Triangle-Free Polyconvex Graphs

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Abstract

The notion of convexity in graphs is based on the one in topology: a set of vertices S is convex if an interval is entirely contained in S when its endpoints belong to S. The order of the largest proper convex subset of a graph G is called the convexity number of the graph and is denoted $\text{con}(G)$. A graph containing a convex subset of one order need not contain convex subsets of all smaller orders. If G has convex subsets of order m for all $1 \leq m \leq \text{con}(G)$, then G is called polyconvex. In response to a question of Chartand and Zhang [3], we show that, given any pair of integers n and k with $2 \leq k < n$, there is a connected triangle-free polyconvex graph G of order n with convexity number k.

Introduction

Convexity in graphs is analogous to topological convexity. Buckley and Harary [1], Harary and Nieminen [4], and Chartrand and Zhang [2] [3] study convexity. This paper resolves a problem of Chartrand and Zhang [3].

To define convexity in graphs, we think of the vertex set $V(G)$ of the graph G as a metric space. Define the distance between two vertices as the length of any shortest path between them. We only consider connected graphs, so this distance is finite. Given two vertices u and v of G, define the *interval* between u and v as

$$I(u, v) = \{u, v\} \cup \{\text{all vertices on shortest paths from } u \text{ to } v\}.$$

As expected, we then call a subset S of $V(G)$ *convex* in G if $I(u, v)$ is contained in S for any two vertices u and v in S. We let the *convexity*

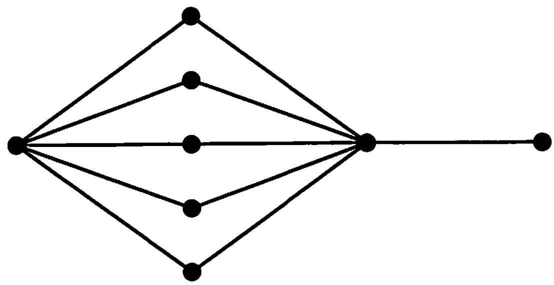

Figure 1: A graph of order 8 and convexity number 7 that has no convex subsets of orders $4, 5$, or 6.

number $\text{con}(G)$ of G be the order of the largest proper convex subset in $V(G)$. We consider only proper subsets since $V(G)$ is always convex in itself.

One of the virtues of the topological definition of convexity is that one may easily find convex subsets of any convex set. Graphs do not share this property. As Figure 1 indicates, it is easy to construct a graph G of order n with no convex subsets of order m for $3 < m < \text{con}(G)$.

If a graph G does have convex subsets of all orders $m \leq \text{con}(G)$, then G is *polyconvex*. Chartrand and Zhang [3] explain how to construct polyconvex graphs of order n and convexity number k for any integers n and k such that $2 \leq k < n$. Their construction yields graphs that are almost complete, which motivates them to ask whether one can find polyconvex graphs of any given order and convexity number that are sparse in the sense that they contain no triangles. In this paper, we show that this is indeed always possible.

Theorem *Given integers n and k with $2 \leq k < n$, there exists a connected graph G of order n and convexity number k which is polyconvex and triangle-free.*

We give the proof in the next section.

Proof of the Theorem

The proof is organized as follows. First we treat the case when $k = 2$, and then we can assume that $k \geq 3$. There are several cases depending on the

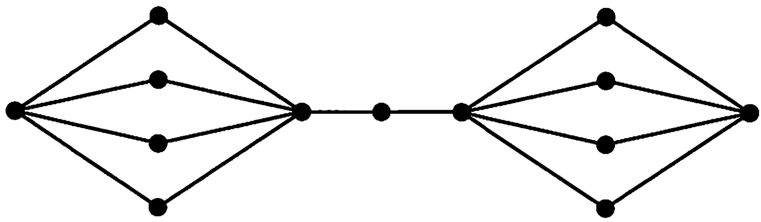

Figure 2: The construction of Case 2 for $n = 14, k = 10$.

relative values of n and k. There are also some exceptional cases that we treat with specific examples. In every case, we give an explicit construction of a graph and describe its proper convex subsets. We leave to the reader the verification that there are no larger convex subsets than the ones that we describe.

We use paths P_n, cycles C_n, and complete bipartite graphs $K_{m,n}$ as building blocks for our constructions. Note that they are all polyconvex. The path P_n with n vertices has convexity number $n-1$; the cycle C_n with n vertices has convexity number $\lceil \frac{n}{2} \rceil$; and $K_{m,n}$ has convexity number 2.

CASE 1: $k = 2$.

Let G be $K_{2,n-2}$. Then any two adjacent vertices form a largest proper convex set. □

Now we may assume that $k \geq 3$.

