Contractible Edges and Bowties in a k-Connected Graph

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Abstract

Let G be a k-connected graph and let F be the simple graph obtained from G by removing the edge xy and identifying x and y in such a way that the resulting vertex is incident to all those edges (other than xy) which are originally incident to x or y. We say that e is *contractible* if F is k-connected. *A bowtie* is the graph consisting of two triangles with exactly one vertex in common. We prove that if a k-connected graph G ($k \geq 4$) has no contractible edge, then there exists a bowtie in G.

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1 Introduction

C. Thomassen [3] proved the following result.

Theorem A *Let G be a k-connected graph with no contractible edges. Then G contains a triangle, i.e., K_3.*

W. Mader [2] proved that there exist many triangles in such graphs.

Theorem B *Let G be a k-connected graph with no contractible edges. Then G contains at least $\frac{|V(G)|}{3}$ triangles.*

Recently Kawarabayashi [1] obtained the following nice extension of Thomassen's theorem. Denote by K_4^- the graph obtained from K_4, i.e., the complete graph on 4 vertices, by removing exactly one edge. In other words, K_4^- is the graph consisting of two triangles with exactly two vertex in common.

Theorem C *Let k be an odd integer, $k \geq 3$. If a k-connected graph G has no contractible edges, then G contains a K_4^-.*

Note that if G is a 2-connected graph and not isomorphic to K_3 or if G is a 3-connected graph and not isomorphic to K_4, then it is well known that G has a contractible edge. Thus Theorem A and Theorem B of the above form are meaningful only for $k \geq 4$. In this paper we prove the next theorem which is also an extension of Thomassen's theorem. *A bowtie* is the graph consisting of two triangles with exactly one vertex in common.

Theorem 1 *Let k be an integer, $k \geq 4$. If a k-connected graph G has no contractible edge, then G contains a bowtie.*

Let G be a finite, undirected graph without loops or multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. Let G be a k-connected graph. Let F be the simple graph obtained from G by removing the edge xy and identifying x and y in such a way that the resulting vertex is incident to all those edges (other than xy) which are originally incident to x or y. We say that e is *contractible* if F is k-connected. A set of points S in a connected graph G is *a cutset* if $G - S$ is not connected. A cutset of cardinality k is simply called *k-cut.* Let S be a k-cut of G. Then a fragment of S is a union of at least one, but not all components of $G - S$. Let A be a vertex subset of G. For a vertex

$x \in G - A$, denote by $N_A(x)$ the set of vertices in A which are adjacent to x in G. Let A and B be two disjoint vertex subsets of G. Then denote by $E(A,\ B)$ the set of edges of G joining a vertex of A and a vertex of B. When no confusion is possible we will not distinguish between a set of vertices and the subgraph that it induces.

2 Preliminary Results

Throughout this section, let G be a k-connected graph containing neither contractible edges nor bowties. In what follows we may assume that G is not isomorphic to K_{k+1}. Thus an edge $e = xy$ in a k-connected graph G is not contractible if and only if in G there is a k-cut containing two vertices x and y. For notational convenience, let $K(a,\ b,\ c)$ denote the subgraph of G which is isomorphic to K_3 and whose vertex set is $\{a,\ b,\ c\}$.(That is, $K(a,\ b,\ c)$ is the triangle consisting of three vertices a, b and c.) Let x be a vertex of G. Then the vertex x is called *a good vertex* if every edge but one incident to x is contained in a triangle of G. Let xz be the edge not contained in a triangle of G. Then the edge xz is called a nice edge of G. More precisely, let e be an edge of G which is not contained in a triangle of G. Then the edge e is called *a nice edge* if (at least) one of its end-vertices is a good vertex. We now observe the following.

Lemma 2 *(a) Let x be a vertex of G such that every edge incident to x is contained in a triangle of G. Then there exists a vertex y in $N_G(x)$ such that $N_G(x) \subseteq N_G(y)$.*
(b) Let x be a good vertex of G and let xz be the nice edge of G. Then there exists a vertex y in $N_G(x) - \{z\}$ such that $N_G(x) - \{z\} \subseteq N_G(y)$.

