Laplacian Eigenvalues and the Excess of a Graph

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Abstract

In this work Γ denotes a finite, simple and connected graph. The k-excess $e_k(H)$, of a set $H \subseteq V(\Gamma)$ is defined as the cardinality of the set of vertices that are at distance greater than k of H, and the k-excess $e_k(h)$ of all h-subsets of vertex is defined as

$$e_k(h) = \max_{H \subset V(\Gamma), |H|=h} \{e_k(H)\}.$$

The k-excess e_k of the graph is obtained from $e_k(h)$ when $h = 1$. Here we obtain upper bounds for $e_k(h)$ and e_k in terms of the Laplacian eigenvalues of Γ.

Keywords: Laplacian eigenvalues, Laplacian matrix, Excess
AMS Subject Classification numbers: 05C50; 05C12; 05A20; 15A42

1 Introduction

Recently, several results bounding metric parameters of a graph using the eigenvalues of either the adjacency matrix or the Laplacian matrix of the graph have been published. In this context, parameters such as mean distance, diameter, radius, isoperimetric number, magnifying constant and excess have been extensively studied. See, for instance, the papers by Alon [1], Biggs [2], Chung, Faber and Manteuffel [3], Delorme and Tillich [6], Fiol, Garriga and the second author of this paper [7, 8, 9, 10], Fiol and Garriga [11, 12, 13], Mohar [15, 16], van Dam and Hammer [5] and the authors [17, 18, 19]. Here we investigate the relation between the excess and the Laplacian spectrum of a graph.

Let $\Gamma = (V, E)$ be a simple and connected graph, of order $|V(\Gamma)| = n$. The *distance* between two vertices $v_i, v_j \in V(\Gamma)$ is denoted by $\partial(v_i, v_j)$. The *Laplacian matrix* of Γ is the matrix $\mathbf{L} = \mathbf{O} - \mathbf{A}$, where $\mathbf{A}$ is the adjacency matrix of Γ and $\mathbf{O} = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n,)$ is the diagonal matrix with vertex degrees

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δ_i on the diagonal. Alternatively, the Laplacian matrix can be defined as $\mathbf{L} = \mathbf{C}\mathbf{C}^{\top}$, where $\mathbf{C}$ is the incidence matrix of an orientation of Γ. A comprehensive survey of the properties and applications of the Laplacian matrix can be found in Mohar [15]. For instance, $\mathbf{L}$ has eigenvalues $\mu_0 = 0 < \mu_1 < \cdots < \mu_b$, and the (simple) eigenvalue $\mu_0 = 0$ has eigenvector $\mathbf{j} = (1, 1, \ldots, 1)$, regardless of the graph being regular or not.

As usual, we identify the Laplacian matrix $\mathbf{L}$ with an endomorphism of the "vertex-space" of Γ, $l^2(V)$ which, for any given indexing of the vertices, is isomorphic to $\mathbb{R}^n$. Thus, for any vertex $v_i \in V(\Gamma)$, e_i will denote the corresponding unit vector of the canonical base of $\mathbb{R}^n$. A polynomial in the vector space of real polynomials with degree at most k, $P \in \mathbb{R}_k[x]$, will operate on $\mathbb{R}^n$ by the rule $Pw := P(\mathbf{L})w$, where $w \in \mathbb{R}^n$.

We define, for any $k = 0, 1, \ldots, D(\Gamma)$, the *k-excess* of a vertex $u \in V(\Gamma)$, denoted by $\mathrm{e}_k(u)$, as the number of vertices which are at distance greater than k from u. That is,

$$\mathrm{e}_k(u) = |\{v \in V : \partial(u, v) > k\}|.$$

Then, trivially, $\mathrm{e}_0(u) = n - 1$, $\mathrm{e}_{D(\Gamma)}(u) = \mathrm{e}_{\epsilon(u)}(u) = 0$ and $\mathrm{e}_k(u) = 0$ if and only if $\epsilon(u) \leq k$, where $\epsilon(u)$ denote the eccentricity of u. The name "*excess*" is borrowed from Biggs [2], in which he gives a lower bound, in terms of the adjacency eigenvalues of a graph, for the excess $\mathrm{e}_r(u)$ of any vertex u in a δ-regular graph with odd girth. The excess of a vertex was studied by Fiol and Garriga [12] using the adjacency eigenvalues of a graph, and the authors [17] using the Laplacian eigenvalues.

