# Latin Interchanges and Direct Products

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ABSTRACT: In this paper we focus on the identification of latin interchanges in latin squares which are the direct product of latin squares of smaller orders. The results we obtain on latin interchanges will be used to identify critical sets in direct products. This work is an extension of research carried out by Stinson and van Rees in 1982.

## 1 Introduction

Since 1978 researchers have been trying to identify those sets of elements of a latin square which uniquely determine the square. Relevant papers on this topic are [1, 2, 3, 4, 5, 7]. In [7] Stinson and van Rees focus on latin squares which may be thought of as the direct product of smaller latin squares. Stinson and van Rees use the structure of the underlying squares to identify, in the direct product, a set of entries which uniquely determine that square. In the present paper we seek to extend Stinson and van Rees results to a more general setting. We present new results

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which use the existence of latin interchanges in the underlying square to identify latin interchanges in the direct product. These results will lead to a construction for critical sets in the direct product. This result is a generalization of Theorem 2.8 of Stinson and van Rees [7] and is related to the work of Cooper, Donovan and Gower [2].

# 2 Background Information

We begin by providing the reader with the necessary background information.

A *latin square* $L$ of order $n$ is an $n \times n$ array with entries chosen from the set $X = \{1, \ldots, n\}$, such that each element of $X$ occurs precisely once in each row and each column. For $i, j, k \in \{1, \ldots, n\}$ the ordered triple $(i, j; k)$ is used to represent the occurrence of element $k$ in cell $(i, j)$ of the latin square. So a latin square may be represented by the set $\{(i, j; k) \mid$ entry $k$ occurs in cell $(i, j)$ of the latin square $L\}$.

A latin square $L$ is said to be *unipotent* if there exists a $z \in \{1, \ldots, n\}$ such that for all $i \in \{1, \ldots, n\}$, $(i, i; z) \in L$. A latin square $L$ is said to be *symmetric* if, for all $i, j \in \{1, \ldots, n\}$, $(i, j; k) \in L$ implies $(j, i; k) \in L$.

Let $M$ and $N$ be two latin squares. The *direct product* of $M$ with $N$ is the array given by the set

$$\{((i_m, i_n), (j_m, j_n); (k_m, k_n)) \mid (i_m, j_m; k_m) \in M \text{ and } (i_n, j_n; k_n) \in N\}.$$

The direct product is usually denoted by $M \times N$. It is easily verified that the array $M \times N$ is also a latin square.

A *partial latin square* $P$ of order $n$ is an $n \times n$ array with entries chosen from the set $\{1, \ldots, n\}$ such that each element of $\{1, \ldots, n\}$ occurs at most once in each row and each column. So $P$ may contain a number of empty cells. At times a partial latin square will be represented by the set $\{(i, j; k) \mid$ entry $k$ occurs in cell $(i, j)$ of the partial latin square $P\}$. In addition, for a partial latin square $P$ we define $R(P) = \{r \mid (r, c; e) \in P\}$, $C(P) = \{c \mid (r, c; e) \in P\}$, $E(P) = \{e \mid (r, c; e) \in P\}$. Then for any two partial latin squares $P_1$ and $P_2$, we say $P_1$ is *isotopically equivalent to* $P_2$ if there exists an ordered triple $(\alpha, \beta, \gamma)$ of one-to-one functions such that $\alpha(R(P_1)) = R(P_2)$, $\beta(C(P_1)) = C(P_2)$, $\gamma(E(P_1)) = E(P_2)$ and $P_2 = \{(\alpha(i), \beta(j); \gamma(k)) \mid (i, j; k) \in P_1\}$.

In the proof of some of the results in this paper it will be useful to follow the lead of Stinson and van Rees, [7] and think of a partial latin square as a partial transversal design. To this end we associate a partial latin square with a set of $|P|$ unordered triples

$$T = \{\{r_i, c_j, e_k\} \mid (i, j; k) \in P\}.$$

Then the 3-set $\{r_i, c_j, e_k\}$ is said to contain the three 2-subsets $\{r_i, c_j\}$, $\{r_i, e_k\}$ and $\{c_j, e_k\}$. Then a collection of 3-sets, $T_1$, is a *trade* if there exists a collection of 3-sets, $T_2$ disjoint from $T_1$ and such that $T_1$ and $T_2$ contain precisely the same 2-subsets. The set $T_2$ is said to be a *disjoint mate* of $T_1$. The collections $T_1 = \{\{r_1, c_1, e_1\}, \{r_1, c_2, e_2\}, \{r_2, c_1, e_2\}, \{r_2, c_2, e_1\}\}$ and $T_2 = \{\{r_1, c_1, e_2\}, \{r_1, c_2, e_1\}, \{r_2, c_1, e_1\}, \{r_2, c_2, e_2\}\}$ provide examples of a trade. A trade $T$ is *minimal* if no proper subset of $T$ is a trade. A partial latin square $P$ is called a *(minimal) latin interchange* if $T$, the set of triples associate with $P$, is a (minimal) trade. The trades listed above provide the following examples of a latin interchange.

| | | |
|---|---|---|
| 1 | 2 | |
| 2 | 1 | |
| | | |

| | | |
|---|---|---|
| 2 | 1 | |
| 1 | 2 | |
| | | |

A latin interchange containing precisely four entries is said to be an *intercalate.* Note that a latin path defined in [5] is a minimal latin interchange. In addition, if $P_1$ and $P_2$ are isotopically equivalent partial latin squares and $P_1$ is a (minimal) latin interchange then $P_2$ is also a (minimal) latin interchange.

