On Potentially C_k-graphic Sequences

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Abstract

In this paper, we characterize the potentially C_k graphic sequence for $k = 3, 4, 5$. These characterizations imply several theorems due to P. Erdös, M. S. Jacobson and J. Lehel [1], R. J. Gould, M. S. Jacobson and J. Lehel[2] and C. H. Lai [5]and [6], respectively.

1 Introduction

An n-term nonincreasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be graphic if it is the degree sequence of a simple graph G of order n and such graph G is referred to as a realization of π. We also denote $\sigma(\pi)$ the sum of all the terms of π. Let H be a graph. A graphic sequence π is said to be potentially H-graphic if it has a realization G containing H as its subgraph.

In [1], Erdös, Jacobson and Lehel considered the following problem about potentially K_k-graphic sequences.: determine the smallest positive even number $\sigma(k, n)$ such that every n-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi) =$

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$d_1+d_2+\cdots+d_n$ at least $\sigma(k,n)$ is potentially K_k-graphic. They gave a lower bound of $\sigma(k,n)$ by the example $\pi_0 = ((n-1)^{k-1}, (k-1)^{n-k+1})$, i.e., $\sigma(k,n) \geq (k-1)(2n-k)+2$, and they further conjectured that the lower bound is the exact value of $\sigma(k,n)$. They also proved the conjecture is true for $k=2$ and $n \geq 6$, i.e., $\sigma(2,n) = 2n$ for $n \geq 6$. The conjecture is confirmed in [2], [7], [8] and [9].

for any $k \geq 4$ and for n sufficiently large. In [2], Gould, Jacobson and Lehel generalized the above probelm: for a given simple graph H, determine the smallest positive even number $\sigma(H,n)$ such that every n-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi) = d_1+d_2+\cdots+d_n$ at least $\sigma(H,n)$ is potentially H-graphic. They determined the values $\sigma(pK_2,n), \sigma(C_4,n)$ where pK_2 is the matching consisted of p edges and C_4 is the cycle of length 4.

In [5] and [6], C. H. Lai determined the values: $\sigma(C_k,n)$ for $k \geq 5$.

Motivated by the above problems, we consider the problem: characterize the potentially C_k-graphic graphic sequences without zero terms. In this paper, we characterize the potential C_k-graphic sequences for $k=3,4,5$. By these charaterizations, the values $\sigma(K_3,n)$ and $\sigma(C_k,n)$ for $k=4,5$ are straitfoward.

2 Lemmas

In order to prove our main results, we need the following results.

Let $\pi = (d_1, d_2, \cdots, d_n)$ be a nonincreasing positive integer sequence. Denote

$$\pi' = (d_1-1, d_2-1, \cdots, d_{d_n}-1, d_{d_n+1}, \cdots, d_{n-1})$$

π' is said to be the residual sequence obtained by laying off d_n from π.

From here on, denote π' the residual sequence obtained by laying off d_n from π and all the graphic sequences have no zero terms.

Lemma 2.1 (D. J. Kleitman and D. L. Wang [4] and Hakimi[3]) π is graphic if and only if π' is graphic.

The following corollary is obvious.

Corollary 2.1 Let H be a simple graph. If π' is potentially H-graphic, then π is potentially H-graphic.

We will use Corollary 2.1 repeatedly in the proofs of our main results.

3 Potentially C_3-graphic Sequences

The main results of this section is the following theorem.

Theorem 3.1 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 3$. Then π is potentially C_3-graphic if and only if $d_3 \geq 2$ except for 2 cases: $\pi = (2^4)$ and $\pi = (2^5)$.

Before we prove theorem 3.1, we first apply the theorem to give a simple proof of the following theorem due to P. Erdös et.al.

Theorem 3.2 (P. Erdös, M. S. Jacobson and J. Lehel [1]) $\sigma(K_3, n) = 2n$ for $n \geq 6$.

Proof : In [1], Erdös et.al. gave the lower bound by the extremal example $D_k = ((n-1)^{k-1}, (k-1)^{n-k+1})$, i.e., $\sigma(K_k, n) \geq (k -$

1)(2n − k) + 2. Therefore $\sigma(K_3, n) \geq 2n$. We only need to show that $\sigma(K_3, n) \leq 2n$. It is enough to show that any graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ with $\sigma(\pi) \geq 2n$ is potentially K_3-graphic. Since $\sigma(\pi) \geq 2n$, we must have that $d_3 \geq 2$. By Theorem 3.1, π is potentially C_3-graphic. □

In order to prove theorem 3.1, we need the follwoing lemma.

Lemma 3.1 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $d_n \geq 2$ and $n \geq 6$. Then π is potentially C_3-graphic.

