PARTIAL DIALLEL CROSS BLOCK DESIGNS *

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ABSTRACT. Partially balanced diallel cross block designs with m associate classes are defined and two general methods of construction are presented. Two-associate class designs based upon group divisible, triangular, and extended group divisible association schemes obtained using the general methods are also given. Tables of designs for no more than 24 parental lines are provided.

1 Introduction

Diallel crosses are commonly used to study the genetic properties of inbred lines in plant and animal breeding experiments. Suppose there are p inbred lines and let a cross between lines i and j be denoted by (i, j) with $i < j = 1, 2, \ldots, p$. Let n_c denote the total number of distinct crosses in the experiment. Our interest lies in comparing the lines with respect to their general combining ability (gca) parameters. The complete diallel cross (CDC) involves all possible crosses among p parental lines with $n_c = p(p-1)/2$, as discussed in detail by Griffing [8] who referred to it as type IV mating design. Gupta and Kageyama [10] gave a method of constructing balanced block designs for CDCs using the nested balanced incomplete block (BIB) designs of Preece [19]. Subsequently, Dey and Midha [6], Das, Dey and Dean [4], Das and Ghosh [2], Prasad, Gupta and Srivastava [18], and Choi and Gupta [1], among others, gave further methods of constructing balanced diallel cross block designs.

Complete diallel crosses involve equal numbers of occurrences of each of the $p(p-1)/2$ distinct crosses. If r_c denotes the number of times that each cross appears in a complete diallel, then the experiment requires $r_c p(p-1)/2$ crosses. When p is large, sometimes it becomes impractical to carry out a balanced or even a partially balanced complete diallel cross. In such situations, only a subset of all possible $p(p-1)/2$ crosses is used in the experiment, which is called a partial

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diallel cross (PDC). Das, Dean and Gupta [3] and Mukerejee [15] gave some PDC block designs. Ghosh and Divecha [7] obtained partially balanced PDC and CDC block designs by forming all pairs of crosses between the treatment labels within each block of a conventional incomplete block design. The purpose of this paper is to define partially balanced partial diallel cross block (PBDCB) designs in a unified way and to give some new general methods of constructing them. The PBDCB block designs are defined in Section 2. Two general methods of construction and some classes of designs based on group divisible, triangular and extended group divisible association schemes are given in Section 3. Finally, tables of designs for $p \leq 24$ are provided in Section 4.

2 Preliminaries

Consider a block design D_b for a diallel cross experiment involving $n_c = pr/2$ distinct crosses laid out in b blocks of k crosses each, each cross replicated r_c times, with each line contributing to r crosses. Let r_{ij} be the number of replications of cross (i,j), $i < j = 1, 2, \ldots, p$,, where

$$r_{ij} = \begin{cases} r_c & \text{if the cross } (i,j) \text{ occurs in } D_b \\ 0 & \text{otherwise} \end{cases} .$$

Then, the total number of crosses n in D_b is given by

$$n = \sum_{\substack{i=1 \\ (i<j)}}^{p} \sum_{j=1}^{p} r_{ij} = r_c n_c = bk.$$

Following Gupta and Kageyama [10], the model for the data is assumed to be

$$Y = \mu 1_n + \Delta_1 g + \Delta_2 \beta + \varepsilon \tag{2.1}$$

where Y is the $n \times 1$ vector of responses, μ is the overall mean, 1_t is the $t \times 1$ vector of 1's, and $g = (g_1, g_2, \ldots, g_p)'$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_b)'$ are the vectors of p gca effects and b block effects respectively; the rectangular matrices Δ_1, Δ_2 are the corresponding design matrices, and ε is the $n \times 1$ vector of independent random errors with zero expectations and constant variance σ^2. The information matrix C for estimating all pairwise comparisons among the gca parameters is then given by

$$C = G - \frac{1}{k} N_b N_b' \tag{2.2}$$

where $G = (g_{ij})$ is a symmetric matrix with $g_{ii} = r$, $g_{ij} = r_{ij}$ for $i < j = 1, 2, \ldots, p$, and N_b is the $p \times b$ line versus block incidence matrix of the design. The matrix N_b is the usual incidence matrix; in the present context, it is obtained by ignoring the crosses, and thus by considering $2k$ lines as the contents of a block. Note that $N_b 1_b = r1_p$, $N_b' 1_p = 2k1_b$.

Now consider two lines in each of the n crosses as the block contents of a design D_c with block size $k = 2$, and let N_c denote the $p \times n$ incidence matrix of the block design thus obtained. Then $G = N_c N_c'$. Thus, the information matrix C of equation (2.2) can be written as

$$C = 2(C_b - C_c) \tag{2.3}$$

where, taking lines as p treatments, C_b and C_c are the usual information matrices for designs with constant block size $2k$ and 2 respectively.