# 3 Direct Products and Latin Interchanges

Let $M$ and $N$ be latin squares of order $m$ and $n$ respectively and $L = M \times N$ be the direct product of $M$ with $N$. Let $IM$ be a latin interchange contained in $M$ and let $IN$ be a latin interchange contained in $N$. Denote the corresponding sets of triples by $TM$ and $TN$ respectively. That is, let

$$\begin{aligned} TM &= \{\{r_{i_m}, c_{j_m}, e_{k_m}\} \mid (i_m, j_m; k_m) \in IM\} \text{ and} \\ TN &= \{\{r_{i_n}, c_{j_n}, e_{k_n}\} \mid (i_n, j_n; k_n) \in IN\}. \end{aligned}$$

Let $IM'$ and $IN'$ denote the partial latin squares associated with the disjoint mates $TM'$ and $TN'$, respectively. The following lemmas establish the existence of latin interchanges in $M \times N$.

**Lemma 1** *Let $IN$ be a latin interchange in $N$. Then for each $(i_m, j_m; k_m) \in M$, the partial latin square*

$$IL = \{((i_m, i_n), (j_m, j_n); (k_m, k_n)) \mid (i_n, j_n; k_n) \in IN\}$$

*is a latin interchange in the direct product $L = M \times N$.*

**Proof:** Since $IN$ is a latin interchange there exists a corresponding trade $TN$. Now consider the following collection of 3-sets:

$$TL = \{\{r_{(i_m, i_n)}, c_{(j_m, j_n)}, e_{(k_m, k_n)}\} \mid \{r_{i_n}, c_{j_n}, e_{k_n}\} \in TN\}.$$

Since $TN$ is a trade and $(i_m, j_m; k_m)$ is a fixed element in $M$ it follows that $TL$ is also a trade. So $IL$ is a latin interchange in $L$. □

Similarly:

**Lemma 2** *Let $IM$ be a latin interchange in $M$. Then for each $(i_n, j_n; k_n) \in N$, the partial latin square*

$$\{((i_m, i_n), (j_m, j_n); (k_m, k_n)) \mid (i_m, j_m; k_m) \in IM\}$$

*is a latin interchange in the direct product $L = M \times N$.*

**Lemma 3** *Let $IM$ and $IN$ be isotopically equivalent latin interchanges. For any triple $(\alpha, \beta, \gamma)$ which provides this equivalence, the following partial latin square in $L = M \times N$ is a latin interchange.*

$$IL = \{((i_m, \alpha(i_m)), (j_m, \beta(j_m)); (k_m, \gamma(k_m))) \mid (i_m, j_m; k_m) \in IM\}$$

**Proof:** It is clear that $IL$ is a subset of $L$. Let

$$TM = \{\{r_{i_m}, c_{j_m}, e_{k_m}\} \mid (i_m, j_m; k_m) \in IM\}$$

be the trade associated with $IM$. Since $IN$ is isotopically equivalent to $IM$ the set

$$TN = \{\{r_{\alpha(i_m)}, c_{\beta(j_m)}, e_{\gamma(k_m)}\} \mid (i_m, j_m; k_m) \in IM\}$$

is the trade associated with $IN$. Further, if $\{r_{i'_m}, c_{j'_m}, e_{k'_m}\} \in TM'$, the disjoint mate of $TM$, then $\{r_{\alpha(i'_m)}, c_{\beta(j'_m)}, e_{\beta(k'_m)}\} \in TN'$, the disjoint mate of $TN$.

We need to show that the two sets

$$TL = \{\{r_{(i_m,\alpha(i_m))}, c_{(j_m,\beta(j_m))}, e_{(k_m,\gamma(k_m))}\} \mid \{r_{i_m}, c_{j_m}, e_{k_m}\} \in TM\}$$

and

$$TL' = \{\{r_{(i'_m,\alpha(i'_m))}, c_{(j'_m,\beta(j'_m))}, e_{(k'_m,\gamma(k'_m))}\} \mid \{r_{i'_m}, c_{j'_m}, e_{k'_m}\} \in TM'\}$$

form a trade $TL$ and its disjoint mate $TL'$.

Assume that this is not the case. Then either $TL$ and $TL'$ are not disjoint or $TL$ contains a particular 2-set while $TL'$ does not.

Now if $\{r_{(i_m,\alpha(i_m))}, c_{(j_m,\beta(j_m))}, e_{(k_m,\gamma(k_m))}\} \in TL \cap TL'$ then $\{r_{i_m}, c_{j_m}, e_{k_m}\} \in TM \cap TM'$ which is a contradiction.

