The metamorphosis of lambda-fold $K_{3,3}$-designs into lambda-fold 6-cycle systems

Elizabeth J. Billington*
Centre for Discrete Mathematics and Computing,
Department of Mathematics,
The University of Queensland,
Queensland 4072
AUSTRALIA
ejb@maths.uq.edu.au

and

Gaetano Quattrocchi†
Department of Mathematics,
University of Catania,
viale A. Doria, 95125 Catania,
ITALY
quattrocchi@dmi.unict.it

Abstract

In this paper necessary and sufficient conditions are given for the metamorphosis of a λ-fold $K_{3,3}$-design of order n into a λ-fold 6-cycle system of order n, by retaining one 6-cycle subgraph from each copy of $K_{3,3}$, and then rearranging the set of all the remaining edges, three from each $K_{3,3}$, into further 6-cycles so that the result is a λ-fold 6-cycle system.

*Research supported by the Australian Research Council grant A69701550.
†Research supported by MURST.

1 Introduction and preliminaries

Let G and H be simple graphs, and let λH denote the graph H with each of its edges replicated λ times. A *λ-fold G-design* of λH is a pair (X, K) where X is the vertex set of H and K is a collection of isomorphic copies of the graph G whose edges partition the edges of λH. If H is a complete graph K_n, we refer to such a λ-fold G-design as one of *order n*. Also if $\lambda = 1$, we drop the term "1-fold".

The graph $K_n \setminus K_m$ has vertex set of order n containing a distinguished subset of order m; the edge-set of $K_n \setminus K_m$ is the same as the edge-set of K_n *but with the* $\binom{m}{2}$ *edges between the m distinguished vertices removed.* Thus this graph is sometimes referred to as a complete graph of order n with a *hole* of size m.

In what follows we shall be concerned with G-designs where G is the complete bipartite graph $K_{3,3}$. It will be convenient to think of this graph as given in Figure 1(b) rather than the traditional Figure 1(a), since Figure 1(b) clearly exhibits a 6-cycle subgraph. We shall use the notation $(1,2,3 : 4,5,6)$ for this graph with 6-cycle subgraph $(1,2,3,4,5,6)$ and three further edges $\{1,4\},\{2,5\},\{3,6\}$. We also use the more common term *6-cycle system* for a G-design where G is a 6-cycle.

Henceforth the graph $K_{3,3}$ will be denoted by W for brevity.

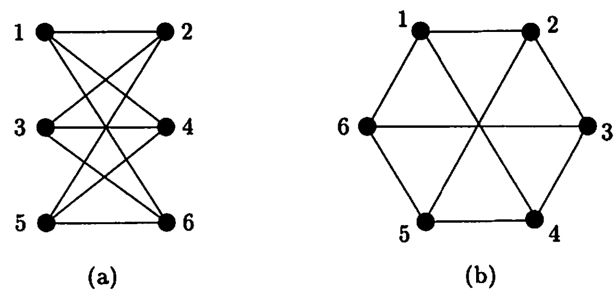


Figure 1. $K_{3,3}$ (henceforth denoted W)

We are interested here in exhibiting a *metamorphosis* of a λ-fold W-design (X, K) into a λ-fold 6-cycle system, (X, C). This metamorphosis requires us to *retain* one 6-cycle subgraph from *each* copy of W in K, and rearrange the $3|K|$ remaining edges (three edges left from each copy of W in K) into further 6-cycles, so that a λ-fold 6-cycle system (X, C) is obtained. For brevity we denote such a W-design with a metamorphosis by $(X, K < C)$.

In this paper we give complete necessary and sufficient conditions for existence of a λ-fold $K_{3,3}$-design of order n having a metamorphosis into a λ-fold 6-cycle system of order n.

For a brief history of further work on the metamorphosis of designs, see the introduction to [2]. Other recent papers on design metamorphosis include [1], [3] and [4].