Following Das and Ghosh [2], we now present the definition of a balanced CDC block design.

Definition 2.1. A diallel cross design D_b will be called a balanced CDC block design with parameters $\{p, n_c, b, r_c, k, \lambda\}$ if kC takes the form

$$kN_cN_c' - N_bN_b' = a\left(I_p - \frac{1}{p}J_p\right)$$

for some positive constant a, where I_p is the identity matrix of order p and $J_p = 1_p 1_p'$.

Now we define partially balanced diallel cross block designs. The definition requires the concept of an m-class association scheme, for which a reference may be made to Raghavarao [20].

Definition 2.2. A PDC block design D_b will be called an m-associate class partially balanced PDC block (PBDCB) design with parameters $\{p, n_c, b, r_c, k, \alpha_1, \alpha_2, \ldots, \alpha_m\}$ if the following holds for a given pair of lines β and γ that are ith associates,

$$k\lambda_{c(\beta,\gamma)} - \lambda_{b(\beta,\gamma)} = \alpha_i$$

where $\lambda_{b(\beta,\gamma)}$ and $\lambda_{c(\beta,\gamma)}$ are the numbers of concurrences of the lines β and γ in designs D_b and D_c respectively, and α_i is a constant independent of the pair of ith associates chosen, $i = 1, 2, \ldots, m$. For CDCs, D_b will be called a partially balanced CDC block (PBCDCB) design.

Note that for finding the number of within-block concurrences of two lines, the lines are taken as the contents of a block. Also, since each of the n_c distinct crosses is replicated r_c times, $\lambda_{c(\beta,\gamma)}$ equals r_c if the cross (β, γ) appears in the design and it is zero otherwise.

For an m-associate class PBDCB design, we can write down the following spectral decomposition

$$kN_cN_c' - N_bN_b' = \sum_{i=1}^{m} \theta_i L_i$$

where the matrices L_i are idempotent, with respective nonzero eigen values θ_i, $i = 1, 2, ..., m$. These idempotent matrices depend only on the association scheme. The eigen values θ_i can be obtained using the approach of John [14, Section 9.5], and the idempotent matrices L_i can be obtained as described by Gupta and Singh [11]. The Moore-Penrose generalized inverse of the matrix C of equation (2.2) is then given by

$$C^+ = k \sum_{i=1}^{m} \frac{1}{\theta_i} L_i.$$

3 Two general methods of construction

Two widely applicable methods of constructing PBDCB designs are presented in this section. The methods are given first in Theorems 3.1 and 3.2. PBDCB designs obtained using the two methods based on specific association schemes are then presented in Theorems 3.3−3.7.

Let D_1 be an m-associate class PBIB design with parameters $v = p$, $b = b_1, r = r_1, k = 2, \lambda_1, \lambda_2, \ldots, \lambda_m$ such that $\lambda_i = 0$ for $i(\neq s) = 1, 2, \ldots, m$, and $\lambda_s = 1$, where $s \in \{1, 2, \ldots, m\}$. For any association scheme, these designs can be obtained by taking all possible distinct pairs of lines that are sth associates. Although D_1 is based upon an m-associate class scheme, it has only two distinct values of the λ parameters. In this sense D_1 is equivalent to a two-associate class PBIB design with a suitably defined association scheme. Though D_1 need not be connected, $N_c N_c'$ is assumed to be of full-rank so that all pairwise comparisons among gca parameters are estimable. We then have the following result.

Theorem 3.1. The existence of an m-associate class PBIB design D_1 with parameters $p, b_1, r_1, k = 2, \lambda_s = 1, \lambda_i = 0,\ i(\neq s) = 1, 2, \ldots, m$, and the existence of a BIB design D_2 with parameters $v = b_1, b_2, r_2, k_2, \lambda$ implies the existence of an m-associate class PBDCB design with parameters $\{p, n_c = b_1, b = b_2, r_c = r_2, k = k_2, \alpha_s = \lambda(b_1 - r_1^2),\ \alpha_i = -r_1^2\lambda,\ i(\neq s) = 1, 2, \ldots, m\}$.

Proof. The $n_c = b_1$ distinct crosses of the PBDCB design are obtained by forming a cross between the two lines in each of the b_1 blocks of D_1. Let these crosses be serially numbered 1 through n_c. A PBDCB design D_b of the theorem is then obtained by replacing the ith treatment of D_2 by the ith cross, $i = 1, 2, \ldots, n_c$.