The *k-excess* of Γ, denoted by e_k, is defined as

$$\mathrm{e}_k = \max_{v_i \in V(\Gamma)} \{\mathrm{e}_k(v_i)\}$$

and the *k-excess* of a subset $H \subseteq V(\Gamma)$, which we denote by $\mathrm{e}_k(H)$, is defined as

$$\mathrm{e}_k(H) = |\{v \in V(\Gamma) : \partial(v, H) > k\}|.$$

Moreover, the *k-excess* of all h-subsets of $V(\Gamma)$ denoted by $\mathrm{e}_k(h)$ is defined as

$$\mathrm{e}_k(h) = \max_{H \subseteq V(\Gamma), |H| = h} \{\mathrm{e}_k(H)\}.$$

The parameter $\mathrm{e}_k(H)$ ($H \subseteq V(\Gamma)$) was studied by the authors in [17] using the so-called H-Laplacian spectra of a subset $H \subseteq V(\Gamma)$. In Section 3 we obtain upper bounds for the k-excess $\mathrm{e}_k(h)$ of all h-subsets of $V(\Gamma)$ and, in particular, for the k-excess e_k of Γ in terms of the Laplacian eigenvalues of Γ.

We begin recalling some known results. Fiol, Garriga and the second author of this paper [8], by using the k-alternating polynomial P_k associated to the mesh of the adjacency eigenvalues of Γ, showed that the number $\sigma_k(h)$ of vertices which are at distance $\geq k$ from a given subset $H \subset V(\Gamma)$ of cardinality $|H| = h$ is bounded above by

$$\sigma_k(h) \leq \left\lfloor \frac{\|\nu\|^2(\|\nu\|^2 - h)}{h(P_{k-1}^2(\lambda) - 1) + \|\nu\|^2} \right\rfloor, \quad 1 \leq k \leq D(\Gamma), \tag{1}$$

where λ is the largest adjacency eigenvalue of Γ and ν is the (positive) eigenvector associated to λ with minimum component 1. Note that $e_{k-1}(h) = \sigma_k(h)$ therefore

$$e_k(h) \leq \left\lfloor \frac{\|\nu\|^2(\|\nu\|^2 - h)}{h(P_k^2(\lambda_0) - 1) + \|\nu\|^2} \right\rfloor. \tag{2}$$

Van Dam [4] showed that if Γ is a graph whose diameter coincides with the number of non null Laplacian eigenvalues, $D(\Gamma) = b$, and if $V_1, V_2 \subset V(\Gamma)$ are sets of vertices of equal cardinality $|V_1| = |V_2| = t$, such that $\partial(V_1, V_2) = b$, then

$$t \leq \left\lfloor \frac{n}{c+1} \right\rfloor \quad \text{where} \quad c = \sum_{j=1}^{b} \prod_{i \neq 0,j}^{b} \frac{\mu_i}{\mid \mu_j - \mu_i \mid}. \tag{3}$$

He also showed that the number $\sigma_b(v)$ of vertices at distance b from an arbitrary vertex v are bounded above by

$$\sigma_b(v) \leq \left\lfloor \frac{n(n-1)}{c^2 + n - 1} \right\rfloor. \tag{4}$$

On the other hand, Haemers [14] showed that if X and Y are disjoint sets of vertices of Γ, such that there is no edge between X and Y, then

$$\frac{\mid X \mid\mid Y \mid}{(n- \mid X \mid)(n- \mid Y \mid)} \leq \left(\frac{\mu_b - \mu_1}{\mu_b + \mu_1}\right)^2. \tag{5}$$

Our main tool throughout this paper are the k-alternating polynomials and the Laplacian polynomials, so we begin recalling some of its main properties.

2 The Tools

In [7] Fiol, Garriga and the second author of this paper defined and studied some properties of the k-alternating polynomials. Those polynomials can be defined as follows: let $\mathcal{M} = \{\mu_1 < \cdots < \mu_b\}$ be a mesh of b real numbers. For any $k = 0, 1, ..., b-1$ let Q_k be the *k-alternating polynomial* associated to $\mathcal{M}$. That is, the polynomial of $\mathbb{R}_k[x]$ with norm $\|Q_k\|_\infty = \max_{1 \leq i \leq b}\{|Q_k(\mu_i)|\} \leq 1$, such that

$$Q_k(\mu) = \sup\{P(\mu) : P \in \mathbb{R}_k[x],\ \|P\|_\infty \leq 1\}$$

where μ is any real number smaller than μ_1. In [7] it was shown that, for any $k = 0, 1, ..., b-1$,

- There is an unique Q_k which, moreover, is independent of the value of $\mu (< \mu_1)$;
- Q_k has degree k;
- $Q_0(\mu) = 1 < Q_1(\mu) < \cdots < Q_{b-1}(\mu)$.