Necessary Conditions

We start with some obvious necessary conditions. Since W is regular of degree 3, the degree, $\lambda(n-1)$, of λK_n must be 0 (mod 3), and since a 6-cycle is regular of degree 2, the degree, $\lambda(n-1)$, must be even. Furthermore, the number of edges in λK_n must be divisible by 9 for a W-design (since W has 9 edges), and divisible by 6 for a 6-cycle system. Thus necessary conditions for existence are:

$K_{3,3}$ or W-design

λ (mod 9)	order n
1,2,4,5,7,8	1 (mod 9)
3,6	0,1 (mod 3)
9	any $n \geq 6$

6-cycle system

λ (mod 6)	order n
1,5	1,9 (mod 12)
2,4	0,1 (mod 3)
3	1 (mod 4)
6	any $n \geq 6$

We require the *intersection* of these conditions for the possible existence of a λ-fold W-design with potential for metamorphosis into a λ-fold 6-cycle system; these are as follows:

λ (mod 18)	order n
1,5,7,11,13,17	1 (mod 36)
2,4,8,10,14,16	1 (mod 9)
3,15	1,9 (mod 12)
6,12	0,1 (mod 3)
9	1 (mod 4)
0	any $n \geq 6$

Consequently in the subsequent sections we deal with the cases λ equal to 1, 2, 3, 6, 9 and 18.

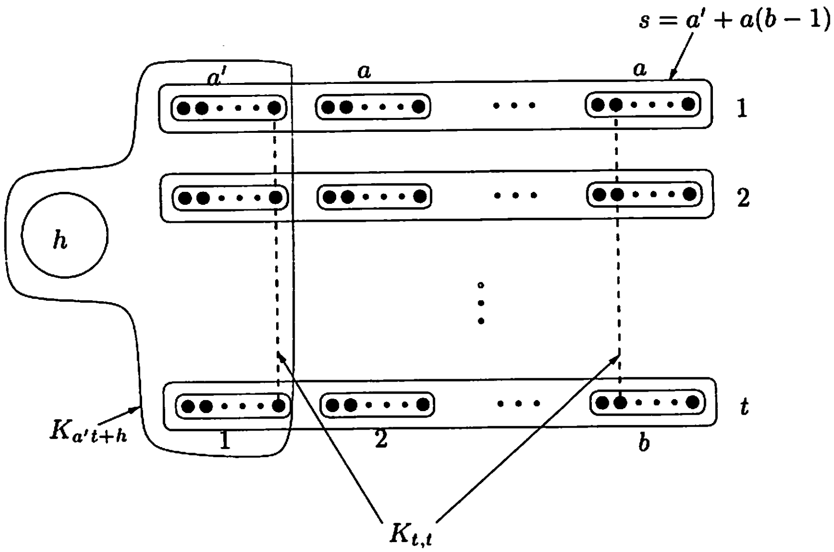
The Construction

For each admissible order n we shall construct a λ-fold W-design of order n, and exhibit a metamorphosis of the W-design into a λ-fold 6-cycle system of order n. The construction we use is as follows.

Let the order be $n = st + h$, where here h will be 0, 1, 2, 5 or 9. Let the vertex set of K_n be $\{\infty_i \mid 1 \le i \le h\} \cup \{(i,j) \mid 1 \le i \le s,\ 1 \le j \le t\}$. Moreover, let $s = a' + a(b-1)$ (where possibly $a' = a$). (We regard the integer s as partitioned into b ***groups***, one of size a' and $b-1$ of size a.) Then if there exist λ-fold W-designs of $K_{a't+h}$, $K_{at+h} \setminus K_h$ and $K_{t,t}$, which each possess a metamorphosis into λ-fold 6-cycle systems, it follows that there is a λ-fold W-design of order K_{st+h} which has a metamorphosis into a λ-fold 6-cycle system. Explicitly, this is obtained from the designs:

- $K_{a't+h}$ (once) on the vertex set $\{\infty_i \mid 1 \le i \le h\} \cup \{(i,j) \mid 1 \le i \le a,\ 1 \le j \le t\}$;
- $K_{at+h} \setminus K_h$ ($b-1$ times) on the vertex sets $\{\infty_i \mid 1 \le i \le h\} \cup \{(i,j) \mid a' + a(\ell-1) + 1 \le i \le a' + a\ell,\ 1 \le j \le t\}$, for $\ell = 1, 2, \ldots, b-1$, where the "hole" K_h is on $\{\infty_i \mid 1 \le i \le h\}$;
- $K_{t,t}$ ($\binom{b-1}{2}a^2 + (b-1)aa'$ times) on the vertex sets $\{\{(i_1,j) \mid 1 \le j \le t\}, \{(i_2,j) \mid 1 \le j \le t\}\}$ for $1 \le i_1 < i_2 \le ab$ and with the proviso that i_1 and i_2 are in different "groups" where the b groups are $\{1, 2, \ldots, a'\}$ and $\{a' + a(\ell-1)+1, \ldots, a' + a\ell\}$ for $1 \le \ell \le b-1$.