Proof First we shall prove part (a). Set $N_G(x) = \{u_1,\ u_2,\ \ldots,\ u_m\}$ where $m \geq k \geq 4$. Now since every edge incident to x is contained in a triangle and since G contains no bowtie, we may assume that G contains $K(x,\ u_1,\ u_2)$ and $K(x,\ u_1,\ u_3)$. Thus it is easily seen that for any $i \geq 4$ the triangle having the edge xu_i must contain the edge xu_1. The proof of part (a) is completed by setting $y = u_1$. Similarly, we can prove part (b) of the lemma. □

If x is a vertex of G such that every edge incident to x is contained in a triangle of G, then by Lemma 4 there exists a vertex y in $N_G(x)$ such that $N_G(x) \subseteq N_G(y)$. Set $N_G(x) = \{y,\ u_2,\ \ldots,\ u_m\}$. The edge xy is called *a central edge* of x and for $i \geq 2$, xu_i is called *a peripheral edge* of x.

Lemma 3 *Let S be a k-cut of G such that S contains an edge $e = xz$ which is not contained in a triangle of G. Let A be a component of $G - S$ such that $|A| \leq 2$. Then $|A| = 2$, say $A = \{a,\ b\}$, satisfying the following:*

(i) The vertex a is adjacent to exactly one of $\{x, z\}$, say x, and the vertex b is adjacent to z and not to x.
(ii) For each vertex u in $S - \{x, z\}$, u is adjacent to both of $\{a, b\}$.
(iii) Both ax and bz are nice edges of G.

Proof Since G is k-connected, the degree of each vertex of G is at least k. Hence if $|A| = 1$, then e have to be contained in a triangle, so that we have $|A| = 2$, say $A = \{a, b\}$. Since $e = xz$ is not contained in a triangle of G, each vertex of $\{a, b\}$ cannot be adjacent to both x and z. If each vertex of $\{a, b\}$ is adjacent to the same vertex of $\{x, z\}$, say x, then $S - \{z\}$ is a cutset of cardinality $k - 1$, contradicting our assumption that G is k-connected. Thus a is adjacent to exactly one of $\{x, z\}$, say x, and b is adjacent to z and not to x. Moreover, for each vertex u in $S - \{x, z\}$, u is adjacent to both of $\{a, b\}$. In order to establish that both ax and bz are nice edges of G, it suffices to show that neither ax nor bz is contained in a triangle of G. Suppose one of ax and bz, say ax, is contained in a triangle of G, say $K(a, x, w)$, then obviously $w \notin \{b, z\}$, so that $w \in S - \{x, z\}$. Since $k \geq 4$, there is a vertex u in $S - \{x, z, w\}$. However, $K(a, x, w) \cup K(a, b, u)$ is a bowtie of G, a contradiction. This completes the proof of the lemma. □

Lemma 4 *Let S_e and S_f be two distinct k-cuts of G such that A_e is a fragment of $G - S_e$ and $B_e = G - S_e - V(A_e)$, and such that A_f is a fragment of $G - S_f$ and*
$B_f = G - S_f - V(A_f)$.
If $A_e \cap A_f = \emptyset$, $A_e \cap B_f = \emptyset$ and $|A_e \cap S_f| \geq k-1$, then either $|A_f| \leq \frac{k}{2}$ or $|B_f| \leq \frac{k}{2}$.

Proof Since $|S_f| = k$ and $|A_e \cap S_f| \geq k - 1$, we have $|(S_f \cap B_e) \cup (S_e \cap S_f)| = |S_f - A_e| \leq 1$. By symmetry, we may assume that $|S_e \cap A_f| \leq \frac{k}{2}$, so that $|(S_f \cap B_e) \cup (S_e \cap S_f) \cup (S_e \cap A_f)| \leq \frac{k}{2} + 1 < k$.