In particular, for $\mu = 0$, we have

$$Q_1(0) = \frac{\mu_b + \mu_1}{\mu_b - \mu_1}; \tag{6}$$

and

$$Q_{b-1}(0) = \sum_{j=1}^{b} \prod_{i \neq 0, j}^{b} \frac{\mu_i}{|\mu_j - \mu_i|}. \tag{7}$$

Moreover, it was shown that Q_k takes $k+1$ *alternating* values ± 1 at the mesh points.

In this work, we also use the *h-Laplacian polynomial* (see [20]) defined as follows: Let $\mu_0 = 0 < \mu_1 < \cdots < \mu_b$ be the Laplacian eigenvalues of $\Gamma = (V, E)$. For each $k = 0, ..., b$, the mapping $\|\ \|_h$: $\mathbb{R}_k[x] \mapsto \mathbb{R}$ defined by

$$\|P\|_h = \max_{H \subseteq V(\Gamma), |H| = h} \{\|P(\mathbf{L})u\|\}, \text{ where, } u = \sum_{v_i \in H} e_i$$

is a norm of $\mathbb{R}_k[x]$. We consider the closed unit ball

$$B_k = \{P \in \mathbb{R}_k[x] : \|P\|_h \leq 1\}.$$

On this compact set, the linear continuous function $P \mapsto P(0)$ attains its maximum at a point $q_k^{(h)}$, which we call *h-Laplacian polynomial.* Notice that, such a point must be on the border of B_k; that is, $\|q_k^{(h)}\|_h = 1$. If $h = 1$ these polynomials will be called Laplacian polynomials and denoted by q_k.

In general, Laplacian polynomials are difficult to calculate. Some particular cases of these polynomials are calculated in [20]. We emphasize the following cases.

- δ-regular graphs:

$$q_1(x) = -\frac{\sqrt{\delta+1}}{\delta+1} x + \sqrt{\delta+1}; \tag{8}$$

- non-regular graphs with minimum degree ρ and maximum degree Δ:

$$q_1(x) = -\frac{2x}{\sqrt{(1+\rho+\Delta)^2 - 4\rho\Delta}} + \frac{1+\rho+\Delta}{\sqrt{(1+\rho+\Delta)^2 - 4\rho\Delta}}; \tag{9}$$

- walk-regular graphs: Let $Spec(\mathbf{L}) = \{0, \mu_1^{m_1}, ..., \mu_b^{m_b}\}$ be the Laplacian spectrum of a walk-regular graph and let $q_r(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_r x^r$ be the Laplacian polynomial of degree r. Then the polynomial q_r can be computed by solving the following optimization problem (see [17, 20])

$$\text{maximize} \quad \alpha_0$$

$$\text{subject to} \quad \alpha_0^2 + \sum_{l=1}^{b} m_l(\alpha_0 + \alpha_1\mu_l + \cdots + \alpha_k\mu_l^k)^2 = n.$$

We recall that a graph is *walk-regular* if for every k the number of walks of length k with both endpoints at v does not depend on v. In other words, any power $\mathbf{A}^k$, of the adjacency matrix, has all its diagonal entries equal to $Tr(\mathbf{A}^k)/n$ (see [6]). This class of walk-regular graphs contains the class of vertex-transitive graphs and the class of distance-regular graphs.

3 Main Results

The main results of this paper are deduced from the following basic theorem.

Theorem 1 *Let Γ be a graph of order n. Let Q_k be the k-alternating polynomial associated to the mesh of the Laplacian eigenvalues of Γ, and let $q_r^{(h)}$ be a h-Laplacian polynomial of degree r, then,*

$$\mathbf{e}_{k+r}(h) \leq \left\lfloor \frac{n\left(n - h^2\left(q_r^{(h)}(0)\right)^2\right)}{h^2\left(q_r^{(h)}(0)\right)^2(Q_k^2(0)-1)+n} \right\rfloor$$