Next assume that the 2-set $\{r_{(i_m,\alpha(i_m))}, e_{(k_m,\gamma(k_m))}\}$ belongs to $TL$. Thus $\{r_{i_m}, e_{k_m}\}$ is a 2-set in $TM$ and $\{r_{\alpha(i_m)}, e_{\gamma(k_m)}\}$ is a 2-set in $TN$. Since $TM$ and $TN$ are trades, both $\{r_{i_m}, e_{k_m}\}$ and $\{r_{\alpha(i_m)}, e_{\gamma(k_m)}\}$ are 2-sets in $TM'$ and $TN'$ respectively. Moreover since $IM$ and $IN$ are isotopically equivalent there exists $c_{j_m}$ such that $\{r_{i_m}, c_{j_m}, e_{k_m}\} \in TM'$ and

$\{r_{\alpha(i_m)}, c_{\beta(j_m)}, e_{\gamma(k_m)}\} \in TN'$. It now follows that $\{r_{(i_m,\alpha(i_m))}, e_{(k_m,\gamma(k_m))}\}$ belongs to $TL'$. Similarly, if the 2-sets $\{r_{(i_m,\alpha(i_m))}, c_{(j_m,\gamma(j_m))}\}$ or $\{c_{(j_m,\alpha(j_m))}, e_{(k_m,\gamma(k_m))}\}$ belong to $TL$ then they also belong to $TL'$. Hence $TL$ is a latin interchange. □

These latin interchanges are used in the next section to construct critical sets.

# 4 Critical Sets and Direct Products

A *critical set*, in a latin square $L$ of order $n$, is a partial latin square $C$ such that,

1. $L$ is the only latin square of order $n$ which has element $k$ in position $(i, j)$, for each $(i, j; k) \in C$;

2. no proper subset of $C$ satisfies 1.

If a partial latin square satisfies condition 1 above it is said to have *unique completion.*

It is clear from the definitions of latin interchanges and critical sets that the following lemma must be satisfied.

**Lemma 4** *Let $L$ be a latin square and $C$ a partial latin square contained in $L$. Then $C$ is a critical set if and only if the following hold:*

1. *for any latin interchange $I$ in $L$, $|C \cap I| \geq 1$;*

2. *for each $(i, j; k) \in C$, there exists a latin interchange $I$ in $L$ such that $I \cap C = \{(i, j; k)\}$.*

The definition of a critical set can be strengthened as follows. Let $L$ be a latin square, of order $n$, based on the set $N$. Let $L$ contain a critical set $C$. The set $C$ is said to be a *strong critical set* if there exists a set $\{P_1, P_2, \ldots, P_m\}$ of $m = n^2 - |C|$ partial latin squares, of order $n$, which satisfy the following properties:

1. $C = P_1 \subset P_2 \subset \ldots \subset P_{m-1} \subset P_m \subset L$;

2. for any $2 \leq i \leq m$, given $P_i = P_{i-1} \cup \{(r, s; t)\}$, then the set $P_{i-1} \cup \{(r, s; t')\}$ is not a partial latin square for any $t' \in N \setminus \{t\}$.

Let $C$ be a critical set in the latin square $L$. $C$ is called *semi-strong critical set* if

$$C_{(a,b,c)} = \{(x_a, x_b; x_c) \mid (x_1, x_2; x_3) \in C\}$$

is a strong critical set in

$$L_{(a,b,c)} = \{(x_a, x_b; x_c) \mid (x_1, x_2; x_3) \in L\}$$

for some $\{a, b, c\} = \{1, 2, 3\}$.

We now use the latin interchanges given in Lemma 3 to extend Stinson and van Rees, [7], Theorem 2.8, in a more general setting.

**Theorem 5** *Let $CN$ and $CM$ be critical sets in $N$ and $M$ respectively such that at least one of $CN$ and $CM$ is either a strong or a semi-strong critical set. Assume that for each pair of triples with $(x_m, y_m; z_m) \in CM$ and $(x_n, y_n; z_n) \in CN$ there exist latin interchanges $IM \subseteq M$ and $IN \subseteq N$ and an ordered triple of one-to-one and onto functions $(\alpha, \beta, \gamma)$ which satisfy the following conditions.*

1. $IM \cap CM = \{(x_m, y_m; z_m)\}$;
2. $IN \cap CN = \{(x_n, y_n; z_n)\}$;
3. $\alpha(R(IM)) = R(IN)$, $\beta(C(IM)) = C(IN)$, $\gamma(E(IM)) = E(IN)$;
4. $IN = \{(\alpha(i_m), \beta(j_m); \gamma(k_m)) \mid (i_m, j_m; k_m) \in IM\}$; *and*
5. $\alpha(x_m) = x_n$, $\beta(y_m) = y_n$, *and* $\gamma(z_m) = z_n$.

*Then the partial latin square*

$$\begin{aligned} CL \quad = \quad & \{((i_m, i_n), (j_m, j_n); (k_m, k_n)) \\ & \mid (i_m, j_m; k_m) \in CM, (i_n, j_n; k_n) \in N\} \cup \\ & \{((i_m, i_n), (j_m, j_n); (k_m, k_n)) \\ & \mid (i_m, j_m; k_m) \in M \setminus CM, (i_n, j_n; k_n) \in CN\} \end{aligned}$$

*is a critical set in $M \times N$.*

**Proof:** For the unique completion of $CL$ see [6]. Now we prove each entry in $CL$ is necessary.

The latin interchanges existing in $N$ together with Lemma 1 are enough to prove the necessity of the elements of the set

$$\{((i_m, i_n), (j_m, j_n); (k_m, k_n)) \mid (i_m, j_m; k_m) \in M \setminus CM, (i_n, j_n; k_n) \in CN\}.$$

For each of the elements of the set

$$\{((i_m, i_n), (j_m, j_n); (k_m, k_n)) \mid (i_m, j_m; k_m) \in CM, (i_n, j_n; k_n) \in N \setminus CN\}$$

Lemma 2 and the latin interchanges in $M$ demonstrate the necessity of these elements for unique completion.

