NOTE ON A CONJECTURE FOR GROUP TESTING

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ABSTRACT. Let $M(d,n)$ denote the minimax number of group tests required for the identification of the d defectives in a set of n items. It was conjectured by Hu, Hwang and Wang that $M(d,n) = n-1$ for $n \leq 3d$, a surprisingly difficult combinatorial problem with very little known. The best known result is $M(d,n) = n-1$ for $n \leq \frac{42}{16}d$ by Du and Hwang. In this note we improve their result by proving $M(d,n) = n-1$ for $d \geq 193$ and $n \leq \frac{43}{16}d$.

1. INTRODUCTION

Consider a population of n items consisting of d defectives and $n-d$ good ones. The problem is to identify these d defectives by means of a sequence of group tests. Each test is on a subset of items with two possible outcomes: a *pure* outcome indicates that all items in the subset are good, and a *contaminated* outcome indicates that at least one item in the subset is defective. The problem has applications in high speed computer networks [4], medical examination [1], [2], quantity searching [3], and statistics [1], etc. The reference book written by Du and Hwang [6] offers a clear picture for the development of group testing and its application. Let $M_T(d,n)$ denote the maximum number of tests required by the algorithm T to identify the d defectives in n items, where the maximum is taken over all possible combinations of the d defectives among the n items. Define

$$M(d,n) = \min_T M_T(d,n).$$

Then $M(d,n)$ is the minimax test number for given d and n. We know that $M(n,n) = M(0,n) = 0$. An algorithm which achieves $M(d,n)$ is called a minimax algorithm for the (d,n) problem.

The question studied by Hu, Hwang and Wang [7] is for what values of n and d is it the case that

$$M(d,n) = n-1,$$

achieved by testing the first $n-1$ items one by one. They proved in [7] that

$$M(d,n) = n-1 \quad \text{for} \quad 0 < d < n \leq \lceil \frac{40d+8}{16} \rceil$$

and

$$M(d,n) < n-1 \quad \text{for} \quad n > 3d \geq 3.$$

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In [7], they conjectured that

$$M(d,n) = n - 1 \quad \text{for} \quad 3d \geq n > d > 0,$$

a longstanding problem with but little progress. The best known result was proved in [5] that

$$M(d,n) = n - 1 \quad \text{for} \quad \lfloor \frac{42d}{16} \rfloor \geq n > d > 0,$$

where $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the largest (smallest) integer not greater (less) than x.

In section 3, we will push Du and Hwang's result a little further by proving

$$M(d,n) = n - 1 \quad \text{for} \quad d \geq 193 \quad \text{and} \quad n \leq \lfloor \frac{43d}{16} \rfloor.$$

2. Some preliminary results

To obtain our main result, we need the following basic lemmas. The proofs of Lemmas 2.2 and 2.3 can be found in [7].

Lemma 2.1. *$M(d,n) \leq n-1$ for $n > d > 0$.*

Proof. The individual testing algorithm needs only $n-1$ tests since the state of the last item can be deduced by knowing the states of the other items and knowing d. □

Lemma 2.2. *$M(d,n) \geq min\{n-1, 2l + \lceil \log_2 \binom{n-l}{d-l} \rceil\}$ for $n > d \geq l > 0$, where $\binom{a}{b} = \frac{a!}{(a-b)!b!}$ denotes the binomial.*

Lemma 2.3. *Suppose that $n - d > 1$. Then $M(d,n) = n-1$ implies $M(d,n-1) = n-2$.*

3. The main result

Theorem . *$M(d,n) = n-1$ for $d \geq 193$ and $n \leq \lfloor \frac{43d}{16} \rfloor$.*

Proof. From Lemma 2.3, it suffices to prove Theorem for $n = \lfloor \frac{43d}{16} \rfloor$.

We decompose the proof into sixteen cases. In each case, we will choose an l and prove $2l + \lceil \log_2 \binom{n-l}{d-l} \rceil \geq n-1$ by showing that $\binom{n-l}{d-l}/2^{n-2l-2} > 1$.

Case 1. $d = 16k$. Then $n = \lfloor \frac{43d}{16} \rfloor = 43k$. Set $l = 7k$. Then $n - l = 36k$, $d - l = 9k$, and $n - 2l - 2 = 29k - 2$. For $k \geq 13$, we will prove $2l + \lceil \log_2 \binom{n-l}{d-l} \rceil = 14k + \lceil \log_2 \binom{36k}{9k} \rceil \geq n - 1 = 43k - 1$ by showing that $\binom{n-l}{d-l}/2^{n-2l-2} = \binom{36k}{9k}/2^{29k-2} > 1$ for $k \geq 13$. The theorem then follows from Lemmas 2.1 and 2.2.

For integer $k \geq 0$, define

$$G(k) = \frac{\binom{n-l}{d-l}}{2^{n-2l-2}} = \frac{\binom{36k}{9k}}{2^{29k-2}}$$

and $f(k) = G(k+1)/G(k)$. Then

$$\begin{aligned} f(k) &= \frac{\prod_{i=1}^{36}(36k+i)}{2^{29}\{\prod_{i=1}^{9}(9k+i)\}\{\prod_{i=1}^{27}(27k+i)\}} \\ &= \frac{36^{36}\prod_{i=1}^{36}(k+\frac{i}{36})}{2^{29}9^{9}27^{27}\{\prod_{i=1}^{9}(k+\frac{i}{9})\}\{\prod_{i=1}^{27}(k+\frac{i}{27})\}} \\ &= \frac{c\prod_{i=1}^{36}(k+\frac{3i}{108})}{\{\prod_{i=1}^{9}(k+\frac{12i}{108})\}\{\prod_{i=1}^{27}(k+\frac{4i}{108})\}} \quad \text{(where } c = \frac{36^{36}}{2^{29}9^{9}27^{27}}) \\ &= \frac{c\prod^{36}_{\substack{i=1 \\ i\not\equiv 0 \pmod 4}}(k+\frac{3i}{108})}{\prod_{i=1}^{27}(k+\frac{4i}{108})}. \end{aligned}$$