Proof: Let $v_1, v_2, \ldots, v_n$ be a labeling of the vertex set $V(\Gamma)$ of Γ. Let $H, S \subset V(\Gamma)$ be such that $|H| = h$, let $|S| = s$, and let

$$u = \sum_{v_i \in H} e_i \quad \text{and} \quad f = \sum_{v_i \in S} e_i$$

be vectors associated to H and S where e_i is the ith unit vector of $\mathbb{R}^n$. Using the following decompositions of the vectors u and f

$$u = \frac{h}{n}\mathbf{j} + z_u; \quad f = \frac{s}{n}\mathbf{j} + z_f, \quad (\text{where} \quad z_u, z_f \in \mathbf{j}^{\perp})$$

we obtain

$$q_r^{(h)}u = \frac{h}{n}q_r^{(h)}(0)\mathbf{j} + q_r^{(h)}z_u \Rightarrow \|q_r^{(h)}u\|^2 = \frac{h^2}{n}\left(q_r^{(h)}(0)\right)^2 + \|q_r^{(h)}z_u\|^2 \leq 1,$$

then

$$\|q_r^{(h)}z_u\| \leq \frac{1}{\sqrt{n}}\sqrt{n - h^2\left(q_r^{(h)}(0)\right)^2}.$$

Moreover,

$$\|f\|^2 = \frac{s^2}{n} + \|z_f\|^2 \Rightarrow \|z_f\| = \sqrt{s - \frac{s^2}{n}}.$$

If for two vertices $v_i, v_j \in V(\Gamma)$ we have $\partial(v_i, v_j) > k$ then $(\mathbf{L}^k)_{ij} = 0$. Hence,

$$\partial(H,S) > k + r \Rightarrow 0 = \langle Q_k q_r^{(h)}u, f\rangle = \frac{hs}{n}Q_k(0)q_r^{(h)}(0) + \langle Q_k q_r^{(h)}z_u, z_f\rangle$$

$$= \frac{hs}{n}Q_k(0)q_r^{(h)}(0) + \langle q_r^{(h)}z_u, Q_k z_f\rangle.$$

Then, by the Cauchy-Schwarz inequality

$$\frac{hs}{n} Q_k(0) q_r^{(h)}(0) \leq \|q_r^{(h)} z_u\| \|Q_k z_f\|$$

$$\leq \|q_r^{(h)} z_u\| \|Q_k\|_\infty \|z_f\|$$

$$= \frac{1}{n} \sqrt{n - h^2 \left(q_r^{(h)}(0)\right)^2} \sqrt{s(n-s)}.$$

So, solving for $e_{k+r}(h) = s$ the result follows. ■

In particular, for $r = 0$ in Theorem 1, we obtain the Laplacian analogous of (2). That is,

Corollary 2 *Let Q_k be the k-alternating polynomial associated to the mesh of the Laplacian eigenvalues of a graph Γ of order n. Then*

$$e_k(h) \leq \left\lfloor \frac{n(n-h)}{h(Q_k^2(0) - 1) + n} \right\rfloor.$$

In the case $k = 0$ we have $Q_0(x) \equiv 1$. Thus, the above corollary leads to the following bound: $e_0(h) \leq n - h$. Obviously, this bound coincides with the exact value of the excess.

As a particular case of the above corollary, when $k = 1$, $h = |X|$ and $e_1(h) = |Y|$, we obtain again the inequality (5) established by Haemers [14] using the eigenvalue interlacing.

In the case $h = 1$ Corollary 2 leads to the following bound for the excess of Γ

$$e_k \leq \left\lfloor \frac{n(n-1)}{Q_k^2(0) + n - 1} \right\rfloor. \tag{10}$$

When $k = b - 1$ in the above corollary we have that all eigenvalues are explicitly involved in the bound of the number $\sigma_b(h)$ of vertices which are at distance b from any subset $H \subset V(\Gamma)$ of cardinality h

$$\sigma_b(h) \leq \left\lfloor \frac{n(n-h)}{h(c^2 - 1) + n} \right\rfloor \quad \text{where} \quad c = \sum_{j=1}^{b} \prod_{i \neq 0,j}^{b} \frac{\mu_i}{|\mu_j - \mu_i|}. \tag{11}$$

Taking $k = b - 1$ in (10) or $h = 1$ in (11) we deduce the inequality (4) given by van Dam [4] by using the eigenvalue interlacing.