Proof: Since $\sigma(\pi) \geq 2n$, every realization of π must contain a cycle. Let G be a realization of π with minimal girth $g(G)$. Then $3 \leq g(G) \leq n$. We only needs to show that $g(G) = 3$.

By way of contradiction, we assume that $g(G) \geq 4$.

we consider the following 3 cases.

Case 1: $g(G) \geq 6$.

Let $v_1v_2v_3 \cdots v_{g(G)}$ be a cycle of length $g(G)$. Then $v_1v_4, v_1v_5, v_2v_4, v_2v_5 \notin E(G)$. Therefore, $G' = G - v_1v_2 - v_4v_5 + v_1v_5 + v_2v_4$ is still a realization of π and $v_2v_3v_4$ is a 3-cycle in G'. It contradicts the minimality of $g(G)$.

Case 2: $g(G) = 5$. Then G has a cycle $C = v_1v_2v_3v_4v_5$ of length 5.

Subcase 1: C is not a connected component of G.

By the assumption, there exits a vertex $u \neq v_i, i = 1, 2, \cdots, 5$ so that u is adjacent to some vertex in C.

Without loss of generality, we may assume that u is adjacent to v_1. Since $g(G) = 5$, $uv_i \notin E(G)$, $i = 2, 3, 4, 5$ and $v_1v_4 \notin E(G)$ otherwise there is a cycle of length 3 or 4 in G. Therefore $G' = G - uv_1 - v_3v_4 + v_1v_4 + uv_3$ is also a realzation of π and G' contains a cycle of length 3, $v_1v_5v_4v_1$. A contradiction.

Subcase 2: C is a connected component of G

By the assumption, for each v_i, $i = 1, 2, 3, 4, 5$ is not adjacent to any vertices outside of C.

Since $n \geq 6$ and $d_n \geq 2$. There exists an edge uv out of the cycle C. Therefore $G' = G - v_1v_2 - v_3v_4 - uv + v_1v_4 + v_2u + v_3v$ is also a realization of π with $g(G') = 3$ since $v_1v_4v_5v_1$ is a 3-cycle in G'. A contradiciton.

Case 3: $g(G) = 4$. Let $C = v_1v_2v_3v_4v_1$ be a 4-cycle in G.

Similar to Case 2, we consider the following two subcases.

Subcase 1: C is not a connected component of G

By the assumption, there exists a vertex u not in C such that u is adjacent to some v_i. Without loss of generality, we assume that u is adjacent to v_1. Since $g(G) = 4$, we must have that $v_1v_3, uv_2 \notin E(G)$. Therefore $G' = G - v_3v_2 - uv_1 + v_1v_3 + uv_2$ is also a realization of π and the girth of G' is 3 since $v_1v_3v_4v_1$ is a 3-cycle in G'. A contradiction.

Subcase 2: C is a connected component of G.

Since $n \geq 6$ and $d_n \geq 2$, there is an edge $uv \in E(G) - C$. Since C is a connected component, $uv_i, vv_i \notin E(G)$. Since $g(G) = 4$ and C is a 4-cycle, $v_1v_3 \notin E(G)$. Therefore $G' = G - v_1v_2 - v_2v_3 - uv + v_1v_3 + v_2u + v_2v$ is also a realization of π. Since $v_1v_3v_4v_1$ is a 3-cycle in G', the girth of G' is 3. A contradition.

Combining Case 1, 2 and 3, $g(G) = 3$. □

Proof of Theorem 3.1: The necessary condition is obvious. Therefore we only need to prove the sufficient condition. Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence satisfying the conditions of the theorem. It suffices to show that π has a realization containing C_3 as its subgraph. We consider the following cases:

Case 1: $n = 3$. It is obvious.

Case 2: $n = 4$. Since $\pi \neq (2^4)$ and $d_3 \geq 2$, we must have that $d_1 = 3$. Then π must be one of the following sequences:

$$(3, 2, 2, 1), (3^2, 2^2), (3^4).$$

It is easy to see that all of them are potentially C_3-graphic

Case 3. $n = 5$. If $d_1 = 2$, since $\pi \neq (2^5)$ and $d_3 \geq 2$, we must have $\pi = (2^3, 1^2)$. It is obvious to see that it is potentially C_3-graphic. If $d_1 = 3$, then π is one of the follwing:

$$(3^4, 2),\ (3^2, 2^3),\ (3^3, 2, 1),\ (3^2, 2, 1^2),\ (3, 2^3, 1).$$

It is easy to see that all of the above sequences are potentially C_3-graphic. If $d_1 = 4$, let G be a realization of π with vertices set $\{v_1, v_2, v_3, v_4, v_5\}$ where $d(v_i) = d_i$, $i = 1, 2, \cdots, 5$. Since $d(v_1) = 4$, v_1 is adjacent to every other vertices. Since $d(v_3) \geq 2$, v_3 is adjacent to an vertex other than v_1, say v_i. Then $v_1 v_3 v_i$ forms a C_3.