This implies that $A_f \cap B_e = \emptyset$ because G is k-connected. We conclude that $|A_f| = |S_e \cap A_f| \leq \frac{k}{2}$, which completes the proof. □

Lemma 5 *Let S be a k-cut of G and A be a component of $G - S$. If $|A| \geq 3$, then $|A| \geq k - 1$.*

Proof Let abc be a path of order 3 in A. Set $B = A - \{a, b, c\}$.

Case 1 There is a vertex x in S such that $x \in N_G(a) \cap N_G(b)$.
There is no vertex y in $S \cup A - \{x, a, b, c\}$ such that $y \in N_G(b) \cap N_G(c)$, since otherwise $K(a, b, x) \cup K(b, c, y)$ is a bowtie of G, contradicting the assumption that G contains no bowtie. Thus $|E(S, \{b, c\})| \leq k + 1$. We see that $|N_B(b) \cup N_B(c)| \geq \deg_G(b) + \deg_G(c) - |E(S, \{b, c\})| - 4$. Hence

$|N_B(b) \cup N_B(c)| \geq k-5$ and so $|A| \geq k-2$. Observe that equality holds only when there are two edges cx and ac in G. Since $k \geq 4$, there must be a vertex w in $N_G(a) - \{x, b, c\}$. If $w \in N_G(b)$, then $K(b, c, x) \cup K(a, b, w)$ is a bowtie of G. If $w \in N_G(c)$, then $K(a, b, x) \cup K(a, c, w)$ is a bowtie of G. Thus we obtain $w \notin N_G(b) \cup N_G(c)$ and therefore the number of vertices of A increases by one, implying $|A| \geq k-1$.

Case 2 There is no vertex x in S such that $x \in N_G(a) \cap N_G(b)$.

There are now two subcases to distinguish.

Case 2a There is no K_4^- in A.

Since there is no vertex x in S such that $x \in N_G(a) \cap N_G(b)$, we have $|E(S, \{a, b\})| \leq k$. we also see that $|N_B(a) \cup N_B(b)| \geq \deg_G(a) + \deg_G(b) - |E(S, \{a, b\})| - 4$, since there is at most one triangle having an edge ab in G. Hence $|N_B(a) \cup N_B(b)| \geq k-4$ and so $|A| \geq k-1$.

Case 2b There is a K_4^- in A.

We may assume that there are two triangles $K(a, b, d)$ and $K(b, c, d)$ in A, where d is a vertex in B. Since there is no vertex x in S such that $x \in N_G(a) \cap N_G(b)$, we obtain $|E(S, \{a, b\})| \leq k$. If there is a vertex u in $N_B(a) \cup N_B(b) - \{d\}$, then $K(b, c, d) \cup K(a, b, u)$ is a bowtie of G. Thus we assume that $N_B(a) \cup N_B(b) - \{d\} = \emptyset$. Hence we see that $|N_B(a) \cup N_B(b)| \geq \deg_G(a) + \deg_G(b) - |E(S, \{a, b\})| - 5$ and so $|A| \geq k-2$. Observe that equality holds only when there is the edge ac in G. Since $k \geq 4$, there must be a vertex w in $N_G(c) - \{a, b, d\}$. If $w \in N_G(a)$, then $K(a, c, w) \cup K(a, b, d)$ is a bowtie of G. If $w \in N_G(b)$, then $K(b, c, w) \cup K(a, b, d)$ is a bowtie of G. Thus we obtain $w \notin N_G(a) \cup N_G(b)$ and hence the number of vertices of A increases by one, implying $|A| \geq k-1$. This completes the proof of the lemma. □

Lemma 6 *Let x be a vertex of G such that every edge incident to x is contained in a triangle of G and let xz be a peripheral edge of x. Let S be a k-cut of G such that S contains the edge xz. If A is a component of $G - S$, then $|A| \geq k-1$.*

Proof By Lemma 2, there is a vertex y in $N_G(x)$ such that $N_G(x) \subseteq N_G(y)$. Set $N_G(x) = \{u_1 = y, u_2 = z, \ldots, u_m\}$ $(m \geq k \geq 4)$, where xy is a central edge of x and xu_i is a peripheral edge for $m \geq i \geq 2$. Letting