The picture below may aid the reader's understanding!



2 The case $\lambda = 1$

Let the order $n \equiv 1 \pmod{36}$ be given by $n = 1 + 3b \cdot 12$ (so $h = 1$, $a' = a = 3$ and $t = 12$ in the construction). Then using the construction, the next two lemmas, which give W-designs of K_{37} and of $K_{12,12}$, each with a metamorphosis into a 6-cycle system, complete this case.

LEMMA 2.1 *There exists a W-design (X, K) of order 37 with a metamorphosis into a 6-cycle system (X, C) of order 37.*

Proof Let $X = \mathbb{Z}_{37}$ and $K = \{(i, 18+i, 2+i : 15+i, 1+i, 12+i), (i, 9+i, 2+i : 6+i, 1+i, 3+i) \mid 0 \le i \le 36\}$. A metamorphosis into a 6-cycle system (X, C) of order 37 is given by

$$C = \{(i, 18+i, 2+i, 15+i, 1+i, 12+i), (i, 9+i, 2+i, 6+i, 1+i, 3+i), (i, 20+i, 12+i, 11+i, 5+i, 15+i) \mid 0 \le i \le 36\}.$$

□

LEMMA 2.2 *There exists a W-design (X, K) of $K_{12,12}$ with a metamorphosis into a 6-cycle system (X, C).*

Proof Let $X = \{\{i_a, i_b, i_c \mid 1 \le i \le 4\}, \{i'_a, i'_b, i'_c \mid 1 \le i \le 4\}\}$. Now K contains 16 copies of $K_{3,3}$ (which is W), on the 16 vertex sets $\{\{i_a, i_b, i_c\}, \{j'_a, j'_b, j'_c\}\}$, for $1 \le i, j \le 4$. In order to describe C, we shall list the 1-factors removed from each of these 16 copies of W (thus implicitly giving the 6-cycle left behind in the metamorphosis), and then (from these 48 edges) list the remaining eight 6-cycles.

The sixteen 1-factors to remove from the sixteen copies of W in K:

$$\begin{array}{llll}
1_a1'_a,\ 1_b1'_b,\ 1_c1'_c; & 1_a2'_a,\ 1_b2'_b,\ 1_c2'_c; & 1_a3'_a,\ 1_b3'_b,\ 1_c3'_c; & 1_a4'_a,\ 1_b4'_b,\ 1_c4'_c; \\
2_a1'_b,\ 2_b1'_c,\ 2_c1'_a; & 2_a2'_c,\ 2_b2'_b,\ 2_c2'_a; & 2_a3'_c,\ 2_b3'_b,\ 2_c3'_a; & 2_a4'_c,\ 2_b4'_b,\ 2_c4'_a; \\
3_a1'_b,\ 3_b1'_c,\ 3_c1'_a; & 3_a2'_c,\ 3_b2'_a,\ 3_c2'_b; & 3_a3'_c,\ 3_b3'_a,\ 3_c3'_b; & 3_a4'_a,\ 3_b4'_b,\ 3_c4'_c; \\
4_a1'_a,\ 4_b1'_b,\ 4_c1'_c; & 4_a2'_a,\ 4_b2'_b,\ 4_c2'_c; & 4_a3'_a,\ 4_b3'_c,\ 4_c3'_b; & 4_a4'_b,\ 4_b4'_c,\ 4_c4'_a.
\end{array}$$

These 48 edges can be formed into the following eight 6-cycles:

$$\begin{array}{lll}
(1_a, 1'_a, 2_c, 2'_a, 3_b, 3'_a), & (1_b, 1'_b, 2_a, 4'_c, 4_b, 2'_b), & (1_c, 1'_c, 4_c, 3'_b, 3_c, 4'_c), \\
(2_a, 2'_c, 4_c, 4'_a, 3_a, 3'_c), & (1_b, 3'_b, 2_b, 1'_c, 3_b, 4'_b), & (1_a, 2'_a, 4_a, 3'_a, 2_c, 4'_a), \\
(1_c, 2'_c, 3_a, 1'_b, 4_b, 3'_c), & (2_b, 2'_b, 3_c, 1'_a, 4_a, 4'_b). &
\end{array}$$

This completes the metamorphosis of the W-design on $K_{12,12}$ into a 6-cycle system on $K_{12,12}$. □

Now using Lemmas 2.1 and 2.2 together with the Construction, we have the following corollary.