Case 4: $n \geq 6$.
We are going to use induction to prove the sufficient condition.

First we prove the sufficient condition for $n = 6$. Assume that $d_3 \geq 2$.

If $d_6 \geq 2$, by Lemma 3.1, π is potentially C_3-graphic. Now we assume that $d_6 = 1$. Let $\pi' = (d'_1, d'_2, d'_3, d'_4, d'_5)$ be the residual sequence obtained by layingoff d_6. If $d_1 \geq 4$, then $d'_3 \geq min\{d_1 - 1, d_2, d_3\} \geq 2$ and $d'_1 \geq 3$. By Case 3, π' is potentially C_3-graphic. Therefore, π is potentially C_3-graphic by Corollary 2.1. Thus we may assume that $2 \leq d_1 \leq 3$. If $d_1 = 2$, then $d_1 = d_2 = d_3 = 2$. Since $\sigma(\pi)$ is even, we must have that $d_5 = 1$ and $d_4 = 2$. Therefore $\pi = (2^4, 1^2)$.

It is easy to see that π is potentially C_3-graphic. Now assume that $d_1 = 3$. If $d_2 = 3$, then $d'_1 = d_2 = 3$ and $d'_3 \geq min\{d_1 - 1, d_2, d_3\} \geq 2$. Thus by case 3, π' is potentially C_3-graphic. Therefore π is potentially C_3-graphic by Corollary 2.1. Thus we may further assume that $\pi = (3, 2, 2, d_4, d_5, 1)$. Since $\sigma(\pi)$ is even, $d_4 + d_5$ must be even. Therefore $\pi = (3, 2^4, 1)$ or $\pi = (3, 2^2, 1^3)$. It is easy to see that they both have realizations containing C_3. Therefore, π is potentially C_3-graphic for $n = 6$.

Now we suppose that the sufficient condition is true for $n-1 \geq 6$. We are going to prove that it is also true for n. It suffices to prove that the graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ with $d_3 \geq 2$ is potentially C_3-graphic by the definition.

If $d_n \geq 2$, by Lemma 3.1, π is potentially C_3-graphic. Therefore we may assume that $d_n = 1$. Let $\pi' = (d_1 - 1, d_2, d_3, \cdots, d_{n-1}) = (d'_1 d'_2, \cdots, d'_{n-1})$ be the resudial sequence obtained by layingoff $d_n = 1$ where $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$. If $d'_3 \leq 1$, then $d_1 = d_2 = d_3 = 2$ and $d_4 = d_5 = \cdots = d_n = 1$. It follows that $\pi = (2^3, 1^{n-3})$. Since $\sigma(\pi)$ is even, $n - 3$ must be even. It follows that π has a realization consisting of a C_3 and $(n-3)/2$ disjoint edges. If $d'_3 \geq 2$, by induction hypothesis, π' is potentially C_3-graphic. Therefore π is potentially C_3-graphic by Corollary 2.1.

This completes the proof. □

4 Potentially C_4-graphic Sequences

The main result of this section is the following:

Theorem 4.1 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence. Then π is potentially C_4-graphic if and only if the following conditions hold:

(1) $d_4 \geq 2$.
(2) $d_1 = n - 1$ implies $d_2 \geq 3$
(3) If $n = 5, 6$, then $\pi \neq (2^n)$.

Proof: First we assume that π is potentially C_4-graphic. Then (1) and (3) are obvious. Assume that $d_1 = n - 1$. Let G be a realization of π which contains a C_4 as its subgraph and let $v_1 \in V(G)$ with degree $d(v_1) = d_1$. Then $G - v_1$ contains a path with length at least 2. Thus $G - v_1$ has a vertex with degree at least 2. Since $d(v_1) = n - 1$, we must have $d_2 \geq 3$.

Now we are going to prove the sufficient condition.

We consider the following Cases:

Case 1. $n = 4$.

If $d_1 = 2$, then $\pi = (2^4)$. It is potentially C_4-graphic. Now assume that $d_1 = 3$. Then $d_2 = 3$. Therefore $\pi = (3^2, 2^2)$ or $\pi = (3^4)$. Obviously, both of them are potentially C_4-graphic..

Case 2. $n = 5$.