So it remains to prove that each element of $CM \times CN$ is necessary. Let $(x_m, y_m; z_m)$ be in $CM$ and $(x_n, y_n; z_n)$ be in $CN$. Then by the initial assumptions, there exist latin interchanges $IM$ and $IN$ such that $IN = \{(\alpha(i_m), \beta(j_m); \gamma(k_m)) \mid (i_m, j_m; k_m) \in IM\}$ and further that $IM \cap CM = \{(x_m, y_m; z_m)\}$ and $IN \cap CN = \{(\alpha(x_m), \beta(y_m); \gamma(z_m))\}$. We need to check that $CL$ intersects the following latin interchange (see Lemma 3)

$$\{((i_m, \alpha(i_m)), (j_m, \beta(j_m)); (k_m, \gamma(k_m))) \mid (i_m, j_m; k_m) \in IM\}$$

in the entry $((x_m, x_n), (y_m, y_n); (z_m, z_n))$ alone. Assume it does not. Then there exists an entry $((u_m, u_n), (v_m, v_n); (w_m, w_n))$ distinct from $((x_m, x_n), (y_m, y_n); (z_m, z_n))$ which belongs to $CL \cap \{((i_m, \alpha(i_m)), (j_m, \beta(j_m)); (k_m, \gamma(k_m))) \mid (i_m, j_m; k_m) \in IM\}$. But then $(u_m, v_m; w_m) \in IM$ and $(u_n, v_n; w_n) = (\alpha(u_m), \beta(v_m); \gamma(w_m)) \in IN$. However from the assumptions $(u_m, v_m; w_m) = (x_m, y_m; z_m)$ and $(u_n, v_n; w_n) = (x_n, y_n; z_n)$, which is a contradiction. This completes the proof. □

We now provide an example of this theorem.

**Example 6** Let $M$ be the latin square representing the cyclic group of order 3, which is given below on the left, and take $CM$ to be the critical set given on the right.

$M$

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 3 | 1 |
| 3 | 1 | 2 |

$CM$

| | | |
|---|---|---|
| 1 | | |
| | | |
| | | 2 |

Then take $N$ to be the latin square representing the cyclic group of order 6, which is given below on the left and take $CN$ to be the critical set given on the right.

$N$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 3 | 4 | 5 | 6 | 1 |
| 3 | 4 | 5 | 6 | 1 | 2 |
| 4 | 5 | 6 | 1 | 2 | 3 |
| 5 | 6 | 1 | 2 | 3 | 4 |
| 6 | 1 | 2 | 3 | 4 | 5 |

$CN$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | | | |
| 2 | 3 | | | | |
| 3 | | | | | |
| | | | | | |
| | | | | | 4 |
| | | | | 4 | 5 |

The set $CL$ in $M \times N$ is given below.

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | | | | 13 | 14 | 15 | | | |
| 2 | 3 | 4 | 5 | 6 | 1 | 8 | 9 | | | | | 14 | 15 | | | | |
| 3 | 4 | 5 | 6 | 1 | 2 | 9 | | | | | | 15 | | | | | |
| 4 | 5 | 6 | 1 | 2 | 3 | | | | | | | | | | | | |
| 5 | 6 | 1 | 2 | 3 | 4 | | | | | | 10 | | | | | | 16 |
| 6 | 1 | 2 | 3 | 4 | 5 | | | | | 10 | 11 | | | | | 16 | 17 |
| 7 | 8 | 9 | | | | 13 | 14 | 15 | | | | 1 | 2 | 3 | | | |
| 8 | 9 | | | | | 14 | 15 | | | | | 2 | 3 | | | | |
| 9 | | | | | | 15 | | | | | | 3 | | | | | |
| | | | | | | | | | | | | | | | | | |
| | | | | | 10 | | | | | | 16 | | | | | | 4 |
| | | | | 10 | 11 | | | | | 16 | 17 | | | | | 4 | 5 |
| 13 | 14 | 15 | | | | 1 | 2 | 3 | | | | 7 | 8 | 9 | 10 | 11 | 12 |
| 14 | 15 | | | | | 2 | 3 | | | | | 8 | 9 | 10 | 11 | 12 | 7 |
| 15 | | | | | | 3 | | | | | | 9 | 10 | 11 | 12 | 7 | 8 |
| | | | | | | | | | | | | 10 | 11 | 12 | 7 | 8 | 9 |
| | | | | | 16 | | | | | | 4 | 11 | 12 | 7 | 8 | 9 | 10 |
| | | | | 16 | 17 | | | | | 4 | 5 | 12 | 7 | 8 | 9 | 10 | 11 |

To verify that Theorem 5 can be applied to this example we need only focus on the entries

$$(1,1;1),(1,2;2),(1,3;3),(2,1;2),(2,2;3),(3,1;3),(5,6;4),(6,5;4),(6,6;5)$$

of $M \times N$. Then the symmetry of the latin square can be used to justify the necessity of the entries in rows 13 to 18 and columns 13 to 18. So first consider $(1,3;3)$ of $M \times N$. We see that $M$ contains the entries

$$\{(1,1;1),(1,2;2),(2,1;2),(2,2;3),(3,1;3),(3,2;1)\}$$

which correspond to the latin interchange:

| | | |
|---|---|---|
| 1 | 2 | |
| 2 | 3 | |
| 3 | 1 | |

The latin square $N$ contains the entries $\{(1,3;3),(1,5;5),(3,3;5),(3,5;1),(5,3;1),(5,5;3)\}$ and they correspond to the latin interchange

| | | | | | |
|---|---|---|---|---|---|
| | | 3 | | 5 | |
| | | | | | |
| | | 5 | | 1 | |
| | | | | | |
| | | 1 | | 3 | |
| | | | | | |

