by Stone [6], the Lindenbaum algebra is isomorphic to a Boolean algebra of sets.

Corollary 3 *The following propositional formula is a tautology:*

$$\bigvee_{i=1}^{n} F_i \leftrightarrow \bigboxplus_{\substack{I \subseteq \{1,\dots,n\} \\ I \neq \emptyset}} \bigwedge_{i \in I} F_i .$$

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Non-isomorphic Minimal Colorings of K_{4n+3}

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Abstract

We prove that the number of nonisomorphic minimal 2-colorings of the edges of K_{4n+3} is at least $2n$ less than the number of nonisomorphic minimal 2-colorings of the edges of K_{4n+2}, where n is a nonnegative integer. Harary explicitly gave all the nonisomorphic minimal 2-colorings of the edges of K_6. In this paper, we give all the nonisomorphic minimal 2-colorings of the edges of K_7.

1 Introduction and background results

Definition 1.1 *If a graph G with 2-coloring of the edges has the minimum number of monochromatic triangles then that coloring of G is said to be a* **minimal coloring**. *We denote by $M(K_3, G)$ the number of monochromatic triangles in a minimal coloring of G.*

A.W.Goodman [1] has proved the following result regarding $M(K_3, K_n)$, where K_n is the complete graph on n vertices. The same result was proved by Sauve [3] in a more elegant and simple way using the method of weights.

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Proposition 1.2 [**A. W. Goodman**]

$$\begin{aligned} M(K_3, K_n) &= \frac{1}{3}\, t\,(t-1)\,(t-2) && \text{if } n = 2t, \\ &= \frac{2}{3}\, t\,(t-1)\,(4t+1) && \text{if } n = 4t+1, \\ &= \frac{2}{3}\, t\,(t+1)\,(4t-1) && \text{if } n = 4t+3. \end{aligned}$$

Definition 1.3 *Let G be a graph in which the edges are colored with two colors, say, red and blue. Let v be any vertex of G. We define the* **degree pair** *of the vertex v as (s,t) where s is the number of red edges incident at v and t is the number of blue edges incident at v.*

Sauve [3] has proved the following result about the degree pairs of the vertices in a minimal coloring of K_n.

Proposition 1.4 [**L. Sauve**] *A 2-coloring of the edges of K_n is a minimal coloring if and only if the degree pair of*

(1) *any vertex is $(t, t-1)$ or $(t-1, t)$, when $n = 2t$;*

(2) *any vertex is $(2t, 2t)$, when $n = 4t+1$;*

(3) *$4t+2$ vertices are $(2t+1, 2t+1)$ and the degree pair of one exceptional vertex is $(2t, 2t+2)$ or $(2t+2, 2t)$, when $n = 4t+3$.*

Definition 1.5 *Let G be a graph. Suppose $C_1(G)$ and $C_2(G)$ are two 2-coloring of the edges of G. $C_1(G)$ and $C_2(G)$ are said to be* **non-isomorphic** *if and only if the graph defined by one color of $C_1(G)$ is not isomorphic to the graph defined by either color of $C_2(G)$.*

2 2-Colorings of K_{4n+3}

In this section we prove that in any minimal coloring of K_{4n+3}, if we remove a suitable vertex we get a minimal coloring of K_{4n+2} and that a minimal coloring of K_{4n+2} can be extended to a minimal coloring of K_{4n+3} in at most one way. We construct all the 4 non-isomorphic colorings of K_7 using the non-isomorphic minimal colorings of K_6 given by Harary [2].

B. Radhakrishnan Nair and A. Vijayakumar [4] have proved the following theorem about the number of monochromatic triangles incident at a vertex v in any 2-edge coloring of a graph G. Their theorem, in our language is as follows.

