A NOTE ON THE CONSTRUCTION OF MAGIC SQUARES OF HIGHER ORDER

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Abstract: Bailey, Cheng and Kipnis [3] developed a method for constructing trend free run orders of factorial experiments called the generalized foldover method (GFM). In this paper, we use the GFM of constructing run orders of factorial experiments to give a systematic method of constructing magic squares of higher order.

AMS 1991 subject classifications: Primary 10 J 06; Secondary 10 A 99.

Keywords and phrases: magic square; generalized foldover method;

ARS COMBINATORIA 64(2002), pp. 289-299

1. INTRODUCTION. An $m \times m$ magic square of order k can be thought of as an $m \times m$ matrix $X = (x_{ij})$ whose entries x_{ij} consist of the integers $1, 2, \cdots, m^2$ such that for $i \neq i'$,

$$\sum_{j=1}^{m} x_{ij}^{\ell} = \sum_{j=1}^{m} x_{i'j}^{\ell} \qquad \text{for } \ell = 1, \cdots, k$$

and for $j \neq j'$,

$$\sum_{i=1}^{m} x_{ij}^{\ell} = \sum_{i=1}^{m} x_{ij'}^{\ell} \qquad \text{for } \ell = 1, \cdots, k.$$

The construction of magic squares has largely been used as a source of mathematical recreation, e.g., see Andrews [2]. However, more recently, magic squares have proven useful for the construction of trend free factorial experimental designs, e.g., see Phillips [8] and Jacroux [7]. A number of methods of constructing magic squares of order one are known, e.g., see Andrews ([1], [2]) and Harmuth ([5], [6]). However, very little seems to be known about constructing magic rectangles of higher order. In this note, we use the generalized foldover method (GFM) of constructing run orders of fractional factorial experiments as given in Bailey, Cheng and Kipnis [3] to give some systematic methods of constructing $m \times m$ magic squares of orders two and three where m is a prime power.

2. THE GFM. In this section, we describe the GFM method for constructing run orders of factorial experiments. Let $m = p^s$ where p is an odd prime and let $n = 2s$. Now consider the set of all $n \times 1$ vectors $x' = (x_1, \cdots, x_n)$ whose entries are $x_i \in \{0, 1, \cdots, p-1\}$. The set of all such vectors V form a vector space over the field of integers $\mathcal{Z} = \{0, 1, \cdots, p-1\}$ under the following operations:

For $\boldsymbol{x} \in V$, $\boldsymbol{y} \in V$,

$$\boldsymbol{x}' + \boldsymbol{y}' = (x_1, x_2, \cdots, x_n) + (y_1, \cdots, y_n) = \boldsymbol{z}' = (z_1, \cdots, z_n) \tag{2.1}$$

where $z_i = x_i + y_i \ (mod\ p)$;

For $\ell \in \mathcal{Z}$ and $\boldsymbol{x} \in V$,

$$\ell \boldsymbol{x}' = \ell(x_1, \cdots, x_n) = \boldsymbol{y}' = (y_1, \cdots, y_n) \qquad \text{where } y_i = \ell x_i \ (mod\ p).$$

One can order the vectors in V into a sequence using an appropriate set of generators $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$ in the following way;

Given a sequence of generators $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$, the first vector in the sequence is $\mathbf{0}$ followed by $\boldsymbol{x}_1$, $2\boldsymbol{x}_1$, $\cdots, (p-1)\boldsymbol{x}_1$. Suppose a sequence U_j of p^j vectors have been generated. Then U_j is followed by $U_j + \boldsymbol{x}_{j+1}, U_j + 2\boldsymbol{x}_{j+1}, \cdots, U_j + (p-1)\boldsymbol{x}_{j+1}$ where $U_j + t\boldsymbol{x}_{j+1}$ is the sum of $t\boldsymbol{x}_{j+1}$ with the vectors in U_j in this same order as in U_j. Once a sequence of generators is chosen, the vectors in V can be sequenced systematically as described above. This method of sequencing the vectors in V is the GFM of construction as described in Bailey, Cheng and Kipnis [3].