To prove the theorem we need to show that

$$k\lambda_{c(\beta,\gamma)} - \lambda_{b(\beta,\gamma)} = \alpha_i$$

where the symbols are as in Definition 2.2. We first find $\lambda_{b(\beta,\gamma)}$ for the PBDCB design of the theorem, when β, γ are sth associates. As $\lambda_s = 1$, the lines β, γ occur together in exactly one block of D_1. Let this particular block be denoted by $g_{\beta\gamma}$. Further, for $r_1 \geq 2$, let $g_\beta(g_\gamma)$ denote the set of $r_1 - 1$ blocks of D_1 in which $\beta(\gamma)$ occurs with a treatment other than $\gamma(\beta)$. Consider the $2r_1 - 1$

blocks $g_{\beta\gamma}$, g_β, and g_γ, and let g_1, g_2 be any two of these blocks. The method of construction of D_b causes g_1, g_2 to appear together in λ blocks of the PBDCB design D_b. Since $g_{\beta\gamma}$ is replicated r_2 times in D_b, the contribution to $\lambda_{b(\beta,\gamma)}$ from $g_{\beta\gamma}$ itself is given by r_2. If $r_1 \geq 2$ then the contributions to $\lambda_{b(\beta,\gamma)}$ may also arise from the other pairs $(g_1, g_2) \in (g_{\beta\gamma}, g_\beta, g_\gamma)$. For determining all such other contributions to $\lambda_{b(\beta,\gamma)}$, it is helpful to consider the following 4 cases separately for each of the other pairs $(g_1, g_2) \in (g_{\beta\gamma}, g_\beta, g_\gamma)$:

(i) both of g_1, g_2 belong to g_β or to g_γ ;

(ii) g_1 belongs to g_β and g_2 belongs to g_γ ;

(iii) g_1 belongs to g_β and $g_2 = g_{\beta\gamma}$;

(iv) g_1 belongs to g_γ and $g_2 = g_{\beta\gamma}$.

For (i), the pair g_1, g_2 does not contribute to $\lambda_{b(\beta,\gamma)}$, as the $r_1 - 1$ blocks of $g_\beta(g_\gamma)$ do not contain $\gamma(\beta)$. For (ii), as each such pair g_1, g_2 appears together in λ blocks of D_b, it contributes λ to $\lambda_{b(\beta\gamma)}$. Since there are $(r_1 - 1)^2$ such pairs of g_1, g_2 possible, the total contribution to $\lambda_{b(\beta,\gamma)}$ under (ii) is given by $\lambda(r_1 - 1)^2$. Under case (iii), in addition to the contribution to $\lambda_{b(\beta,\gamma)}$ from $g_{\beta\gamma}$ itself which has been considered already, each such pair contributes λ to $\lambda_{b(\beta,\gamma)}$. As there are $r_1 - 1$ such pairs, the total additional contribution to $\lambda_{b(\beta,\gamma)}$ under (iii) is then given by $\lambda(r_1 - 1)$. Similarly, the contribution to $\lambda_{b(\beta,\gamma)}$ under (iv) is also given by $\lambda(r_1 - 1)$. Thus,

$$\begin{aligned} \lambda_{b(\beta,\gamma)} &= r_2 + \lambda(r_1 - 1)^2 + 2\lambda(r_1 - 1) \\ &= \lambda(r_1^2 - 1) + r_2, \text{ if}(\beta,\gamma) \text{ are } s\text{th associates.} \end{aligned} \tag{3.4}$$

Next, we find $\lambda_{b(\beta,\gamma)}$ when β, γ are not sth associates, i.e. any two treatments for which $\lambda_i = 0,\ i(\neq s) = 1, 2, \ldots, m$. Then, along similar lines it can be verified that the total concurrence $\lambda_{b(\beta,\gamma)}$ in D_b for any two such treatments is given by

$$\lambda_{b(\beta,\gamma)} = r_1^2\lambda, \qquad \text{if } (\beta,\gamma) \text{ are } i\text{th associates, } i(\neq s) = 1, 2, \ldots, m. \tag{3.5}$$

Finally, for determining the concurrences $\lambda_{c(\beta,\gamma)}$, note that each of the n_c distinct crosses is replicated r_2 times giving a total of $n = r_2 n_c$ crosses of D_b. Thus,

$$\left.\begin{aligned} \lambda_{c(\beta,\gamma)} &= r_2 \quad \text{if } (\beta,\gamma) \text{ are } s\text{th associates} \\ &= 0 \quad \text{if } (\beta,\gamma) \text{ are } i\text{th associates, } i(\neq s) = 1, 2, \ldots, m \end{aligned}\right\}. \tag{3.6}$$