If $d_5 \geq 3$, then π' satisfies the assumption. Thus π' is potentially C_4-graphic. Therefore, π is potentially C_4-graphic by Corollary 2.1. Now we assume that $d_5 \leq 2$. If $d_5 = 2$, consider π'. If $d_2 \geq 3$, then π' satisfies the assumption. Thus π' is potentially C_4-graphic. If $d_2 = 2$, then $\pi = (d_1, 2^4)$. Since $\sigma(\pi)$ is even, we have that $d_1 = 2$, *or*, 4. It is impossible. If $d_5 = 1$, then $d_1 \geq 3$ since $d_4 \geq 2$. Therefore π' satisfies the assumption. Thus π is potentially C_4-graphic by Corollary 2.1.

Case 3. $n = 6$.

If $d_6 \geq 4$, then π' is potentially C_4-graphic. If $d_6 = 3$, then $\pi' = (d_1 - 1, d_2 - 1, d_3 - 1, d_4, d_5)$ satisfies the assumption. Thus π'

is potentally C_4-graphic. Therefore π is potentially C_4-graphic by Corollary 2.1. If $d_6 = 2$, then $\pi' = (d_1 - 1, d_2 - 1, d_3, d_4, d_5)$ satisfies that $d'_4 \geq 2$ by condition (3). If $\pi' = (2^5)$, then $\pi = (3^2, 2^4)$. It is easy to see that π is potentially C_4-graphic. If $d'_1 = 4$ and $d'_2 \leq 2$, then $d'_2 = 2$. Therefore $\pi' = (4, 2^k, 1^{5-k-1})$ where $k \geq 3$. Since $\sigma(\pi')$ is even, $k = 4$. Thus $\pi' = (4, 2^4)$. Therefore, $\pi = (5, 3, 2^4)$. It is easy to see that π is potentially C_4-graphic. If $d_6 = 1$, then $\pi' = (d_1 - 1, d_2, d_3, d_4, d_5)$. If $d'_4 = 1$, then $\pi = (2^4, 1^2)$. Obviously, π is potentially C_4-graphic. Now we assume that $d'_4 \geq 2$. If $\pi' \neq (2^5), or, (4, 2^4)$, then π' is potentially C_4-graphic. Therefore π is potentially C_4-graphic by Corollary 2.1. If $\pi' = (2^5)$, then $\pi = (3, 2^4, 1)$. Clearly π is potentially C_4-graphic. If $\pi' = (4, 2^4)$, then $\pi = (5, 2^4, 1)$, it is impossible by (2).

Case 4: $n \geq 7$.

We are going to use induction to prvoe this case.

First we show the case for $n = 7$. If $d_7 \geq 3$, by Case 3, we are done. We assume that $d_7 \leq 2$. If $d_7 = 2$, then $d'_4 \geq 2$. If $d_7 = 1$, then $d'_4 \geq 2$ otherwise $\pi = (2^4, 1^3)$. Therefore, $d'_4 \geq 2$ if $d_7 \leq 2$. If $d'_1 = 5$ and $d'_2 = 2$, then $\pi' = (5, 2^4, 1)$. Therefore $d_1 = 6$ and $d_2 = 2$. It contradicts the assumption. Thus if $\pi' \neq (2^6)$, then π' is potentially C_4-graphic. Therefore π is potentially C_4-graphic by Corollary 2.1. If $\pi' = (2^6)$, then $\pi = (3^2, 2^5)$ or $\pi = (3, 2^5, 1)$. Clearly they are potentially C_4-graphic. It follows that the sufficient is true for $n = 7$

Now we assume that the sufficient condition is true for $n-1 \geq 7$. We are going to prove that it is true for n. Let $\pi = (d_1, \cdots, d_n)$ be a graphic sequence with n terms and satisfying the condtions (1) and (2). We only need to show that π is potentially C_4-graphic. If $d_n \geq 3$, then π' satisfies the assumption. By induction hypothesis, π' is potentially C_4-graphic. Therefore π is potentially C_4-graphic by

Corollary 2.1. If $d_n = 2$, then $\pi' = (d_1-1, d_2-1, d_3, \cdots, d_{n-1})$. Then $d'_4 \geq 2$. If $d'_1 = n-2$, then either $d_1 = n-1$ or $d_1 = d_2 = n-2 = d_3$. In the later case, π' satisfies the assumption. By the induction hypothesis, π' is potentially C_4-graphic. Therefore π is potentially C_4-graphic by Corollary 2.1. If $d_1 = n-1$, then either π' satistisfies the assumption or $\pi = (n-1, 3, 2^{n-2})$. It is easy to see that π is potentially C_4-graphic. Now we assume that $d_n = 1$. If $d'_4 = 1$, then $\pi = (2^4, 1^{n-4})$. Then n is even. It is easy to see π is potentially C_4-graphic. Now assume that $d'_4 \geq 2$. If $d'_1 = n-2$ and $d'_2 = 2$, then $d_1 = n-1$ and $d_2 = 2 \leq 3$. It contradicts the assumption. Therefore π' is potentially C_4-graphic. Hence π is potentially C_4-graphic by Corollary 2.1. □