$B = G - S - A$, we observe that $N_A(x) \neq \emptyset$ and $N_B(x) \neq \emptyset$, since S is a k-cut of a k-connected graph G. This implies that $y \in S$, so that S contains the triangle $K(x, y, z)$. We may assume that $u_3 \in N_A(x)$ and $u_4 \in N_B(x)$. Since G is k-connected, the degree of each vertex of G is at least k. Hence if $|A| = 1$, then $K(x, z, u_3) \cup K(x, y, u_4)$ is a bowtie of G. If $|A| = 2$, say $A = \{a, u_3\}$, then a is adjacent to at least one vertex of x and y, since $\deg_G(a) \geq k$. If there is the edge ax, then $K(a, x, u_3) \cup K(x, y, z)$ is a bowtie of G. If there is the edge ay, then $K(a, y, u_3) \cup K(x, y, z)$ is a bowtie of G. Thus assume that $|A| \geq 3$. It follows from Lemma 5 that $|A| \geq k - 1$, establishing the lemma. □

Lemma 7 *Let x be a good vertex of G and let xz be a nice edge incident to x. Let S be a k-cut of G such that S contains the edge xz. If A is a component of $G - S$, then $|A| \geq k - 1$.*

Proof By Lemma 2, there is a vertex y in $N_G(x) - \{z\}$ such that $N_G(x) - \{z\} \subseteq N_G(y)$. Set $N_G(x) = \{u_1 = y, u_2 = z, \ldots, u_m\}$ where $m \geq k \geq 4$. Letting $B = G - S - A$, we observe that $N_A(x) \neq \emptyset$ and $N_B(x) \neq \emptyset$, since S is a k-cuts of a k-connected graph G. This implies that $y \in S$, so that S contains three vertices x, y and z. We may assume that $u_3 \in N_A(x)$. If $|A| \geq 3$, then it follows from Lemma 5 that $|A| \geq k - 1$. Therefore assume that $|A| \leq 2$. Since S contains the edge $e = xz$ which is not contained in a triangle of G, it follows from Lemma 3 that $|A| = 2$, say $A = \{u_3, b\}$, satisfying the following:

(i) The vertex u_3 is adjacent to x and not to z and b is adjacent to z and not to x.

(ii) For each vertex u in $S - \{x, z\}$, u is adjacent to both of $\{u_3, b\}$.

However, $K(u_3, b, u_4) \cup K(u_3, x, y)$ is a bowtie of G, a contradiction. This completes the proof of the lemma.

□

3 Proof of Theorem 1

Now armed with Lemmas 2-7, we are in the position to give a proof of Theorem 1. We start with the following simple case.

Case 1 There exists a vertex x in G such that every edge incident to x is contained in a triangle of G.

As in Lemma 4, let xy be a central edge of x and so $N_G(x) \subseteq N_G(y)$. Set $N_G(x) = \{y, u_2, \ldots, u_m\}$. For each peripheral edge e of x, let S_e be a k-cut of G containing the edge e and let A_e be a component of $G - S_e$. Choose e, S_e and A_e in such a way that among all such edges, k-cuts and components, A_e is a component minimal with respect to inclusion. Set $B_e = G - S_e - A_e$. We may assume that $e = xu_2$. $N_A(x) \neq \emptyset$, say u_3, since S is a k-cut of a k-connected graph G. Setting $xu_3 = f$, let S_f be a k-cut of G containing the edge f and let A_f be a fragment of $G - S_f$. Set $B_f = G - S_f - A_f$. Note that by Lemma 6, $|A_e|$, $|B_e|$, $|A_f|$, $|B_f| \geq k-1$. We claim that

(i) $A_e \cap A_f = \emptyset$ or $B_e \cap B_f = \emptyset$ and
(ii) $A_e \cap B_f = \emptyset$ or $B_e \cap A_f = \emptyset$.