These two latin interchanges are isotopically equivalent. Therefore the partial latin square

$$\{((1,1),(1,3);(1,3)),((1,1),(2,5);(2,5)),((2,3),(1,3);(2,5)),\\ ((2,3),(2,5);(3,1)),((3,5),(1,3);(3,1)),((3,5),(2,5);(1,3))\},$$

in $M \times N$ (which corresponds to the set $\{(1,3;3),(1,11;11),(9,3;11),(9,11;13),(17,3;13),(17,11;3)\}$, when using the symbols 1 to 18), is a latin interchange and it is easy to check that it intersects $CL$ in the entry $(1,3;3)$ alone. Similar arguments verify that the necessity of the remaining elements. However it should be mentioned that care should be taken with entry $(1,1;1)$. Here the latin interchanges $\{(1,1;1),(1,3;3),(2,3;1),(2,2;3),(3,2;1),(3,3;3)\}$ in M and $\{(1,1;1),(1,5;5),(3,5;1),(3,3;5),(5,3;1),(5,1;5)\}$ in N should be used.

For a more general example, represent the cyclic group $C_n$ by the latin square $BC_n = \{(i,j;i+j) \mid 0 \le i,j \le n-1\}$, where addition is modulo $n$. Then if $n$ is even the partial latin square

$$\begin{aligned}\mathcal{E}_n &= \{(i,j;i+j) \mid i=0,\ldots,\frac{n}{2}-1 \text{ and } j=0,\ldots,\frac{n}{2}-1-i\} \cup \\ &\quad \{(i,j;i+j) \\ &\quad \mid i=\frac{n}{2}+1,\ldots,n-1 \text{ and } j=(\frac{3n}{2}-i),\ldots,n-1\},\end{aligned}$$

is a critical set in $BC_n$. If $n$ is odd the partial latin square

$$\begin{aligned}\mathcal{O}_n &= \{(i,j;i+j) \mid i=0,\ldots,\frac{n-3}{2} \text{ and } j=0,\ldots,\frac{n-3}{2}-i\} \cup \\ &\quad \{(i,j;i+j) \\ &\quad \mid i=\frac{n-1}{2}+1,\ldots,n-1 \text{ and } j=(\frac{3n-1}{2}-i),\ldots,n-1\},\end{aligned}$$

is a critical set in $BC_n$. When $n$ is even, it is easy to see that for each $(x,y;z) \in \mathcal{E}_n$ there exists an intercalate $I = \{(x,y;z),(x,y+n/2;z+n/2),(x+n/2,y;z+n/2),(x+n/2,y+n/2;z)\}$ such that $I \cap \mathcal{E}_n = \{(x,y;z)\}$. If $n$ is odd, then for each $(x,y;z) \in \mathcal{O}_n$, where $0 \le x \le \frac{n-3}{2}$, there exists a latin interchange $I = \{(x,y;z),(x,y+\frac{n-1}{2};z+\frac{n-1}{2}),(x+\frac{n-1}{2},y+i;z+i+\frac{n-1}{2}),(x+\frac{n+1}{2},y+i;z+i+\frac{n+1}{2}) \mid i=0,\ldots,\frac{n-1}{2}\}$ such that $I \cap \mathcal{O}_n = \{(x,y;z)\}$. Similarly, one can see that if $\frac{n+1}{2} \le x \le n-1$ then there exists a latin interchange $J$ which is isotopically equivalent to $I$ and is such that $J \cap \mathcal{O}_n = \{(x,y;z)\}$.

These observations suggest the next two corollaries.

**Corollary 7** *Let $m$ and $n$ be positive even integers. Then the partial latin* *square*

$$\{((i_m,i_l),(j_m,j_l);(k_m,k_l)) \mid (i_m,j_m;k_m) \in \mathcal{E}_m, (i_l,j_l;k_l) \in BC_n\} \; \bigcup$$
$$\{((i_m,i_l),(j_m,j_l);(k_m,k_l)) \mid (i_m,j_m;k_m) \in BC_m \setminus \mathcal{E}_m, (i_l,j_l;k_l) \in \mathcal{E}_n\}$$

*is a critical set in* $BC_m \times BC_n$.

When $m = 2$, Stinson and van Rees's result (see [7]) is a special case of this result.

**Corollary 8** *Let* $m$ *be an odd positive integer. Then the partial latin square*

$$\{((i_m,i_l),(j_m,j_l);(k_m,k_l)) \mid (i_m,j_m;k_m) \in \mathcal{O}_m, (i_l,j_l;k_l) \in BC_m\} \; \bigcup$$
$$\{((i_m,i_l),(j_m,j_l);(k_m,k_l)) \mid (i_m,j_m;k_m) \in BC_m \setminus \mathcal{O}_m, (i_l,j_l;k_l) \in \mathcal{O}_m\}$$

*is a critical set in* $BC_m \times BC_m$.

We would like to conclude this section with an example which produces a uniquely completable set but one which is not critical. In this example the conditions 1 to 5 of Theorem 5 are not satisfied and so not all the elements of the resulting partial latin square are necessary for unique completion.