By Theorem 4.1 , we give a simple proof of the following theorem due to R.J.Gould et.al:

Theorem 4.2 (R. J. Gould, M. S. Jacobson, J.Lehel[2])
For $n \geq 4$,

$$\sigma(C_4, n) = \begin{cases} 3n-1, & \textit{if } n \textit{ is odd,} \\ \\ 3n-2, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof: In [2], by taking the extremal examples $\pi = ((2k)^1, 2^{2k})$ when $n = 2k+1$ and $\pi = ((2k+1)^1, 2^{2k}, 1)$ when $n = 2k$, R. J. Gould et. al. presented a lower bound for $\sigma(C_4, n)$, i.e.,

$$\sigma(C_4, n) \geq \begin{cases} 3n-1, & \textit{if } n \textit{ is odd,} \\ \\ 3n-2, & \textit{if } n \textit{ is even.} \end{cases}$$

Now we are going to show that

$$\sigma(C_4, n) \leq \begin{cases} 3n-1, & \textit{if } n \textit{ is odd,} \\ \\ 3n-2, & \textit{if } n \textit{ is even.} \end{cases}$$

Let π be a graphic sequence with $\sigma(\pi) \geq 3n-2$. It suffices to show that π is potentially C_4-graphic. If $d_4 = 1$, then $\sigma(\pi) = d_1+d_2+d_3+(n-3)$ and $d_1+d_2+d_3 \leq 6+(n-3) = n+3$. Therefore $\sigma(\pi) \leq 2n \leq 3n-2$. Thus $d_4 \geq 2$. Clearly, if $n = 5, 6$, $\pi \neq (2^n)$. Assume that $d_1 = n-1$. If $d_2 \leq 2$, then $\sigma(\pi) \leq n-1+2(n-1) = 3n-3 \leq 3n-2$. Therefore π satisfies the conditions (1)-(3) in Theorem 4.1, thus π is potentially C_4-graphic. □

5 Potentially C_5-graphic sequences

The main result of this section is the following:

Theorem 5.1 Graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ is potentially C_5-graphic if and only if π satisfies the following conditions:
(1) $d_5 \geq 2$ and $\pi \neq (2^n)$ for $n = 6, 7$.
(2) For $i = 1, 2$, $d_1 = n - i$ implies $d_{4-i} \geq 3$.
(3) If $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$, then $d_1 + d_2 \leq n + k - 2$.
Proof: We assume that π is potentially C_5-graphic. It is obvious that (1) holds. Assume $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$ with $d_1+d_2 \geq n+k$. Since π is graphic, $d_1 + d_2 \leq 2k + n - k - 2 + 2 = n + k$. Therefore $d_1 + d_2 = n + k$. Hence π has a unique realization. It is easy to see that the realization is of girth 4. Hence (3) holds. Let G be a realization of π which contains a cycle C_5 and let $v_1 \in V(G)$ with degree $d(v_1) = d_1$. Then $G - v_1$ contains a path with length at least 3. Thus $G - v_1$ contains at least two vertices with degree at least 2. Therefore $d_1 = n - i, i = 1, 2$ implies $d_{4-i} \geq 3$. Thus (2) holds. Now we are going to prove the sufficient condition. It is enough to show the following three lemmas.

Lemma 5.1 Graphic Sequence $\pi = (d_1, d_2, \cdots, d_5)$ is potentially

C_5-graphic if π satisfies the following conditions:
(1) $d_5 \geq 2$;
(2) If $d_1 = 4$, then $d_3 \geq 3$.

Proof: If $d_5 \geq 3$, then by Theorem 4.1 , π' is potentially C_4-graphic. Since $d_5 \geq 3$, π is potentially C_5-graphic. Now assume that $d_5 = 2$. If $d_1 = 2$, then $\pi = (2^5)$. Obviously it is potentially C_5-graphic. If $d_1 = 3$, then $\pi = (3^2, 2^3)$ or $\pi = (3^4, 2)$. It is easy to see that they are potentially C_5-graphic. Now assume that $d_1 = 4$, then $d_3 \geq 3$ by (2). Since $d_5 = 2$, we have $d_3 = 3$. Therefore π is $(4^2, 3^3, 2)$ or $(4, 3^2, 2^2)$. It is easy to see that they are potentially C_5-graphic. □

Lemma 5.2 For $n = 6, 7$, $\pi = (d_1, d_2, \cdots, d_n)$ is potentially C_5-graphic if π satisfies the following conditions:
(1) $d_5 \geq 2$ and $\pi \neq (2^n)$.
(2) For $i = 1, 2$, $d_1 = n - i$ implies $d_{4-i} \geq 3$.
(3) $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$ implies $d_1 + d_2 \leq n + k - 2$.