For purposes of this paper, we will also be interested in partitioned versions of vectors $\boldsymbol{x}' = (\boldsymbol{x}_1', \boldsymbol{x}_2') \in V$ where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are $s \times 1$ subvectors of $\boldsymbol{x}$. We note that in any sequence containing all of the vectors in V, if

$x'=(x'_1,x'_2) \in V$, then both of the subvectors x_1 and x_2 will occur as subvectors of other vectors in the sequence p^s times. Hence, after sequencing all the vectors in V, if we assign a position integer to each vector $x \in V$, it follows that the set of all possible subvectors x_1 partitions the integers $1, \cdots, p^{2s}$ into subsets corresponding to the positions in the sequence of vectors from V in which they occur. Similarly, the set of all possible subvectors x_2 provides a partition of the position integers $1, 2, \cdots, p^{2s}$. Now, for any sequence of the vectors in V, we can associate an $m \times m$ matrix whose entries are the integers $1, \cdots, p^{2s}$. In particular, for each possible subvector x_1 we associate a row i of the square matrix and for each possible subvector x_2 we associate a column j and assign the position number where $x'=(x'_1,x'_2)$ occurs in the sequence to the (i,j) element of the matrix. The problem of constructing an $m \times m$ magic rectangle matrix becomes one of finding an appropriate sequence of the vectors in V which yields the magic rectangle of the desired order. With this in mind, we now state a result concerning the GFM method of sequencing vectors in V which is an application of Theorem 4.1 of Bailey, Cheng and Kipnis [3].

Theorem 2.1. Let V be as described above where $m = p^s$ and p is an odd prime. Further, suppose a sequence is generated using the GFM and the generators $x'_i = (x'_{i1}, x'_{i2})$, $i = 1, \cdots, 2s$ where the x_{ij} are $s \times 1$ subvectors. Then the $m \times m$ square containing the integers $1, 2, \cdots, p^{2s}$ associated with the sequence as described above is a magic square of order k provided the following conditions hold:

(i) $x_1, \cdots, x_{2s}$ is a linearly independent set of vectors in V.

(ii) For any $s \times 1$ vector $y' = (y_1, \cdots, y_s)$ where $y_i \in \{0, 1, \cdots, p-1\}$, $y'x_{i1} \neq 0 \pmod p$ for at least $k+1$ generators out of $x_1, \cdots, x_{2s}$,

(iii) For any $s \times 1$ vector $y' = (y_1, \cdots, y_s)$ where $y_i \in \{0, 1, \cdots, p-1\}$, $y'x_{i2} \neq 0 \pmod p$ for at least $k+1$ generators out of $x_1, \cdots, x_{2s}$.

From the previous theorem we see that the problem of constructing magic rectangles of various orders using the GFM of construction now becomes one of finding appropriate sets of generators for the GFM. In the following corollary we provide a set of generators for constructing $m \times m$ magic rectangles of order two where $m = p^2$ and $p \geq 3$ is any odd prime.

Corollary 2.2. Suppose $m = p^2$ where p is an odd prime and let $x_1' = (1,0,1,0)$, $x_2' = (0,1,0,1)$, $x_3' = (1,1,2,2)$ and $x_4' = (1,2,2,1)$. Then the $m \times m$ rectangle generated from the GFM using x_1, x_2, x_3 and x_4 as the generators is a magic rectangle of order two.

Proof. In order to establish this corollary, the conditions of Theorem 2.1 must be verified. It is easily seen that the generators x_1, x_2, x_3 and x_4 are linearly independent in V, hence condition (i) of Theorem 2.1 is satisfied. To verify condition (ii) of Theorem 2.1, consider the subvectors $x_{11}' = (1,0), x_{21}' = (0,1), x_{31}' = (1,1)$ and $x_{41}' = (1,2)$ of x_1, x_2, x_3 and x_4 and let $y' = (y_1, y_2)$ be any vector where $y_1, y_2 \in \{0, 1, \cdots, p-1\}$. If either $y_1 = 0$ or $y_2 = 0$, then we have that $y'x_{i1} \neq 0$ $(mod\ p)$ for at least three vectors out of x_{11}, x_{21}, x_{31} and x_{41}. On the other hand, if $y_1 \neq 0$ and $y_2 \neq 0$, then y is nonorthogonal to x_{11} and x_{12} and can be orthogonal to at most one of x_{31} and x_{41} since x_{31} and x_{41} are linearly independent. Thus any vector $y' = (y_1, y_2)$ is nonorthogonal to at least three vectors out of x_{11}, x_{21}, x_{31} and x_{41}. Condition (iii) of Theorem 2.1 is similarly verified, thus we have the desired result.

Example 2.3. Consider the construction of a 9×9 magic rectangle of order two. Then $m = 3^2$ and the generators x_1, x_2, x_3 and x_4 of Corollary 2.2 produces using the GFM the following sequence of vectors:

1 – (0, 0, 0, 0)
2 – (1, 0, 1, 0)
3 – (2, 0, 2, 0)
4 – (0, 1, 0, 1)
5 – (1, 1, 1, 1)
6 – (2, 1, 2, 1)
7 – (0, 2, 0, 2)
8 – (1, 2, 1, 2)
9 – (2, 2, 2, 2)
10 – (1, 1, 2, 2)
11 – (2, 1, 0, 2)
12 – (0, 1, 1, 2)
13 – (1, 2, 2, 0)
14 – (2, 2, 0, 0)
15 – (0, 2, 1, 0)
16 – (1, 0, 2, 1)
17 – (2, 0, 0, 1)
18 – (0, 0, 1, 1)
19 – (2, 2, 1, 1)
20 – (0, 2, 2, 1)
21 – (1, 2, 0, 1)
22 – (2, 0, 1, 2)
23 – (0, 0, 2, 2)
24 – (1, 0, 0, 2)
25 – (2, 1, 1, 0)
26 – (0, 1, 2, 0)
27 – (1, 1, 0, 0)