**Example 9** Let both $M$ and $N$ be the latin square given below on the left. Likewise let both $CM$ and $CN$ be the critical set given on the right.

$M = N$

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 1 | 4 | 3 |
| 3 | 4 | 1 | 2 |
| 4 | 3 | 2 | 1 |

$CM = CN$

| | | | |
|---|---|---|---|
| 1 | 2 | | |
| | | 4 | |
| | | | 2 |
| | 3 | | |

So $M \times N$ will be the direct product of $M$ with itself. In $CM$ each of the entries $(1,1;1), (2,3;4), (3,4;2), (4,2;3)$ occurs in an intercalate which intersects $CM$ in this entry alone. It can also be seen that the set $M \setminus \{(1,1;1), (2,3;4), (3,4;2), (4,2;3)\}$ is a latin interchange which intersects $CM$ in the entry $(1,2;2)$ alone.

The partial latin square $CL$ in $M \times M$, is given below.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | | | 13 | 14 | | |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 | | | 12 | | | | 16 | |
| 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 | | | | 10 | | | | 14 |
| 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 | | 11 | | | | 15 | | |
| 5 | 6 | | | 1 | 2 | | | 13 | 14 | 15 | 16 | 9 | 10 | | |
| | | 8 | | | | 4 | | 14 | 13 | 16 | 15 | | | 12 | |
| | | | 6 | | | | 2 | 15 | 16 | 13 | 14 | | | | 10 |
| | 7 | | | | 3 | | | 16 | 15 | 14 | 13 | | 11 | | |
| 9 | 10 | | | 13 | 14 | | | 1 | 2 | | | 5 | 6 | 7 | 8 |
| | | 12 | | | | 16 | | | | 4 | | 6 | 5 | 8 | 7 |
| | | | 10 | | | | 14 | | | | 2 | 7 | 8 | 5 | 6 |
| | 11 | | | | 15 | | | | 3 | | | 8 | 7 | 6 | 5 |
| 13 | 14 | | | 9 | 10 | 11 | 12 | 5 | 6 | | | 1 | 2 | | |
| | | 16 | | 10 | 9 | 12 | 11 | | | 8 | | | | 4 | |
| | | | 14 | 11 | 12 | 9 | 10 | | | | 6 | | | | 2 |
| | 15 | | | 12 | 11 | 10 | 9 | | 7 | | | | 3 | | |

Since $C_m$ is strong it is easy to see that this partial latin square has a unique completion. However not all entries in this partial latin square are necessary. If we take elements $(1,2;2)$ and $(3,4;2)$ of $CM$ we see that they do not satisfy conditions of Theorem 5 and so this theorem does not apply to this partial latin square. Further one can show that the entry $((1,3),(2,4);(2,2))$, which may be thought of as the entry $(9,14;6)$ in the above partial latin square, is not necessary for unique completion. Therefore, the above partial latin square is not a critical set in $M \times M$.

# 5 More on latin interchanges and direct products

The initial condition placed on the critical sets $CN$ and $CM$ in Theorem 5 are quite strong. However it is not clear which, if any, of these conditions can be relaxed, (see Example 9). To understand this problem we must investigate fully the occurrence of latin interchanges in the direct product $M \times N$. One interesting question is "Do all latin interchanges in $M \times N$ correspond to latin interchanges in $M$ or $N$ or the direct product of these latin interchanges?" We do not know the answer to this question, however we feel that the following results shed a little light on this question. For these results we need to place certain restrictions on the entries of $M$ and $N$.

**Property 1** *Assume that there exists a symbol $x \in \{1, \dots, n\}$ such that*

$(a,b;x),(c,d;x) \in IN$ *and* $(a,d;x),(c,b;x) \in IN'$, *where* $a,b,c,d \in \{1,\dots,n\}$ *with* $a \neq c$.

**Lemma 10** *Let* $M$ *and* $N$ *be latin squares. Suppose* $IM \subseteq M$ *and* $IN \subseteq N$ *are latin interchanges and* $IN$ *satisfies Property* 1. *For any fixed entry* $r_m \in E(IM)$ *the set*

$$\begin{array}{rc} \{((i_m,i_n),(j_m,j_n);(r_m,k_n)) \mid (i_m,j_m;r_m) \in IM & \text{and} \\ (i_n,j_n;k_n) \in IN \setminus \{(a,b;x),(c,d;x)\}\} & \cup \\ \{((i_m,a),(j_m,b);(k_m,x)),((i_m,c),(j_m,d);(k_m,x)) & \mid \\ (i_m,j_m;k_m) \in IM \text{ and } k_m \neq r_m\} & \end{array}$$

*is a latin interchange in* $M \times N$.

**Proof:** To verify the existence of this latin interchange we will use Lemmas 1 and 2 to identify appropriate latin interchanges in the product. Then the successive substitution of these latin interchanges will verify that the above set has a disjoint mate and hence is a latin interchange.

¿From Lemma 1 we know that for each $(i_m,j_m;r_m) \in IM$ the set of entries

$$\{((i_m,i_n),(j_m,j_n);(r_m,k_n)) \mid (i_n,j_n;k_n) \in IN\}$$

is a latin interchange in $M \times N$.