If $A_e \cap A_f \neq \emptyset$ and $B_e \cap B_f \neq \emptyset$, then $|(S_e \cap A_f) \cup (S_e \cap S_f) \cup (S_f \cap A_e)| = |(S_e \cap B_f) \cup (S_e \cap S_f) \cup (S_f \cap B_e)| = k$, since $|(S_e \cap A_f) \cup (S_e \cap S_f) \cup (S_f \cap A_e)| + |(S_e \cap B_f) \cup (S_e \cap S_f) \cup (S_f \cap B_e)| = |S_e| + |S_f| = 2k$ and since G is k-connected. Notice that $(S_e \cap A_f) \cup (S_e \cap S_f) \cup (S_f \cap A_e)$ is a k-cut containing the edge f. It is easily seen that $|A_e| > |A_e \cap A_f|$, contradicting the assumption that A_e is minimal. So we have $A_e \cap A_f = \emptyset$ or $B_e \cap B_f = \emptyset$. Reasoning in a similar way we obtain $A_e \cap B_f = \emptyset$ or $B_e \cap A_f = \emptyset$ as claimed. In the following argument we will use only (i) and (ii), and will not use the assumption that A_e is a (minimal) component. Therefore, by symmetry we can assume that $A_e \cap A_f = \emptyset$, $A_e \cap B_f = \emptyset$ and so $A_e = A_e \cap S_f$. By Lemma 6, $|A_e \cap S_f| = |A_e| \geq k-1$. Thus by Lemma 4, we have either $|A_f| \leq \frac{k}{2}$ or $|B_f| \leq \frac{k}{2}$, which contradicts the fact that $|A_f| \geq k-1$ and $|B_f| \geq k-1$. This completes the proof of Case 1.

Now assume that for each vertex x of G there is at least one edge e incident to x such that e is not contained in a triangle of G.

Case 2 There exists a good vertex x in G.

Let $e = xz$ be the nice edge incident to x, i.e., the edge not contained in a triangle of G. Let S_e be a k-cut containing the edge e and let A_e be a component of $G - S_e$. Choose e, S_e and A_e in such a way that among all such edges, k-cuts and components, A_e is a component minimal with respect to inclusion. Set $B_e = G - S_e - A_e$. Two@cases are distinguished, depending on whether or not A_e contains a vertex w incident to a nice edge.

Case 2a A_e contains a vertex w incident to a nice edge.

Let f be a nice edge incident to w. (Note that w is not necessarily a good vertex.) We use the same notation as in Case 1. By Lemma 7, $|A_e|$, $|B_e|$, $|A_f|$, $|B_f| \geq k-1$. Thus by the same argument as in Case 1, we can derive a contradiction.

Case 2b A_e contains no vertex incident to a nice edge.

Notice that $|B_e| \geq k-1$ and that S_e is a k-cut containing an edge e such that e is not contained in a triangle of G. Therefore, for each edge e not contained in a triangle of G, let S_e be a k-cut of G containing the edge e and let A_e be a component of $G - S_e$. Choose e, S_e and A_e in such a way that among all such edges, k-cuts and components,

(i) A_e contains no vertex incident to a nice edge and $|B_e| \geq k-1$, where $B_e = G - S_e - A_e$ and
(ii) subject to (i), A_e is a component minimal with respect to inclusion.

(We give a proof along the same lines as that given in Case 1. In order to see this, we use the same notation.) Let w be a vertex in A_e and f be an edge incident to w which is not contained in a triangle of G. Let S_f be a k-cut of G containing the edge f. A_f is a fragment of $G - S_f$ and set $B_f = G - S_f - A_f$. Using the minimal property of A_e, by the same argument as in Case 1, we obtain
(i) $A_e \cap A_f = \emptyset$ or $B_e \cap B_f = \emptyset$ and
(ii) $A_e \cap B_f = \emptyset$ or $B_e \cap A_f = \emptyset$.