Also, since D_2 is a BIB design, we have

$$\lambda(b_1 - 1) = r_2(k_2 - 1). \tag{3.7}$$

Hence, using (3.4), (3.5), (3.6) and (3.7) we have

$$\begin{aligned} k\lambda_{c(\beta,\gamma)} - \lambda_{b(\beta,\gamma)} &= k_2 r_2 - \lambda(r_1^2 - 1) - r_2 \\ &= r_2(k_2 - 1) - \lambda(r_1^2 - 1) \\ &= \lambda(b_1 - r_1^2) \qquad \text{if } (\beta,\gamma) \text{ are } s\text{th associates} \\ &= -r_1^2\lambda \qquad \text{otherwise.} \end{aligned}$$

Hence the theorem.

As mentioned previously, since D_1 has only two distinct values of the parameters λ_i, it is equivalent to a two-associate class PBIB design. The m-associate class PBDCB designs of Theorem 3.1 also have two distinct values of the corresponding parameters, that is the α parameters. Therefore, these PBDCB designs are also equivalent to two-associate class PBDCB designs with appropriately defined association scheme.

We now present a method of construction using α-resolvable PBIB designs.

Theorem 3.2. The existence of an α-resolvable m-associate class PBIB design D_1 with parameters $p, b_1, r_1, k = 2, \lambda_i = 0$ or $1, i = 1, 2, \ldots, m$ implies the existence of an m-associate class PBDCB design with parameters $\{p,\ n_c = b_1,\ b = r_1/\alpha,\ r_c = 1,\ k = \alpha p/2,\ \alpha_i = k\lambda_i - \alpha r_1, i = 1, 2, \ldots, m\}$.

Proof. In this case also, the crosses of the PBDCB design are obtained by forming a cross between the two lines in each of the b_1 blocks of D_1. Then, the $k = \alpha p/2$ crosses belonging to the jth α-replication set of D_1 constitute the jth block of the PBDCB design, $j = 1, 2, \cdots, r_1/\alpha$. It can be verified that $\alpha_i = k\lambda_i - \alpha r_1$ in the PBDCB design thus obtained, $i = 1, 2, \ldots, m$. Hence the theorem.

Example 3.1. Let D_1 be a resolvable group divisible (GD) design having parameters $p = 6,\ b_1 = 12,\ r_1 = 4,\ k = 2,\ \lambda_1 = 0,\ \lambda_2 = 1$, with the following replication sets:

$$\begin{aligned} &\text{1st replication set} : (1,3), (2,5), (4,6) \\ &\text{2nd replication set} : (1,4), (2,6), (3,5) \\ &\text{3rd replication set} : (1,5), (2,4), (3,6) \\ &\text{4th replication set} : (1,6), (2,3), (4,5) \end{aligned}$$

Then by taking each replication set as one block, Theorem 3.2 yields a PBDCB design with parameters $p = 6,\ n_c = 12,\ b = 4,\ r_c = 1,\ k = 3,\ \alpha_1 = -4$, $\alpha_2 = -1$.

We now present some GD, triangular, and extended group divisible (EGD) PBDCB designs.

3.1 GD designs

For GD designs, $p = mn$ lines are assigned to m groups of size n each, where m, n are positive integers. Then, a GD design D_1 with parameters $p = mn$,

$b_1 = mn(n-1)/2, r_1 = n-1, k = 2, \lambda_1 = 1, \lambda_2 = 0$ can always be constructed. Thus, we have the following from Theorem 3.1.

Theorem 3.3. The existence of a BIB design D_2 with parameters $v = mn(n-1)/2, b_2, r_2, k_2, \lambda$, where $n \geq 2$, implies the existence of a GD PBDCB design with parameters $\{p, n_c = mn(n-1)/2, b = b_2, r_c = r_2, k = k_2, \alpha_1 = \lambda(n-1)\{n(m-2)+2\}/2, \alpha_2 = -(n-1)^2\lambda\}$.

Example 3.2. For $m = 2, n = 3$, take D_1 as the GD design with parameters $p = b_1 = 6, r_1 = k = 2, \lambda_1 = 1, \lambda_2 = 0$, and D_2 as the BIB design with parameters $v = 6, b_2 = 10, r_2 = 5, k_2 = 3, \lambda = 2$. Theorem 3.3 then yields a GD PBDCB design with parameters $\{p = 6, n_c = 6, b = 10, r_c = 5, k = 3, \alpha_1 = 4, \alpha_2 = -8\}$.