Proof: We first prvoe the case for $n = 6$. Notice that if $d_6 \geq 3$, then π' satisfies the conditions (1) and (2) in Lemma 5.1. Hence by Lemma 5.1, π' is potentially C_5-graphic. Therefore π is potentially C_5-graphic by Corollary 2.1. Thus we may assume that $d_6 \leq= 2$.

We consider the following two cases:

Case 1: $d_6 = 1$. Then $d_1 \geq 3$ otherwise $\sigma(\pi)$ is odd. If $d_3 \geq 3$, then by Lemma 5.1, π' is potentially C_5-graphic. Now we assume that $d_3 = 2$. Then $\pi = (d_1, d_2, 2^3, 1)$. Then by conditions (3) and (2), $d_1 \leq n - 2 = 4$ and $d_1 + d_2 \leq n + k - 2 = 6 + 3 - 2 = 7$. Since $d_1 \geq 3$, $5 \leq d_1 + d_2 \leq 7$. Since $\sigma(\pi)$ is even, $d_1 + d_2 = 5$, or 7. Thus $\pi = (3, 2^4, 1)$ or $(4, 3, 2^3, 1)$. It is easy to see that they are potentially C_5-graphic.

Case 2: $d_6 = 2$. Then by (1), $d_1 \geq 3$. If $d_5 \geq 3$ or $d_3 \geq 4$, then by Lemma 5.1, π' is potentially C_5-graphic. Therefore π is potentially

C_5-graphic by Corollary 2.1. Thus we may assume that $d_5 = 2$ and $d_3 \leq 3$. If $d_3 = 2$, then $\pi = (d_1, d_2, 2^4)$. Hence by (2) and (3), we have that $d_1 + d_2 \leq n + k - 2 = 6 + 4 - 2 = 8$ and $d_1 \leq 4$. Thus $\pi = (4^2, 2^4)$ or $(3^2, 2^4)$. It is easy to see both of them are potentially C_5-graphic. If $d_3 = 3$, then we may further assume that $d_2 = 3$ otherwise by Lemma 5.1, π' is potentially C_5-graphic. Then π is one of the following sequences:

$$(5, 3^3, 2^2), (4, 3^2, 2^2), (4, 3^4, 2), (3^4, 2^2).$$

It is easy to see that they are potentially C_5-graphic.

Now we prove the case for $n = 7$.

Similarly, we may assume that $d_7 \leq 2$. If $d_3 \geq 4$ or $d_5 \geq 3$, then by the case for $n = 6$, π' is potentially C_5-graphic. Therefore we may further assume that $d_3 \leq 3$ and $d_5 = 2$. We consider the following two cases;

Case 1. $d_7 = 1$. Then we may assume that $d_4 = 2$ otherwise π' is potentially C_5-graphic. If $d_3 = 3$, then we may assume that $d_1 = d_2 = d_3 = 3$ otherewise π' is potentially C_5-graphic. Therefore $\pi = (3^3, 2^3, 1)$ and it is potentially C_5-graphic. If $d_3 = 2$, then by (2) and (3), $d_1 \leq 5$ and $d_1 + d_2 \leq n + k - 2 \leq 7 + 4 - 2 = 9$. Then π is one of the following sequences:

$$(5, 4, 2^4, 1), (4, 3, 2^4, 1), (3, 2^5, 1), (5, 3, 2^3, 1^2),$$

$$(4^2, 2^3, 1^2), (4, 2^4, 1^2), (3^2, 2^3, 1^2), (2^5, 1^2).$$

It is easy to see that they are all potentially C_5-graphic.