Next let $u,v \in \{1,\dots,n\}$ be such that $(a,b;u),(c,d;v) \in IN'$. Then it follows that for each $(i_m,j_m;r_m) \in IM$, $((i_m,a),(j_m,b);(r_m,u)) \in L$ and $((i_m,c),(j_m,d);(r_m,v)) \in L$, where $L = ((M \times N) \setminus A) \cup B$ and

$$\begin{aligned} A &= \{((i_m,i_n),(j_m,j_n);(r_m,k_n)) \\ &\quad \mid (i_m,j_m;r_m) \in IM \text{ and } (i_n,j_n;k_n) \in IN\}, \\ B &= \{((i_m,i_n),(j_m,j_n);(r_m,k'_n)) \\ &\quad \mid (i_m,j_m;r_m) \in IM \text{ and } (i_n,j_n;k'_n) \in IN'\}. \end{aligned}$$

Next using Lemma 2 and the latin interchange $IM$, with disjoint mate $IM'$, the sets

$$\begin{aligned} &\{((i_m,a),(j_m,b);(r_m,u)),((i_m,a),(j_m,b);(s_m,x)) \\ &\mid (i_m,j_m;r_m) \in IM,\ s_m \neq r_m\}, \text{ and} \\ &\{((i_m,c),(j_m,d);(r_m,v)),((i_m,c),(j_m,d);(s_m,x)) \\ &\mid (i_m,j_m;r_m) \in IM,\ s_m \neq r_m\} \end{aligned}$$

form latin interchanges in $L$.

Now if we replace these sets by their corresponding disjoint mates and focus on the subsquare corresponding to $(s_m,t_m;r_m) \in IM$, for some fixed cell $(s_m, t_m)$, we see that the set

$$\{((s_m, a), (t_m, b); (r'_m, x)), ((s_m, a), (t_m, d); (r_m, x)),$$
$$((s_m, c), (t_m, b); (r_m, x)), ((s_m, c), (t_m, d); (r'_m, x))\},$$

where $x$ is as in Property 1 and $r'_m$ is the symbol in cell $(s_m, t_m)$ of $IM'$, forms an intercalate and can be replaced by the set of entries

$$\{((s_m, a), (t_m, b); (r_m, x)), ((s_m, a), (t_m, d); (r'_m, x)),$$
$$((s_m, c), (t_m, b); (r'_m, x)), ((s_m, c), (t_m, d); (r_m, x))\}.$$

Repeat this process for each cell $(s_m, t_m)$ of $IM$ which contains the symbol $r_m$. Notice that at this point for all $i_m, j_m$ such that $(i_m, j_m; r_m) \in IM$, the cells $((i_m, a), (j_m, b))$ and $((i_m, c), (j_m, d))$ contain the entries $(r_m, x)$ and so the successive substitution of this series of latin interchanges is a partial latin square which is the disjoint mate of the set given above. The result is now immediate. □

We now provide an example of this theorem.

**Example 11** Let $M$ and $N$ be the latin squares representing the cyclic groups of order 3 and 5, respectively, and let $IM \subseteq M$ and $IN \subseteq N$ be two latin interchanges which are given below.

$M$

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 3 | 1 |
| 3 | 1 | 2 |

$IM$

| | | |
|---|---|---|
| | 2 | 3 |
| | 3 | 1 |
| | 1 | 2 |

$N$

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 4 | 5 | 1 |
| 3 | 4 | 5 | 1 | 2 |
| 4 | 5 | 1 | 2 | 3 |
| 5 | 1 | 2 | 3 | 4 |

$IN$

| | | | | |
|---|---|---|---|---|
| | 2 | | 4 | |
| | | | | |
| | 4 | 5 | 1 | |
| | 5 | 1 | 2 | |
| | | | | |

Then $IN$ satisfies Property 1 with $(1, 2; 2)$ and $(4, 4; 2)$ giving the triads $(a, b; x)$ and $(c, d; x)$. Note that entries, row and column numbers in $N$ and $IN$ are in $\{1, 2, 3, 4, 5\}$. Choose $r_m$ to be 3 in $IM$ which occurs in two places. Now applying Lemma 10 leads to the following latin interchange.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 7 | | | | | | | 14 | |
| | | | | | | | | | | | | | | |
| | | | | | | | | | | | 14 | 15 | 11 | |
| | | | | | | | | 7 | | | 15 | 11 | | |
| | | | | | | | | | | | | | | |
| | | | | | | | | 14 | | | 2 | | | |
| | | | | | | | | | | | | | | |
| | | | | | | 14 | 15 | 11 | | | | | | |
| | | | | | | 15 | 11 | | | | | | 2 | |
| | | | | | | | | | | | | | | |
| | | | | | | 2 | | | | | 7 | | | |
| | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | |
| | | | | | | | | 2 | | | | | 7 | |
| | | | | | | | | | | | | | | |

In [1], Cooper, Donovan and Gower identified a series of latin interchanges in the direct product of $C_2$, the cyclic group of order 2, with $C_n$, the cyclic group of order $n$ where $n$ is odd. An example of their family of latin interchange is given below. The latin square $C_2 \times C_5$, is given on the left and the latin interchange on the right.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 3 | 4 | 5 | 1 | 7 | 8 | 9 | 10 | 6 |
| 3 | 4 | 5 | 1 | 2 | 8 | 9 | 10 | 6 | 7 |
| 4 | 5 | 1 | 2 | 3 | 9 | 10 | 6 | 7 | 8 |
| 5 | 1 | 2 | 3 | 4 | 10 | 6 | 7 | 8 | 9 |
| 6 | 7 | 8 | 9 | 10 | 1 | 2 | 3 | 4 | 5 |
| 7 | 8 | 9 | 10 | 6 | 2 | 3 | 4 | 5 | 1 |
| 8 | 9 | 10 | 6 | 7 | 3 | 4 | 5 | 1 | 2 |
| 9 | 10 | 6 | 7 | 8 | 4 | 5 | 1 | 2 | 3 |
| 10 | 6 | 7 | 8 | 9 | 5 | 1 | 2 | 3 | 4 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | | | | | | 8 | 9 | |
| | | | | | | 8 | 9 | 10 | |
| | | | | | | | | | |
| | | | | 3 | | 10 | | 7 | |
| | | | | | | | | | |
| | | | | | | | | | |
| | 8 | | 10 | 6 | | 3 | | | |
| | | | 6 | 7 | | | | | |
| | 10 | | 7 | | | | | 2 | |
| | | | | | | | | | |