There exists a series of BIB design with parameters $v = 6(t+1)$, $b = 2(t+1)(6t+5), r = 6t+5, k = 3, \lambda = 2$, where t is an integer [5, p. 120]. Taking a design belonging to this series as D_2 with $m = t+1$ and $n = 4$ in Theorem 3.3, we have the following.

Corollary 3.1 There exists a GD PBDCB design with parameters $\{p = 4(t+1), n_c = 6(t+1), b = 2(t+1)(6t+5), r_c = 6t+5, k = 3, \alpha_1 = 6(2t-1), \alpha_2 = -18\}$.

Similarly, using D_1 as the GD design with parameters $p = mn$, $b_1 = n^2m(m-1)/2, r_1 = n(m-1), k = 2, \lambda_1 = 0, \lambda_2 = 1$ in Theorem 3.1, we have the following.

Theorem 3.4. The existence of a BIB design D_2 with parameters $v = mn^2(m-1)/2, b_2, r_2, k_2, \lambda$ implies the existence of a GD PBDCB design with parameters $\{p, n_c = mn^2(m-1)/2, b = b_2, r_c = r_2, k = k_2, \alpha_1 = -\lambda n^2(m-1)(m-2)/2, \alpha_2 = -\lambda n^2(m-1)^2\}$.

3.2 Triangular designs

Triangular designs have $p = n(n-1)/2$ lines, where n is an integer greater than 2. Then for $n \geq 3$, taking all distinct pairs of lines that are first associates yields a triangular design D_1 with parameters $v = p = n(n-1)/2$, $b_1 = n(n-1)(n-2)/2, r_1 = 2(n-2), k = 2, \lambda_1 = 1, \lambda_2 = 0$. Using this triangular design in Theorem 3.1, we have the following.

Theorem 3.5. The existence of a BIB design D_2 with parameters $v = n(n-1)(n-2)/2$, b_2, r_2, k_2, λ, where $n \geq 3$, implies the existence of a triangular PBDCB design with parameters $\{p, n_c = n(n-1)(n-2)/2, b = b_2, r_c = r_2, k = k_2, \alpha_1 = \lambda(n-2)(n^2-9n+16)/2, \alpha_2 = -4(n-2)^2\lambda\}$.

Similarly for $n \geq 4$, there also exists a triangular design D_1 with parameters $v = p = n(n-1)/2, b_1 = n(n-1)(n-2)(n-3)/8, r_1 = (n-2)(n-3)/2, k = 2, \lambda_1 = 0, \lambda_2 = 1$ obtained by interchanging the roles of the first and the second associates. Thus, we have the following.

Theorem 3.6. The existence of a BIB design D_2 with parameters $v = n(n-1)(n-2)(n-3)/8$, b_2, r_2, k_2, λ, where $n \geq 4$, implies the existence of a triangular PBDCB design with parameters $\{p, n_c = n(n-1)(n-2)(n-3)/8,\ b = b_2,\ r_c = r_2,\ k = k_2,\ \alpha_1 = \lambda(n-2)(n-3)(9n-n^2-12)/8, \alpha_2 = -\lambda(n-2)^2(n-3)^2/4\}$.

3.3 Extended group divisible (EGD) designs

Hinkelmann and Kempthorne [13] defined the EGD association scheme as a generalization of the GD association scheme. In an EGD design,

$$p = \prod_{i=1}^{f} m_i,$$

where the parameters $m_i,\ i = 1, 2, \ldots, f$, and f are positive integers. Further, the lines are labeled using f-digit numbers $a_1 a_2 \ldots a_f$, where $a_i = 0, 1, \ldots, m_i - 1,\ i = 1, 2, \ldots, f$. Let $x = (x_1, x_2, \ldots, x_f),\ x_i = 0$ or $1,\ i = 1, 2, \ldots, f$. Then, two treatments in the EGD scheme are x-associates where $x_i = 1$ if the ith factor occurs at the same level in both the treatments and $x_i = 0$ otherwise. Let $\lambda(x)$ denote the number of times two treatments which are x-associates occur together within blocks of the design. Note that $\lambda(x)$ depends only on x and is independent of the specific pair of the x-associates chosen. The EGD scheme was earlier considered by Nair and Rao [16] and Shah [21], and has been referred to as the binary number association scheme by Paik and Federer [17]. A detailed study of the EGD scheme is due to Hinkelmann [12]. Clearly, a total of $2^f - 1$ distinct values of $\lambda(x)$ are possible in an EGD design. An EGD design in which only one of these values is non-zero is a first-order design, see Gupta [9]. It is easy to verify that for an EGD design, the number of x-associates of any treatment is given by

$$n(x) = \prod_{i=1}^{f} (m_i - 1)^{1-x_i}.$$

Let D_1 be a first-order EGD design with parameters

$$p = \prod_{i=1}^{f} m_i, b_1 = pn(x)/2, r_1 = n(x), k = 2.$$

A first-order EGD design D_1 can be constructed for each of the distinct values of $x_0 = (x_{10}, x_{20}, \ldots, x_{f0}),\ x_{i0} = 0$ or 1, giving a total of $2^f - 1$ such first-order designs. For each of these $2^f - 1$ designs, we have the following.