Case 2: $d_7 = 2$. Then $d_1 \geq 3$. If $d_3 = 3$, then similarly to Case 1, we may assume that $d_2 = d_3 = 3$. Therefore π is one of the following sequences: $(3^4, 2^3), (5, 3^3, 2^3), (6, 3^2, 2^4), (4, 3^2, 2^4)$ and it is easy to see that they are potentially C_5-graphic. If $d_3 = 2$, then by (2) and (3),

$d_1 \le 5$ and $d_1+d_2 \le n+k-2 = 7+5-2 = 10$. Then π is one of the following sequences: $(5^2, 2^5), (5, 3, 2^5), (4^2, 2^5), (4, 2^6), (3^2, 2^5)$ and it is easy to see that they are potentially C_5-graphic. □

Lemma 5.3 For $n \ge 8$, $\pi = (d_1, d_2, \cdots, d_n)$ is potentially C_5-graphic if π satisfies the following conditions:
(1)$d_5 \ge 2$;
(2) For $i = 1, 2$, $d_1 = n - i$ implies $d_{4-i} \ge 3$.
(3) $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$ implies $d_1 + d_2 \le n + k - 2$.
Proof: We are going to prove this lemma by induction on n.

We first prove it for $n = 8$. We may assume that $d_8 \le 2$ otherewise by Lemma 5.2, π' is potentially C_5-graphic. Similar to the proof of Lemma 5.2, we may further assmue that $d_3 \le 3$ and $d_5 = 2$. We consider the following two cases:
Case 1. $d_8 = 1$. If $d_3 = 3$, then we may assume that $d_1 = d_2 = d_3 = 3$ and $d_4 = 2$ otherwise π' is potentially C_5-graphic by Lemma 5.2. Therefore $\pi = (3^3, 2^4, 1)$ or $(3^3, 2^2, 1^3)$ and it is easy to see that they are potentially C_5-graphic. Now we assume that $d_3 = 2$. Then $\pi = (d_1, d_2, 2^k, 1^{8-k-2})$. Then by (2) and (3), $d_1 \le 6$ and $d_1 + d_2 \le n + k - 2$. If $d_1 = 2$, then $\pi = (2^6, 1^2)$. It is easy to see that it is potentially C_5-graphic. Thus we may further assume that $d_1 \ge 3$. Then $\pi' = (d_1 - 1, d_2, 2^k, 1^{n-1-k-2})$ and $d_1 - 1 + d_2 \le n + k - 2 - 1 = n - 1 + k - 2$. If $\pi' = (2^7)$, then $\pi = (3, 2^6, 1)$. It is easy to see that it is potentially C_5-graphic. Now we assume that $\pi' \ne (2^7)$. Since $d_1 \le 6$, π' satisfies the condition (2) in Lemma 5.2. If $\pi' \ne (2^7)$, then it satisfies the conditions (1), (2) and (3) in Lemma 5.2. By Lemma 5.2, π' is potentially C_5-graphic. Therefore π is potentially C_k-graphic by Corollary 2.1.
Case 2. $d_8 = 2$. If $d_3 = 2$, then $\pi = (d_1, d_2, 2^6)$. By (2) and (3), $d_1 \le 6$ and $d_1 + d_2 \le n + k - 2 = 8 + 6 - 2 = 12$. If $d_2 = 2$,

then $\pi = (4, 2^7)$ or (2^8). It is easy to see that they are potentially C_5-graphic. Thus we may further assume that $d_2 \geq 3$. Then $\pi' = (d_1 - 1, d_2 - 1, 2^5)$. If $\pi' = (2^7)$, then $\pi = (3^2, 2^6)$. It is easy to see that it is potentially C_5-graphic. Therefore we may assume that $\pi' \neq (2^7)$. Thus π' satisfies the condition (1) in Lemma 5.2. Since $d_1 \leq 6$, π' satisfies the condition (2) in Lemma 5.2. Since $d_1' + d_2' = d_1 - 1 + d_2 - 1 \leq 12 - 2 = 10 = 7 + 5 - 2$. Thus π' satisfies the condition (3) in Lemma 5.2. Therefore by Lemma 5.2, π' is potentially C_5-graphic. Therefore π is potentially C_k-graphic by Corollary 2.1. If $d_3 = 3$, then we may assume that $d_2 = 3$ otherwise π' is potentially C_5-graphic. Thus π is one of the following sequences:

$$(3^4, 2^4), (5, 3^3, 2^4), (7, 3^3, 2^4), (6, 3^2, 2^5), (4, 3^2, 2^5).$$

It is easy to see that they are all potentially C_5-graphic.