28 – (1, 2, 2, 1)
29 – (2, 2, 0, 1)
30 – (0, 2, 1, 1)
31 – (1, 0, 2, 2)
32 – (2, 0, 0, 2)
33 – (0, 0, 1, 2)
34 – (1, 1, 2, 0)
35 – (2, 1, 0, 0)
36 – (0, 1, 1, 0)
37 – (2, 0, 1, 0)
38 – (0, 0, 2, 0)
39 – (1, 0, 0, 0)
40 – (2, 1, 1, 1)
41 – (0, 1, 2, 1)
42 – (1, 1, 0, 1)
43 – (2, 2, 1, 2)
44 – (0, 2, 2, 2)
45 – (1, 2, 0, 2)
46 – (0, 1, 0, 2)
47 – (1, 1, 1, 2)
48 – (2, 1, 2, 2)
49 – (0, 2, 0, 0)
50 – (1, 2, 1, 0)
51 – (2, 2, 2, 0)
52 – (0, 0, 0, 1)
53 – (1, 0, 1, 1)
54 – (2, 0, 2, 1)

55 – (2, 1, 1, 2)
56 – (0, 1, 2, 2)
57 – (1, 1, 0, 2)
58 – (2, 2, 1, 0)
59 – (0, 2, 2, 0)
60 – (1, 2, 0, 0)
61 – (2, 0, 1, 1)
62 – (0, 0, 2, 1)
63 – (1, 0, 0, 1)
64 – (0, 2, 0, 1)
65 – (1, 2, 1, 1)
66 – (2, 2, 2, 1)
67 – (0, 0, 0, 2)
68 – (1, 0, 1, 2)
69 – (2, 0, 2, 2)
70 – (0, 1, 0, 0)
71 – (1, 1, 1, 0)
72 – (2, 1, 2, 0)
73 – (1, 0, 2, 0)
74 – (2, 0, 0, 0)
75 – (0, 0, 1, 0)
76 – (1, 1, 2, 1)
77 – (2, 1, 0, 1)
78 – (0, 1, 1, 1)
79 – (1, 2, 2, 2)
80 – (2, 2, 0, 2)
81 – (0, 2, 1, 2)

The 9×9 magic rectangle of order two associated with this sequence is given in the following array:

	00	01	02	10	11	12	20	21	22
00	1	52	67	75	18	33	38	62	23
01	70	21	46	36	78	12	26	41	56
02	49	64	7	15	30	81	59	20	45
10	39	63	24	2	53	68	73	16	31
11	27	42	57	71	5	47	34	76	10
12	60	21	45	50	65	8	13	28	79
20	74	19	32	37	61	22	3	54	69
21	35	77	11	25	40	55	72	6	48
22	14	29	80	58	19	43	51	66	9

We note that the sum of integers in any row or column of the above array is 369 whereas the sum of squares of the integers in any row or column of the above array is 20049.

We now give two additional corollaries.

Corollary 2.4. Suppose $m = p^3$ where $p \geq 7, p \neq 11$, is a prime and let

$$x'_1=(1, 0, 0, 1, 0, 0); \quad x'_2=(0, 1, 0, 0, 1, 0); \quad x'_3=(0, 0, 1, 0, 0, 1);$$
$$x'_4=(1, 1, 1, 2, 2, 2); \quad x'_5=(1, 2, 3, 2, 1, 3); \quad x'_6=(1, 4, 2, 2, 4, 1).$$

Then the $m \times m$ rectangle formed from the GFM using x_1, x_2, x_3, x_4, x_5

and x_6 as the generators is a magic rectangle of order three.