Theorem 3.7. The existence of a BIB design D_2 with parameters $v = pn(x_0)/2$, $b_2, r_2,\ k_2, \lambda$ implies the existence of an EGD PBDCB design with parameters

$\{p = \prod_{i=1}^{f} m_i, \quad n_c = pn(x_0)/2, \quad b = b_2, \ r_c = r_2, \ k = k_2,$
$\alpha(x_0) = \lambda n(x_0)[p - 2n(x_0)]/2, \alpha(x) = -\{n(x_0)\}^2\lambda$ for $x \neq x_0\}$, where $x_0 = (x_{10}, x_{20}, \ldots, x_{f0})$, $x_{i0} = 0$ or 1, $i = 1, 2, \ldots, f$.

Since the designs of Theorem 3.7 have only two distinct values of the α parameters, these designs are equivalent to two-associate class PBDCB designs.

4 Table of designs

We now give GD, triangular, and EGD PBDCB designs for $p \leq 24$ obtained using Theorems 3.2–3.7. The designs are presented in Tables 1–4. As noted earlier, since the parameters α_i of a PBDCB design have two distinct values, the designs are equivalent to two-associate class PBDCB designs. For a two-associate class PBDCB design we have

$$\text{var}(\hat{g}_i - \hat{g}_j) = \begin{cases} \theta_1\sigma^2, & \text{if lines } i \text{ and } j \text{ are } s\text{th associates} \\ \theta_2\sigma^2, & \text{otherwise} \end{cases},$$

where s is as in Theorem 3.1, and θ_1, θ_2 are constants. Further,

$$\text{eff}(\hat{g}_i - \hat{g}_j) = \begin{cases} e_1, & \text{if lines } i \text{ and } j \text{ are } s\text{th associates} \\ e_2, & \text{otherwise} \end{cases},$$

where eff$(\hat{g}_i - \hat{g}_j)$ denotes the efficiency of the design for estimating the elementary contrast $g_i - g_j$ relative to an appropriate randomized complete block design. The efficiencies e_1 and e_2 were computed using equation (16) of Singh and Hinkelmann [22]. These two efficiencies of PBDCB designs are also presented in the tables.

The parameters of the BIB designs D_2 used in constructing the PBDCB designs of Tables 1–3 are given by $v = n_c$, b, $r = r_c$, k, λ with $\lambda = r_c(k-1)/(n_c - 1)$. In Tables 1 and 2, the column labeled as $D_1(m,n)$ gives the values of m and n for GD designs D_1 used in Theorems 3.3 and 3.4. For $p \leq 24$, the EGD designs obtained using Theorem 3.7 were found to be equivalent to GD designs. Thus EGD PBDCB designs are not listed separately as these designs are included in Table 1. GD designs D_1 used in constructing the designs of Table 4 also have $\lambda_s = 1$ and $\lambda_i (i \neq s) = 0$, $i = 1, 2$, and the values of m, n, and s are given in the column labeled as $D_1(m, n, s)$.