Now we assume that the lemma is true for $n - 1 \geq 8$. We are going to show it is true for n. Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence satisfying the conditions (1)-(3). We only need to show that it is potentially C_5-graphic. If $d_n \geq 3$, then by induction hypothesis π' is potentially C_5-graphic and therefore π is potentially C_5-graphic by Corollary 2.1. Thus we may assume that $d_n \leq 2$. Similarly, we may further assume that $d_{d_n+3} = 2$ and $d_3 \leq 3$. We consider the following two cases:

Case 1. $d_n = 1$. Then $d_4 = 2$. If $d_1 = 2$, then $\pi = (2^k, 1^{n-k})$. Since $k \geq 5$ and $n \geq 9$, π is potentially C_5-graphic. Thus we further assume that $d_1 \geq 3$. If $d_3 = 3$, then we may assume that $d_1 = d_2 = d_3 = 3$. Then $\pi' = (3^2, 2^{k+1}, 1^{n-k-4})$ satisfies (1),(2) and (3) and therefore π' is potentially C_5-graphic and hence π is potentially C_5-graphic. If $d_3 = 2$, then $\pi = (d_1, d_2, 2^k, 1^{n-k-2})$. Therefore $\pi' = (d_1 - 1, d_2, 2^k, 1^{n-k-2})$. By (3), $d_1 + d_2 \leq n + k - 2$. Thus $d_1 - 1 + d_2 \leq$

$n-1+k-2$. Obviously, π' satisfies (1)-(3). So π' is potentially C_5-graphic and hence π is potentially C_5-graphic.

Case 2. $d_n = 2$. Then $d_5 = 2$. Therfore $\pi = (d_1, d_2, d_3, d_4, 2^{n-4})$. If $d_3 = 2$, then $\pi = (d_1, d_2, 2^{n-2})$. By (2) and (3), $d_1 + d_2 \leq n + n - 2 - 2 = 2(n-2)$ and $d_1 \leq n-2$. If $d_2 \geq 4$ or $d_1 \leq n-3$ and $d_2 \geq 3$, then π' satisfies (1)-(3) and therefore by induction hyperthesis it is potentially C_5-graphic and hence π is potentially C_5-graphic. If $\pi = (n-2, 3, 2^{n-2})$, it is easy to see it is potentially C_5-graphic. Now assume that $d_2 = 2$. If $d_1 = 2$, then π is potentially C_5-graphic since $n \geq 9$. If $d_1 \geq 3$, then by (3)and (2), $d_1 - 1 + 2 \leq n - 3 + 2 - 2 = n-3 \leq n-1+n-3-2 = 2n-5$ and $d_1 - 1 \leq n-3-1 = (n-1)-3$. It follows that π' is potentially C_5-graphic and hence π is potentially C_5-graphic. Now assume that $d_3 = 3$. If $d_4 = 3$, then we may assume that $d_1 = 3$ otherwise π' is potentially C_5 graphic. Thus $\pi = (3^4, 2^{n-4})$. It is potentially C_5-graphic since $n \geq 9$. Now we assume that $d_4 = 2$. Then $\pi = (d_1, d_2, 3, 2^{n-3})$. Similarly we may assume that $d_2 = 3$. If $d_1 = n-1$, then π is potentially C_5-graphic. If $d_1 \leq n-2$, then $d_1' = d_1 - 1 \leq (n-1)-2$ and $d_1' + d_2' = d_1 - 1 + 3 \leq n-2-1+3 = n \leq n-1+n-3-2 = 2n-6$. Thus π' is potentially C_5-graphic and therefore π is potentially C_5-graphic.

Therefore π is potentially C_5-graphic. □

Now we are going to use theorem 5.1 to give a simple proof of a theorem due to C.H. Lai:

Theorem 5.3 (C. H. Lai [5]) $\sigma(C_5, n) = 4n - 4$ for $n \geq 5$.

Proof: Take $\pi = ((n-1)^2, 2^{(n-2)})$. Then π has unique realization and the realization doesn't contain a cycle of length 5. Therefore $\sigma(C_5, n) \geq \sigma(\pi) + 2 = 4n - 4$. Now we will show that $\sigma(C_5, n) \leq 4n - 4$. Let π be a graphic sequence with n terms and with $\sigma(\pi) \geq$

$4n-4$. It suffices to show that π is potentially C_5-graphic. If $d_5 = 1$, then $d_1 + d_2 + d_3 + d_4 \leq 4 \times 3 + n - 4 = n + 8$. Therefore $\sigma(\pi) \leq n+8+(n-4) = 2n+4 \leq 4n-4$ since $n \geq 5$. A contradiction. Thus $d_5 \geq 2$. If $d_3 = 2$, then $4n-4 \leq \sigma(\pi) \leq d_1 + d_2 + d_3 * (n-2) \leq 2(n-1) + 2(n-2) = 4n-6$. The contradiction shows that $d_3 \geq 3$. Therefore π satisfies the conditions in theorem 5.1. By Theorem 5.1, π is potentially C_4-graphic. Thus $\sigma(C_5, n) = 4n-4$ for $n \geq 5$. □

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