Define a_i and b_i for $i = 1, 2, \ldots, 27$, such that $A = \{a_1 < a_2 < \cdots < a_{27}\} = \{\frac{3i}{108} \mid i = 1, 2, \ldots, 36 \text{ and } i \not\equiv 0 \pmod 4\}$ and $B = \{b_1 < b_2 < \cdots < b_{27}\} = \{\frac{4i}{108} \mid i = 1, 2, \ldots, 27\}$. Clearly,

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_i < b_i < a_{i+1} < b_{i+1} < \cdots < a_{27} < b_{27}.$$

Thus, for $i = 2, 3, \ldots, 27$, we have that $\frac{k+a_i}{k+b_{i-1}} > 1$. Therefore

$$f(k) = \frac{c\prod_{i=1}^{27}(k+a_i)}{\prod_{i=1}^{27}(k+b_i)} > \frac{c(k+a_1)}{k+b_{27}} = \frac{c(k+\frac{3}{108})}{k+1}.$$

Set $h(k) = \frac{c(k+\frac{3}{108})}{k+1}$. Since $c = \frac{36^{36}}{2^{29}9^{9}27^{27}} \doteqdot 1.153$ and $h(x)$ is increasing, we obtain that $h(k) > 1$ for $k \geq 7$. This implies that $G(k)$ is increasing for $k \geq 7$. Furthermore, $G(13) \doteqdot 1.08948 > 1$, hence we have $G(k) > 1$ for $k \geq 13$.

As the proofs for the other fifteen cases are identical to case 1 with different parameter values, we only give the values of the parameters in each case without further details.

Case 2. $d = 16k+1$, $n = 43k+2$, $l = 7k+1$ and $G(k) = \frac{\binom{36k+1}{9k}}{2^{29k-2}} > 1$ for $k \geq 10$.

Case 3. $d = 16k+2$, $n = 43k+5$, $l = 7k+1$ and $G(k) = \frac{\binom{36k+4}{9k+1}}{2^{29k+1}} > 1$ for $k \geq 11$.

Case 4. $d = 16k+3$, $n = 43k+8$, $l = 7k+1$ and $G(k) = \frac{\binom{36k+7}{9k+2}}{2^{29k+4}} > 1$ for $k \geq 12$.

Case 5. $d = 16k+4$, $n = 43k+10$, $l = 7k+2$ and $G(k) = \frac{\binom{36k+8}{9k+2}}{2^{29k+4}} > 1$ for $k \geq 9$.

Case 6. $d = 16k+5$, $n = 43k+13$, $l = 7k+3$ and $G(k) = \frac{\binom{36k+10}{9k+2}}{2^{29k+5}} > 1$ for $k \geq 11$.

Case 7. $d = 16k+6$, $n = 43k+16$, $l = 7k+3$ and $G(k) = \frac{\binom{36k+13}{9k+3}}{2^{29k+8}} > 1$ for $k \geq 12$.

Case 8. $d = 16k+7$, $n = 43k+18$, $l = 7k+3$ and $G(k) = \frac{\binom{36k+15}{9k+4}}{2^{29k+10}} > 1$ for $k \geq 9$.

Case 9. $d = 16k+8$, $n = 43k+21$, $l = 7k+4$ and $G(k) = \frac{\binom{36k+17}{9k+4}}{2^{29k+11}} > 1$ for $k \geq 10$.

Case 10. $d = 16k+9$, $n = 43k+24$, $l = 7k+4$ and $G(k) = \frac{\binom{36k+20}{9k+5}}{2^{29k+14}} > 1$ for $k \geq 11$.

Case 11. $d = 16k+10$, $n = 43k+26$, $l = 7k+5$ and $G(k) = \frac{\binom{36k+21}{9k+5}}{2^{29k+14}} > 1$ for $k \geq 8$.

Case 12. $d = 16k+11$, $n = 43k+29$, $l = 7k+5$ and $G(k) = \frac{\binom{36k+24}{9k+6}}{2^{29k+17}} > 1$ for $k \geq 10$.

Case 13. $d = 16k+12$, $n = 43k+32$, $l = 7k+5$ and $G(k) = \frac{\binom{36k+27}{9k+7}}{2^{29k+20}} > 1$ for $k \geq 11$.

Case 14. $d = 16k+13$, $n = 43k+34$, $l = 7k+6$ and $G(k) = \frac{\binom{36k+28}{9k+7}}{2^{29k+20}} > 1$ for $k \geq 8$.

Case 15. $d = 16k+14$, $n = 43k+37$, $l = 7k+7$ and $G(k) = \frac{\binom{36k+30}{9k+7}}{2^{29k+21}} > 1$ for $k \geq 9$.

Case 16. $d = 16k+15$, $n = 43k+40$, $l = 7k+7$ and $G(k) = \frac{\binom{36k+33}{9k+8}}{2^{29k+24}} > 1$ for $k \geq 10$.

□

Remark . From the proof of Theorem, we actually prove that $M(d, \lfloor \frac{43d}{16} \rfloor) = \lfloor \frac{43d}{16} \rfloor - 1$ for $d \geq 138$ and $d \notin \{$ 139, 140, 142, 143, 144, 145, 146, 147, 149, 150, 152, 153, 155, 156, 159, 160, 162, 163, 165, 166, 169, 172, 176, 179, 182, 192$\}$.

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