Proof. To establish this corollary, we must again verify that the conditions of Theorem 2.1 are satisfied. To begin, it is easy to verify that x_1, x_2, x_3, x_4, x_5 and x_6 are a linearly independent set in V, thus condition (i) of Theorem 2.1 is satisfied. To verify condition (ii) of Theorem 2.1, consider the subvectors $x'_{11} = (1,0,0), x'_{21} = (0,1,0), x'_{31} = (0,0,1), x'_{41} = (1,1,1), x'_{51} = (1,2,3)$, and $x'_{61} = (1,4,2)$ of x_1, x_2, x_3, x_4, x_5 and x_6 and let $y' = (y_1, y_2, y_3)$ be any vector where $y_1, y_2, y_3 \in \{0, 1, 2, \cdots, p-1\}$. If only one of the components in y is nonzero, then y is clearly nonorthogonal to at least four vectors out of $x_{11}, x_{21}, x_{31}, x_{41}, x_{51}$ and x_{61}. If all components of y are nonzero, then y is nonorthogonal to x_{11}, x_{21} and x_{31} and y can be orthogonal to at most two of the vectors out of x_{41}, x_{51} and x_{61} because these three vectors form a linearly independent set modulo p for any $p \geq 7$. Finally, if y has two components which are nonzero, say y_1 and y_2, then y is nonorthogonal to x_{11} and x_{21} and is also nonorthogonal to at least two subvectors out of x_{41}, x_{51} and x_{61} because any pair of 2×1 vectors obtained by taking the first two components of x_{41}, x_{51} or x_{61} are again linearly independent (since $p \geq 7$). A similar argument holds if any other pair of components of y are nonzero and condition (ii) of Theorem 2.1 is satisfied. To verify condition (iii) of Theorem 2.1, consider the subvectors $x'_{12} = (1,0,0), x'_{22} = (0,1,0), x'_{32} = (0,0,1), x'_{42} = (2,2,2), x'_{52} = (2,1,3)$, and $x'_{62} = (2,4,1)$ of x_1, x_2, x_3, x_4, x_5 and x_6 and let $y' = (y_1, y_2, y_3)$ be as defined above. If y has one component nonzero, then clearly y is orthogonal to at least four vectors out of $x_{12}, x_{22}, x_{32}, x_{42}, x_{52}$ and x_{62} whereas if y has all three of its components nonzero, then y is non-orthogonal to x_{12}, x_{22} and x_{32} and can be at most orthogonal to two vectors out of x_{42}, x_{52} and x_{62} since these three vectors form a linearly independent set for all values of $p \geq 7, p \neq 11$. Finally, if y has two components nonzero, say y_2 and y_3, then y is nonorthogonal to x_{22} and x_{32} and also nonorthogonal to at least two subvectors out of x_{42}, x_{52} and x_{62} because any pair of 2×1 subvectors obtained by taking the last two components from x_{42}, x_{52}, or x_{62} are again linearly independent for any value of $p \geq 7, p \neq 11$. A similar argument holds if any other pair of components in y are nonzero. Thus condition (iii) is satisfied and we obtain the desired result from Theorem 2.1.

Corollary 2.5. Suppose $m = p^4$ where $p \geq 13$ is a prime and let

$$
\begin{aligned}
x'_1 &= (1, 0, 0, 0, 1, 0, 0, 0); & x'_2 &= (0, 1, 0, 0, 0, 1, 0, 0); \\
x'_3 &= (0, 0, 1, 0, 0, 0, 1, 0); & x'_4 &= (0, 0, 0, 1, 0, 0, 0, 1); \\
x'_5 &= (1, 1, 1, 1, 2, 2, 2, 2); & x'_6 &= (1, 2, 3, 4, 2, 1, 3, 4); \\
x'_7 &= (1, 4, 2, 3, 2, 4, 1, 3); & x'_8 &= (1, 3, 4, 2, 2, 3, 4, 1).
\end{aligned}
$$

Then the $m \times m$ rectangle formed from the GFM using $x_1, x_2, x_3, x_4, x_5, x_6, x_7$, and x_8 as the generators is a magic rectangle of degree four.

Proof. Similar to the proof of Corollary 2.4.

Comment. We note that when using the GFM to generate a $p^s \times p^s$ magic rectangle, the maximal order of any such rectangle is s. This follows from the fact that if

$$x'_1=(x'_{11},x'_{12}),\ \ x'_2=(x'_{21},x'_{22}),\ \ \cdots\ x'_{2s}=(x'_{2s,1},x'_{2s,2})$$

are the generators of the sequence, one can always find a vector y which is orthogonal to $x_{11}, x_{21}, \cdots, x_{s-1,1}$. Hence it will be nonorthogonal to at most $s+1$ of the vectors $x_{11}, \cdots, x_{2s,1}$ and the conclusion follows from Theorem 2.1. Thus the magic rectangles given in Corollaries 2.2, 2.4 and 2.5 are of maximal order at least with regards to the GFM method of construction.

With regard to constructing $m \times m$ magic rectangles using the GFM, there appears to be a number of open questions:

1. Is there a systematic method for finding generators for the GFM which will always yield an $m \times m$ magic rectangle of degree s where $m = p^s$ and $p \geq 2$ is an odd prime?

2. If $m = p^s$ where p is an odd prime, is s the maximal order of an $m \times m$ magic rectangle?

These and other questions are currently under research.

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