CASE 2: $\frac{3}{2}k - \frac{1}{2} \geq n \geq k+1$, $k \geq 3$.

When $k = 3$, then n must equal 4. When $k = 4$, then n must equal 5. In these two cases, let G be the path P_4 or the path P_5.

Now let $k \geq 5$. Let G_1 and G_2 be two copies of $K_{2,n-k}$. Let G_3 be a path of length $2k-n-3$; note that $2k-n-3$ is at least 0 since $2k-3 \geq \frac{3}{2}k-\frac{1}{2} \geq n$. Identify one end vertex of G_3 with a vertex in the bipartite class of order 2 of G_1, and identify the other end with the corresponding vertex in G_2. In case G_3 is a path of length 0, then we have identified a vertex from G_1 with a vertex from G_2. See Figure 2 for an example of this construction. Notice that if $n = k+1$, we get the path with n vertices.

We form convex subsets of orders 1 through $2k-n-2$ by taking a subpath of appropriate length from G_3. We form convex subsets of orders $2k-n-1$ and $2k-n$ by taking all the vertices of G_3 and adding at most

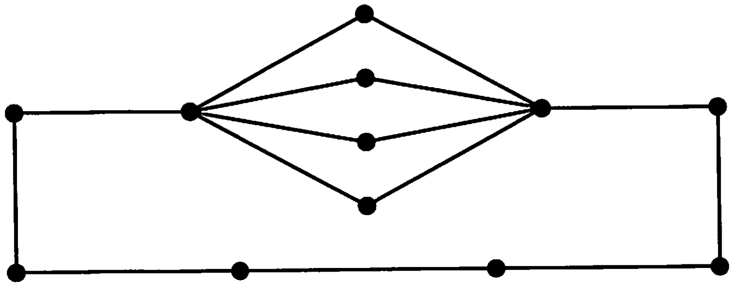

Figure 3: The construction of Case 3 for $n = 12, k = 8$.

one vertex from each of the partite classes of order $n - k$ of G_1 and G_2. We form convex subsets of orders $n - k + 2$ through $k - 1$ by taking all the vertices of G_1 together with an appropriate number of adjacent vertices of G_3. We form a convex subset of order k by taking all the vertices of G_1 and G_3 and one more vertex from the partite class of order $n - k$ of G_2. Since $(n - k + 2) - (2k - n) \leq 1$, we have found convex subsets of all orders between 1 and k. □

CASE 3: $2k - 1 \geq n \geq \frac{3}{2}k$, $k \geq 3$.

Let G_1 be a copy of $K_{2,2k-n}$. We use the condition $2k - 1 \geq n$ to guarantee that the second partite class of G_1 has at least one vertex. Let G_2 be a path of length $2n - 2k - 1$. Identify the end vertices of G_2 with the vertices in the partite class of order 2 of G_1. Figure 3 shows an example of this construction. Notice that when n is equal to $2k - 1$, we obtain the cycle on n vertices.

Vertices belonging to subpaths of G_2 form convex sets of order 1 through $n - k + 1$. We obtain convex subsets of orders $2 + 2k - n$ through k by taking all the vertices of G_1 and adding suitable numbers of neighboring vertices of G_2. Since $(2 + 2k - n) - (n - k + 1) \leq 1$, we have found convex subsets of all orders between 1 and k. □

CASE 4: $n \geq 2k$, $k \geq 3$.

Let $m = \lfloor \frac{n-2k+4}{2} \rfloor$. Note that m is at least two since $n \geq 2k$. Start with a copy of $K_{m,m}$ or $K_{m,m+1}$, depending on whether $n - 2k + 4$ is even or odd. Replace one edge of this graph with $k - 2$ paths of length 3. Let G_1 be the subgraph consisting of the union of these paths.

Let v be one of the vertices of G_1 of degree $k - 2$. Let S be the set of vertices consisting of v, all the neighbors of v in G_1, and one more neighbor of v. Then S is a convex subset of order k. Convex subsets of all smaller orders occur as subsets of S containing v.

When $k = 3$ and $n = 7$, then this construction does not work. Here we take the graph $K_{3,3}$ and replace one edge with a path of length 2. Then the three vertices on this path of length 2 form a largest convex subset. □

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References

[1] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, CA (1990).

[2] G. Chartrand and P. Zhang, The convexity number of a graph, preprint.

[3] G. Chartrand and P. Zhang, Convex sets in graphs, Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999), *Congr. Numer.* **136** (1999), 19–32.

[4] F. Harary and J. Nieminen, Convexity in graphs, *J. Differential Geom.* **16** (1981), no. 2, 185-190.