First suppose that $B_e \cap A_f = \emptyset$ and $B_e \cap B_f = \emptyset$. Then $B_e \subseteq B_e \cap S_f$ and so $|B_e \cap S_f| = |B_e| \geq k-1$. By Lemma 4, either $|A_f| \leq \frac{k}{2}$ or $|B_f| \leq \frac{k}{2}$, say $|A_f| \leq \frac{k}{2}$. By Lemma 3 and Lemma 5, $|A_f| = 2$ and w is incident to a nice edge of G, since f is contained in S_f and since w is an end-vertex of f such that f is not contained in a triangle of G. This contradicts the assumption that w is not incident to a nice edge of G.

Next suppose that either $A_e \cap A_f = \emptyset$ and $B_e \cap A_f = \emptyset$ or $A_e \cap B_f = \emptyset$ and $B_e \cap B_f = \emptyset$. By symmetry we may assume that $A_e \cap A_f = \emptyset$ and $B_e \cap A_f = \emptyset$. Hence $A_f \subseteq A_f \cap S_e$. If $|A_f| \leq 2$, then it follows from Lemma 3 that by the same argument as above we see that w is incident to a nice edge of G. Thus assume that $|A_f| \geq 3$. By Lemma 5, we have $|A_f| \geq k-1$. By Lemma 4, either $|A_e| \leq \frac{k}{2}$ or $|B_e| \leq \frac{k}{2}$. Since $|B_e| \geq k-1$, we have $|A_e| \leq \frac{k}{2}$. By Lemmas 3 and 5, we know that $|A_f| = 2$ and w is incident to a nice edge of G, a contradiction.

Finally suppose that $A_e \cap A_f = \emptyset$ and $A_e \cap B_f = \emptyset$. Hence $A_e \subseteq A_e \cap S_e$.

If $|A_e| \leq 2$, then it follows from Lemma 3 that by the same argument as above we see that w is incident to a nice edge of G. Thus assume that $|A_e| \geq 3$. By Lemma 5, we have $|A_e| \geq k-1$. By Lemma 4, either $|A_f| \leq \frac{k}{2}$ or $|B_f| \leq \frac{k}{2}$. By symmetry we may assume that $|A_f| \leq \frac{k}{2}$. By Lemma 3 and Lemma 5, we see that $|A_f| = 2$ and w is incident to a nice edge of G, a contradiction.

Case 3 There exists no good vertex in G, i.e., there exists no nice edge in G.

Recall that for each vertex x of G there is at least one edge e incident to x such that e does not contained in a triangle of G. For each edge e not contained in a triangle of G, let S_e be a k-cut of G containing the edge e and let A_e be a component of $G - S_e$. Choose e, S_e and A_e in such a way that among all such edges, k-cuts and components, A_e is a component minimal with respect to inclusion. Set $B_e = G - S_e - A_e$. Let f be an edge not contained in a triangle of G such that f is incident to a vertex of A_e. Let S_f be a k-cut of G containing the edge f and let A_f be a fragment of $G - S_f$. Set $B_f = G - S_f - A_f$. By the same argument as in Case 1, we see that

(i) $A_e \cap A_f = \emptyset$ or $B_e \cap B_f = \emptyset$ and
(ii) $A_e \cap B_f = \emptyset$ or $B_e \cap A_f = \emptyset$.

As in Case 1, by symmetry we can assume that $A_e \cap A_f = \emptyset$, $A_e \cap B_f = \emptyset$ and so $A_e = A_e \cap S_f$. If $|A_e| \leq 2$, then it follows from Lemma 3 that there is a nice edge of G. Thus assume that $|A_e| \geq 3$. By Lemma 5, we have $|A_e| \geq k-1$. By Lemma 4, either $|A_f| \leq \frac{k}{2}$ or $|B_f| \leq \frac{k}{2}$. By symmetry we may assume that $|A_f| \leq \frac{k}{2}$. Again it follows from Lemma 3 and Lemma 5 that $|A_f| = 2$ and there is a nice edge of G, which completes the proof of Theorem 1. □

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