Theorem 2.1 *Let G be a complete graph on n vertices in which the edges are colored with two colors, say red and blue and r be the number of red edges in G. For any vertex v of G, let $d(v)$ = the number of red edges incident at v, $N(v)$ = the set of all vertices which are joined to v by red color and $T(v)$ = the number of monochromatic triangles which are incident at v. Then*

$$T(v) = \sum_{u \in N(v)} d(u) \;-\; r \;+\; \frac{1}{2}\,[n - d(v) - 1]\,[n - d(v) - 2].$$

Theorem 2.2 *Any minimal coloring of K_{4n+3} is an extension of a minimal coloring of K_{4n+2} and the number of non-isomorphic minimal colorings of K_{4n+3} is at least $2n$ less than the number of non-isomorphic minimal colorings of K_{4n+2}.*

Proof : Consider K_{4n+3}, where n is a nonnegative integer. Any 2-coloring of the edges of K_{4n+3} will be a minimal coloring if and only if the degree pairs of $4n+2$ vertices are (2n+1,2n+1) and the degree pair of one exceptional vertex, say P is $(2n, 2n+2)$ or $(2n+2, 2n)$ (1.4). The number of monochromatic triangles that lie on the vertex P is $2n^2$ (2.1) and using Goodman's formula (1.2) we get

$$M(K_3, K_{4n+3}) - M(K_3, K_{4n+2}) = 2n^2.$$

So, by removing this vertex P from K_{4n+3} we get a minimal coloring of K_{4n+2}. In other words, any minimal coloring of K_{4n+3} is obtained precisely by extending a minimal coloring of K_{4n+2}.

We consider a minimal coloring of K_{4n+2} and find all possible extensions of this to a minimal coloring of K_{4n+3}. In any minimal coloring of K_{4n+2}, the degree pair of any vertex is $(2n, 2n+1)$ or $(2n+1, 2n)$ (1.4). Hence to extend a minimal coloring of K_{4n+2} to a minimal coloring of K_{4n+3}, we have to add a new vertex P and join this with the vertices of K_{4n+2} in such a way that the degree pair of P is $(2n, 2n+2)$ or $(2n+2, 2n)$ and the degree pairs of all other vertices are $(2n+1, 2n+1)$. It is clear that there is at most one extension possible. In fact, an extension is possible only when exactly $2n$ vertices of K_{4n+2} have degree pair $(2n, 2n+1)$ or exactly $2n$ vertices of K_{4n+2} have degree pair $(2n+1, 2n)$. If $C_1(K_{4n+3})$ and $C_2(K_{4n+3})$ are two minimal colorings of K_{4n+3} which are extensions of two non-isomorphic minimal colorings of K_{4n+2}, then $C_1(K_{4n+3})$ and $C_2(K_{4n+3})$ are also non-isomorphic, for an exceptional vertex must be mapped to an exceptional vertex under any isomorphism.

We claim that there exist $2n$ non-isomorphic minimal colorings of K_{4n+2} which are not extendable to a minimal coloring of K_{4n+3}.

For each integer k such that $0 \le k \le 2n+1$, we construct a minimal coloring C_k of K_{4n+2} such that C_i is non-isomorphic to C_j, for $i \ne j$. Suppose $u_1, u_2, \ldots, u_{2n+1}$ and $v_1, v_2, \ldots v_{2n+1}$ are the vertices of K_{4n+2} and the edges are colored with red and blue. The red edges of C_k are precisely u_iu_j, for $1 \le i \le 2n+1, 1 \le j \le 2n+1, i \ne j$, v_iv_j for $1 \le i \le 2n+1, 1 \le j \le 2n+1, i \ne j$, and if $k > 0$, u_iv_i, for $1 \le i \le k$. It is easy check that C_k is a minimal coloring of K_{4n+2}, for each $0 \le k \le 2n+1$, Also C_ks are all mutually non-isomorphic, because in these colorings all the monochromatic triangles are of red color and the number of red edges are distinct.

The minimal colorings

$$C_0, C_1, \ldots, C_{n-1}, C_{n+2}, C_{n+3}, \ldots, C_{2n+1}$$

of K_{4n+2} are not extendable to get minimal colorings of K_{4n+3}. These are $2n$ in number. Hence the theorem. □

Remark: We note that in Theorem 2.2, the colorings C_u and C_{u+1} are extendable to get minimal colorings of K_{4u+3} and hence there exists at least 2 non-isomorphic minimal colorings of K_{4u+3}.