In the next three lemmas we refine this construction and give general methods for constructing latin interchanges of this nature. Therefore $M$ will be restricted to a latin square of order 2, and so will be denoted by $C_2$. It should also be mentioned that the latin interchanges have been presented using the $(i, j; k)$ notation to facilitate recognition of patterns and the proofs have been omitted for brevity.

**Lemma 12** *Let $N$ be a symmetric and unipotent latin square of order $n$ then $(i, i; z) \in N$ for all $i \in \{1, \ldots, n\}$. Moreover, suppose that $IN$ is a latin interchange in $N$ satisfying Property 1. Then the partial latin square consisting of the triples*

$$\{((1,i),(2,j);(2,k)),((2,j),(1,i);(2,k)) \mid (i,j;k) \in IN \setminus \{(a,b;x)\}\} \cup$$
$$\{((1,a),(1,a);(1,z)),((1,c),(1,c);(1,z)),((2,b),(2,b);(1,z)),$$
$$((2,d),(2,d);(1,z))\}$$

*is a latin interchange in $C_2 \times N$.*

Let $JN$ also be a latin interchange in $N$, where $JN$ and $IN$ are distinct. However $JN$ must satisfy the following property.

**Property 2** *For a given symbol $y \in \{1, \ldots, n\}$ assume that $(a, e; y)$, $(f, d; y) \in JN$, and $(a, d; y), (f, e; y) \in JN'$, where $a, d$ are as in Property 1 and $e, f \in \{1, \ldots, n\}$ with $a \neq f$.*

Further, $IN$ and $JN$ together must satisfy:

**Property 3** *For $y$ as in Property 2, $(c, b; y) \in IN$, $(c, d; y), (a, b; u) \in IN'$ and $(a, e; u) \in JN'$ for some $u \in \{1, \ldots, n\}$.*

**Lemma 13** *If $N$ is a latin square of order $n$ and $IN$ and $JN$ are latin interchanges in $N$ where $IN$ satisfies Properties 1 and 3 and $JN$ Properties*

2 *and* 3*, then the partial latin square consisting of the triples*

$$\begin{array}{r}\{((1,i),(1,j);(1,k)) \mid (i,j;k) \in IN \setminus \{(a,b;x),(c,d;x)\}\} \quad \bigcup \\ \{((1,a),(2,e);(2,y)),((1,c),(2,d);(2,x))\} \quad \bigcup \\ \{((2,a),(1,b);(2,x)),((2,f),(1,d);(2,y))\} \quad \bigcup \\ \{((2,i),(2,j);(1,k)) \mid (i,j;k) \in JN \setminus \{(a,e;y)\}\},\end{array}$$

*is a latin interchange in* $C_2 \times N$.

Alternatively if $IN$ and $JN$ satisfy the following properties, the direct product will contain the latin interchange given below.

**Property 4** *For a given symbol* $y \in \{1,\ldots,n\}$ *assume that* $(g,e;y)$, $(f,h;y) \in JN$, *and* $(g,h;y),(f,e;y) \in JN'$, *where* $e,f,g,h \in \{1,\ldots,n\}$ *with* $g \neq f$.

**Property 5** *Let* $a,b,c,d$ *be as in Property* 1 *and let* $e,f,g,h$ *be as in Property* 4. *There exist* $u,v \in \{1,...,n\}$ *such that* $(a,b;u),(c,d;v) \in IN'$, *and* $(g,e;u),(f,h;v) \in JN'$,

In addition $N$ must satisfy:

**Property 6** *For* $a,b,c,d$ *as in Property* 1, $e,f,g,h$ *as in Property* 4 *and for some* $w,z \in \{1,\ldots,n\}$, $(a,e;w),(f,d;w),(c,h;z),(g,b;z) \in N$.

**Lemma 14** *If* $N$ *is a latin square of order* $n$ *and* $IN$ *and* $JN$ *are latin interchanges in* $N$ *which satisfy the appropriate Properties* 1, 4, 5 *and* 6. *Then the partial latin square consisting of the triples*

$$\{((1,i),(1,j);(1,k)) \mid (i,j;k) \in IN \setminus \{(a,b;x),(c,d;x)\}\} \bigcup$$

$$\{((1,a),(2,e);(2,w)),((1,c),(2,h);(2,z))\} \bigcup$$

$$\{((2,g),(1,b);(2,z)),((2,f),(1,d);(2,w))\} \bigcup$$

$$\{((2,i),(2,j);(1,k)) \mid (i,j;k) \in JN \setminus \{(g,e;y),(f,h;y)\}\},$$

*is a latin interchange in* $C_2 \times N$.

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