Considering $e_k(h) = h$ in Corollary 2, we generalize the bound (3) given by van Dam [4]. That is, we show that the cardinality $h = |H_1| = |H_2|$ of two subsets $H_1, H_2 \subset V(\Gamma)$ such that $\partial(H_1, H_2) > k$ satisfies

$$h \leq \left\lfloor \frac{n}{Q_k(0) + 1} \right\rfloor. \tag{12}$$

The bound (12) is tight for different values of k, as we can see in the following families of graphs:

1. First, we shall need the following binary graph operations. Let Γ_1 and Γ_2 be vertex disjoint graphs. Denote by $\Gamma_1 \cup \Gamma_2$ their union, and let $\Gamma_1 * \Gamma_2$ be their join (obtained from $\Gamma_1 \cup \Gamma_2$ by joining every vertex of Γ_1 with every vertex of Γ_2). The Laplacian characteristic polynomial $\phi(\Gamma, \mu)$ of the resulting graph can be expressed in terms of the Laplacian characteristic polynomial of Γ_1 and Γ_2 as follows:

$$\phi(\Gamma_1 \cup \Gamma_2, \mu) = \phi(\Gamma_1, \mu)\phi(\Gamma_2, \mu),$$

$$\phi(\Gamma_1 * \Gamma_2, \mu) = \frac{\mu(\mu - n_1 - n_2)}{(\mu - n_1)(\mu - n_2)}\phi(\Gamma_1, \mu - n_2)\phi(\Gamma_2, \mu - n_1),$$

where $|V(\Gamma_i)| = n_i$, for $i = 1, 2$. Consider the class of graphs of the form $\Gamma = (\Gamma_1 \cup \Gamma_1) * \Gamma_2$, such that $n_2 \leq 2n_1$. By the above formulas we have that $\mu_b(\Gamma) = 2n_1 + n_2$ and $\mu_1(\Gamma) = n_2$. Thus, by (6) and (12) we have that the cardinality $h = |H_1| = |H_2|$ of two subsets $H_1, H_2 \subset V(\Gamma)$ such that $\partial(H_1, H_2) > 1$ satisfies $h \leq n_1$.

2. A graph Γ is *r-antipodal* if the relation "$u \sim v$ if and only if $\partial(u, v) = D(\Gamma)$" is an equivalence relation and each equivalence class has exactly r vertices. For r-antipodal distance regular graphs we have $Q_{b-1}(0) = 2n/r - 1$ (see [9]), then for two subsets $H_1, H_2 \subset V(\Gamma)$ of cardinality h such that $\partial(H_1, H_2) = b$ we have $h \leq r/2$.

3. For the class of Laplacian k-boundary graphs of diameter $k + 1$ we have $Q_k(0) = n - 1$ (see [20]), so that (12) gives $h \leq 1$, that is, in any such graph every diametral vertex has a unique opposite vertex.

For $h = 1$ in Theorem 1 we obtain a bound for the excess of Γ

Corollary 3 *Let Q_k be the k-alternating polynomial associated to the mesh of the Laplacian eigenvalues of a graph Γ of order n, and let q_r be a Laplacian polynomial of degree r, then,*

$$\mathrm{e}_{k+r} \leq \left\lfloor \frac{n\left(n - (q_r(0))^2\right)}{(q_r(0))^2 (Q_k^2(0) - 1) + n} \right\rfloor .$$

As a particular case of this corollary, the following results are obtained:
When $k = 1$ we have

$$\mathrm{e}_{r+1} \leq \left\lfloor \frac{n\left(n - (q_r(0))^2\right)(\mu_b - \mu_1)}{4(\mu_b - \mu_1)(q_r(0))^2 + n(\mu_b - \mu_1)} \right\rfloor . \tag{13}$$

When $r = 1$ and Γ is a non-regular graph whose minimum degree and maximum degree are respectively ρ and Δ, by (9) we have

$$e_{k+1} \le \left\lfloor \frac{n\left((1+\rho+\Delta)^2(n-1) - 4n\rho\Delta\right)}{(1+\rho+\Delta)^2(Q_k^2(0)+n-1) - 4n\rho\Delta} \right\rfloor . \tag{14}$$

When $r = 1$ and Γ is a δ-regular graph, by (8) we have

$$e_{k+1} \le \left\lfloor \frac{n(n-\delta-1)}{(\delta+1)(Q_k^2(0)-1)+n} \right\rfloor . \tag{15}$$

For $k = 0$ in Theorem 1 we obtain a bound for the excess of all h-subsets of $V(\Gamma)$:

Corollary 4 *Let $q_r^{(h)}$ be a h-Laplacian polynomial of degree r. Then*

$$e_r(h) \le \left\lfloor n - h^2\left(q_r^{(h)}(0)\right)^2 \right\rfloor .$$

As a particular case of this corollary, when $h = 1$, we have

$$e_r \le \left\lfloor n - (q_r(0))^2 \right\rfloor . \tag{16}$$

To discuss the tightness of this bound, we consider as example the Petersen graph whose spectrum is $Spec(\mathbf{L}) = \{0, 2^5, 5^4\}$, from which we get $q_0(x) \equiv 1$, $q_1(x) = -x/2 + 2$ and $q_2(x) = \sqrt{10}/10(x-2)(x-5)$. Therefore (16) gives $e_0 \le 9$, $e_1 \le 6$ and $e_2 \le 0$. These bounds coincides with the exact values of the excess.

Several applications of these results to bound the mean distance of a graph, bandwidth and vertex separator sets can be found in [18, 19].

Some of the obtained results are generalizations of well-known results and some others are tight bounds. All eigenvalues are involved in the bounds. Thus, the drawback of these results is that the bounds are not explicit.

References

[1] N. Alon, Eigenvalues and expanders, *Combinatorica* **6** (1986) 85-89.

[2] N. Biggs, Girth, valency and excess, *Linear Algebra Appl.* **31** 55-59.

[3] F.R.K. Chung, V. Faber and Manteuffel, An upper bound on the diameter of a graph from eigenvalues associated with its Laplacian, *SIAM J. Disc. Math.* 7 No. 3 (1994) 443-457.

[4] E. R. van Dam, Bounds on special subsets in graphs, eigenvalues and association schemes. *J. Algebraic Combin.* **7** (3) (1998) 321-332.

[5] E. R. van Dam and W. H. Haemers, Eigenvalues and the Diameter of graphs, *Linear Multilinear Algebra* **39** (1995) 33-44.

[6] C. Delorme and J.P. Tillich, Eigenvalues, eigenspaces and distances to subsets, *Discrete Mathematics* 165-166 (1997) 161-184.

[7] M.A. Fiol, E. Garriga and J.L.A. Yebra, On a class of polynomials and its relation with the spectra and diameters of graphs, *J. Combin. Theory Ser. B* **67** (1996) 48-61.

[8] M.A. Fiol, E. Garriga and J.L.A. Yebra, The alternating polynomial and their relation with the spectra and conditional diameters of graphs, *Discrete Mathematics* **167/168** (1997) 297-307.

[9] M.A. Fiol, E. Garriga and J.L.A. Yebra, From Regular Boundary Graphs to Antipodal distance regular graphs, *J. Graph Theory* **27** (1998) 123-140.

[10] M.A. Fiol, E. Garriga and J.L.A. Yebra, Locally pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* **68** (1996) 179-205.

[11] M.A. Fiol and E. Garriga , The alternating and adjacency polynomials, and their relation with the spectra and diameters of graphs. *Discrete Appl. Math.,* **87** (1-3) (1998) 77-97.

[12] M.A. Fiol and E. Garriga , From local adjacency polynomials to locally pseudo-distance-regular-graphs. *J. Combin. Theory Ser. B* **71** (1997) 162-183.

[13] M.A. Fiol and E. Garriga , On the algebraic theory of pseudo-distance-regularity around a set. *Linear Algebra Appl.,* **298** (1-3) (1999) 115-141.

[14] W.H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* 226/228 (1995) 593-616.

[15] B. Mohar, The Laplacian spectrum of Graphs, in *Graph Theory, Comb. and Applications* (1991) 871-898.

[16] B. Mohar, Isoperimetric Numbers of Graphs, *J. Combin. Theory Ser. B* **47** (1989) 274-291.

[17] J. A. Rodríguez y J. L. A. Yebra, Cotas espectrales del exceso y la excentricidad de un conjunto de vértices de un grafo. *EAMA-97* (Sevilla,1997) 383-390.

[18] J. A. Rodríguez and J. L. A. Yebra, Bounding the diameter and the mean distance of a graph from its eigenvalues: Laplacian versus adjacency matrix methods. *Discrete Mathematics* **196** (1999) 267-275.

[19] J.A. Rodríguez, A. Gutiérrez and J. L. A. Yebra, On spectral bounds for cutsets. *Discrete Mathematics.* To appear.

[20] J. A. Rodríguez, *Cotas de diversos parámetros de un grafo a partir de los autovalores de su Matriz Laplaciana.* Ph.D. Thesis, Universitat Politècnica de Catalunya, 1997.