Theorem 2.3 *There exists precisely four non-isomorphic minimal colorings of K_7.*

Proof : Suppose we color the edges by red and blue. Harary [2] has constructed all the 6 non-isomorphic minimal colorings of K_6 which are given below in Figures 1 through 6 with only the red edges present.

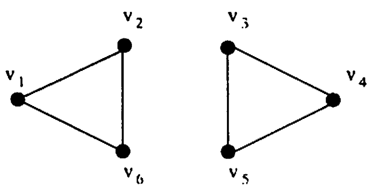

FIGURE 1

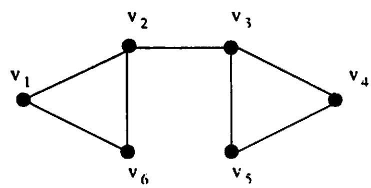

FIGURE 2

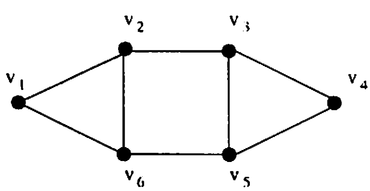

FIGURE 3

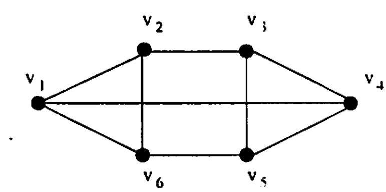

FIGURE 4

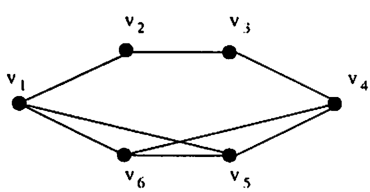

FIGURE 5

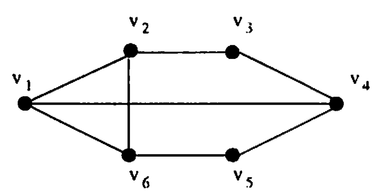

FIGURE 6

By Theorem (2.3) there are at most 4 non-isomorphic minimal colorings of K_7. It is easy to see that the colorings given in figures 2, 3, 5 and 6 are extendable to a minimal coloring of K_7. These are given below in Figures 7 through 10 with only the red edges present.

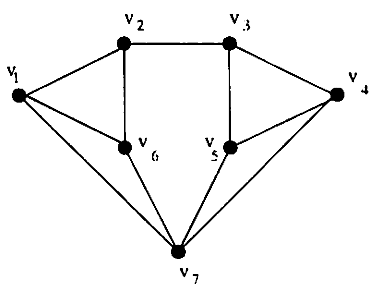

FIGURE 7

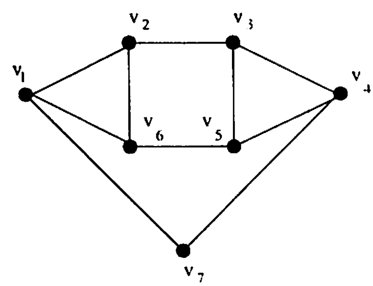

FIGURE 8

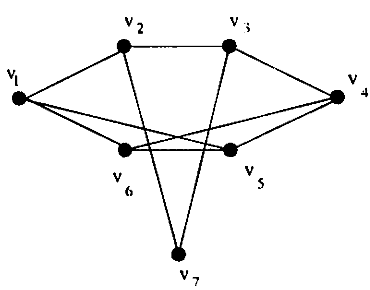

FIGURE 9

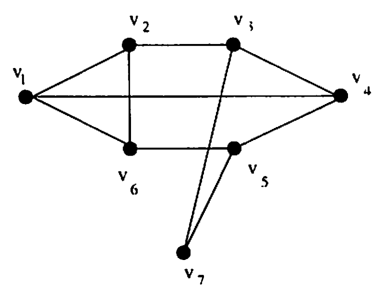

FIGURE 10

Hence the proof. □

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