Table 1. GD PBDCB designs obtained using Theorem 3.3

p	n_c	b	r_c	k	α_1	α_2	e_1	e_2	$D_1(m,n)$
6	6	15	5	2	2	-4	0.375	0.500	2,3
6	6	10	5	3	4	-8	0.500	0.667	2,3
6	6	6	5	5	8	-16	0.600	0.800	2,3
6	6	15	10	4	12	-24	0.563	0.750	2,3
8	12	44	11	3	6	-18	0.566	0.679	2,4
8	12	22	11	6	15	-45	0.707	0.849	2,4
9	9	12	4	3	5	-4	0.429	0.571	3,3
9	9	36	8	2	5	-4	0.321	0.429	3,3
9	9	18	8	4	15	-12	0.482	0.643	3,3
9	9	12	8	6	25	-20	0.536	0.714	3,3
9	9	9	8	8	35	-28	0.563	0.750	3,3
9	9	18	10	5	25	-20	0.514	0.686	3,3
10	20	38	19	10	36	-144	0.799	0.914	2,5
12	12	44	11	3	16	-8	0.400	0.533	4,3
12	12	33	11	4	24	-12	0.450	0.600	4,3
12	12	22	11	6	40	-20	0.500	0.667	4,3
12	18	102	17	3	18	-18	0.518	0.621	3,4
12	18	34	17	9	72	-72	0.690	0.828	3,4
12	30	58	29	15	70	-350	0.850	0.944	2,6
14	42	82	41	21	120	-720	0.881	0.961	2,7
15	15	35	7	3	11	-4	0.385	0.513	5,3
15	15	15	7	7	33	-12	0.495	0.659	5,3
15	15	15	8	8	44	-16	0.505	0.673	5,3
15	15	35	14	6	55	-20	0.481	0.641	5,3
15	30	58	29	15	196	-224	0.780	0.891	3,5
16	24	184	23	3	30	-18	0.497	0.596	4,4
16	24	46	23	12	165	-99	0.683	0.820	4,4
16	56	56	11	11	14	-98	0.850	0.915	2,8
16	56	70	15	12	21	-147	0.857	0.923	2,8
18	45	99	11	5	20	-25	0.696	0.773	3,6
18	45	55	11	9	40	-50	0.773	0.859	3,6
18	45	45	12	12	60	-75	0.797	0.885	3,6
20	40	40	13	13	96	-64	0.750	0.857	4,5
20	30	290	29	3	42	-18	0.485	0.582	5,4
21	21	30	10	7	51	-12	0.474	0.632	7,3
21	21	42	12	6	51	-12	0.461	0.614	7,3
21	21	35	15	9	102	-24	0.491	0.655	7,3
24	36	420	35	3	54	-18	0.478	0.574	6,4

Table 2. GD PBDCB designs obtained using Theorem 3.4

p	n_c	b	r_c	k	α_1	α_2	e_1	e_2	$D_1(m,n)$
6	12	44	11	3	-8	-32	0.214	0.606	3,2
6	12	33	11	4	-12	-48	0.307	0.682	3,2
6	12	22	11	6	-20	-80	0.437	0.758	3,2
8	24	46	23	12	-132	-396	0.630	0.893	4,2
9	27	39	13	9	-36	-144	0.437	0.791	3,3
9	27	27	13	13	-54	-216	0.533	0.822	3,3
10	40	40	13	13	-96	-256	0.624	0.913	5,2
12	48	94	47	24	-368	-1472	0.611	0.861	3,4
12	54	106	53	27	-702	-2106	0.716	0.925	4,3

Table 3. Triangular PBDCB designs obtained using Theorems 3.5 and 3.6

p	n_c	b	r_c	k	α_1	α_2	e_1	e_2
Theorem 3.5								
10	30	58	29	15	-84	-504	0.846	1.000
15	60	118	59	30	145	-2900	0.883	0.993
Theorem 3.6								
10	15	35	7	3	6	-9	0.536	0.357
10	15	15	7	7	18	-27	0.689	0.459
10	15	15	8	8	24	-36	0.703	0.469
10	15	35	14	6	30	-45	0.670	0.446
15	45	99	11	5	9	-36	0.771	0.617
15	45	55	11	9	18	-72	0.857	0.685
15	45	45	12	12	27	-108	0.883	0.707

Table 4. GD PBDCB designs obtained using Theorem 3.2

p	n_c	b	r_c	k	α_1	α_2	e_1	e_2	α	$D_1(m,n,s)$
6	12	4	1	3	-4	-1	1.000	0.833	1	3,2,2
6	12	2	1	6	-8	-2	1.000	0.833	2	3,2,2
8	12	3	1	4	2	-6	0.778	0.933	1	2,4,1
8	24	6	1	4	-6	-2	1.000	0.933	1	4,2,2
9	27	3	1	9	-12	-3	1.000	0.857	2	3,3,2
10	40	8	1	5	-8	-3	1.000	0.964	1	5,2,2
10	40	4	1	10	-16	-6	1.000	0.964	2	5,2,2
12	18	3	1	6	3	-3	0.733	0.880	1	3,4,1
12	30	5	1	6	1	-5	0.880	0.978	1	2,6,1
12	48	8	1	6	-8	-2	1.000	0.880	1	3,4,2
12	54	9	1	6	-9	-3	1.000	0.943	1	4,3,2
12	60	10	1	6	-10	-4	1.000	0.978	1	6,2,2
14	84	6	1	14	-24	-10	1.000	0.985	2	7,2,2
15	75	5	1	15	-20	-5	1.000	0.897	2	3,5,2
16	56	7	1	8	1	-7	0.918	0.989	1	2,8,1
16	112	14	1	16	-28	-12	1.000	0.989	2	8,2,2
18	45	5	1	9	4	-5	0.850	0.944	1	3,6,1
18	108	6	1	18	-24	-6	1.000	0.911	2	3,6,2
18	162	8	1	18	-32	-14	1.000	0.992	2	9,2,2
18	135	15	1	9	-15	-3	1.000	0.981	1	6,3,2
20	30	3	1	10	7	-3	0.704	0.844	1	5,4,1
20	150	15	1	10	-15	-5	1.000	0.960	1	4,5,2
20	180	18	1	10	-18	-8	1.000	0.993	1	10,2,2
20	160	8	1	20	-16	-6	1.000	0.974	2	5,4,2
22	220	10	1	22	-40	-18	1.000	0.995	2	11,2,2
24	36	3	1	12	9	-3	0.697	0.836	1	6,4,1
24	60	5	1	12	7	-5	0.836	0.929	1	4,6,1
24	84	7	1	12	5	-7	0.896	0.965	1	3,8,1
24	192	8	1	24	-32	-8	1.000	0.929	2	3,8,2
24	216	18	1	12	-18	-6	1.000	0.965	1	4,6,2
24	252	21	1	12	-21	-9	1.000	0.990	1	8,3,2

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References

[1] K.C. Choi and S. Gupta (2000). On constructions of optimal complete diallel crosses. *Utilitas Math.* **58**, 153-160.

[2] A. Das and D.K. Ghosh (1999). Balanced incomplete block diallel cross designs. *Statistics and Applications* **1**, 1-16.

[3] A. Das, A.M. Dean and S. Gupta (1998). On optimality of some partial diallel cross designs, *Sankhyā* **B 60**, 511-524.

[4] A. Das, A. Dey and A.M. Dean (1998). Optimal designs for diallel cross experiments, *Statist. & Prob. Letters* **36**, 427-436.

[5] A. Dey (1986). *Theory of Block Designs.* Wiley-Eastern, New Delhi.

[6] A. Dey and C.K. Midha (1996). Optimal block designs for diallel crosses, *Biometrika* **83**, 484-489.

[7] D.K. Ghosh and J. Divecha (1997). Two associate class partially balanced incomplete block designs and partial diallel crosses. *Biometrika* **84**, 245-248.

[8] B. Griffing (1956). Concepts of general and specific combining ability in relation to diallel crossing systems. *Aust. J. Bio. Sci.* **9**, 463-493.

[9] S. Gupta (1987). Generating generalized cyclic designs with factorial balance. *Commun. Statist. – Theor. Meth.* **16**, 1885-1900.

[10] S. Gupta and S. Kageyama (1994). Optimal complete diallel crosses. *Biometrika* **81**, 420-424.

[11] S. Gupta and M. Singh (1989). Analysis of PBIB designs using association matrices. *Metrika* **36**, 1-6.

[12] K. Hinkelmann (1964). Extended group divisible partially balanced incomplete block designs. *Ann. Math. Stat.* **35**, 681-695.

[13] K. Hinkelmann and O. Kempthorne (1963). Two classes of group divisible block designs. *Biometrika* **50**, 281-291.

[14] P.W.M. John (1980). *Incomplete Block Designs.* Marcel Dekker, New York.

[15] R. Mukerjee (1997). Optimal partial diallel crosses. *Biometrika* **84**, 939-948.

[16] K.R. Nair and C.R. Rao (1948). Confounding in asymmetric factorial experiments. *J. Roy. Statist. Soc.* **B10**, 109-131.

[17] U.B. Paik and W.T. Federer (1977). Analysis of binary number association scheme partially balanced designs. *Commun. Statist. – Theor. Meth.* **6**, 895-932 .

[18] R. Prasad, V.K. Gupta and R. Srivastava (1999). Universally optimal block designs for diallel crosses. *Statistics and Applications* **1**, 35-52.

[19] D.A. Preece (1967). Nested balanced incomplete block designs. *Biometrika* **54**, 479-486.

[20] D. Raghavarao (1971). *Constructions and Combinatorial Problems in Design of Experiments.* Wiley.

[21] B.V. Shah (1959). A generalization of partially balanced incomplete block designs. *Ann. Math. Stat.* **30**, 1041-1050.

[22] M. Singh and K. Hinkelmann (1998). Analysis of partial diallel crosses in incomplete blocks. *Biom. J.* **40**, 165-181.