COROLLARY 2.3 *There exists a $K_{3,3}$-design of order* 1 *(mod* 36*) which has a metamorphosis into a* 6*-cycle system.*

3 The case $\lambda = 2$

When $\lambda = 2$ the order n is 1 (mod 9), so let $n = 1 + 3 \cdot 3b$; we use the construction with $h = 1$, $a' = a = 3$, $s = 3b$, $t = 3$.

LEMMA 3.1 *There exists a* 2*-fold W-design (X, K) of order* 10 *with a metamorphosis into a* 2*-fold* 6*-cycle system (X, C) of order* 10.

Proof A suitable design is given by (X, K, C) where $X = \mathbb{Z}_{10}$, $K = \{(i, 5+i, 6+i : 3+i, 9+i, 8+i) \mid 0 \le i \le 9\}$, and $C = \{(i, 5+i, 6+i, 3+i, 9+i, 8+i) \mid 0 \le i \le 9\} \cup \{(2i, 2+2i, 5+2i, 1+2i, 3+2i, 6+2i) \mid 0 \le i \le 4\}$. □

LEMMA 3.2 *There exists a* 2*-fold W-design of $2K_{3,3}$ with a metamorphosis into a* 2*-fold* 6*-cycle system.*

Proof A suitable design is given by (X, K, C) where

$$X = \{\{1,2,3\},\{1',2',3'\}\}, K = \{(1,1',2:2',3,3'),(1,1',3:3',2,2')\}$$

and $C = \{(1,1',2,2',3,3'),(1,1',3,3',2,2'),(1,2',3,1',2,3')\}$. □

From Lemmas 3.1 and 3.2 and the Construction we now have:

COROLLARY 3.3 *There exists a* 2*-fold $K_{3,3}$-design of order* 1 *(mod* 9*) which has a metamorphosis into a* 2*-fold* 6*-cycle system.*

4 The case $\lambda = 3$

Here the order n is 1 or 9 (mod 12); we use the construction with $h = 1$ or 9, $t = 6$, $a' = a = 2$ and $s = 2b$.

LEMMA 4.1 *There exists a* 3*-fold W-design (X, K) of $3K_{6,6}$ with a metamorphosis into a* 3*-fold* 6*-cycle system (X, C) of $3K_{6,6}$.*

Proof Trivially, $3K_{6,6}$ can be decomposed into 12 copies of $W = K_{3,3}$. Retaining a 6-cycle from each copy of W leaves three disjoint edges, so we have altogether 36 edges to form into a further six 6-cycles. We may choose these 36 edges so that they are all distinct, and so that they precisely cover $K_{6,6}$. This graph $K_{6,6}$ is then well-known to have a decomposition into six 6-cycles, completing the metamorphosis of $3K_{6,6}$. □

LEMMA 4.2 *There exists a 3-fold W-design with a metamorphosis into a 3-fold 6-cycle system, (X, K, C), of order 13.*

Proof Let $X = \mathbb{Z}_{13}$, $K = \{(i, 1+i, 3+i : 5+i, 9+i, 4+i), (i, 3+i, 6+i : 4+i, 10+i, 7+i) \mid 0 \le i \le 12\}$, $C = \{((i, 1+i, 3+i, 5+i, 9+i, 4+i), (i, 3+i, 6+i, 4+i, 10+i, 7+i), (i, 5+i, 11+i, 6+i, 2+i, 1+i) \mid 0 \le i \le 12\}$. □

LEMMA 4.3 *There exists a 3-fold W-design with a metamorphosis into a 3-fold 6-cycle system, (X, K, C), of order 21.*

Proof Let $X = \{(i, j) \mid 0 \le i \le 6,\ 1 \le j \le 3\}$,

$$\begin{aligned} K = \{ & ((i,3),(2+i,3),(5+i,2) : (6+i,1),(6+i,3),(4+i,1)), \\ & ((5+i,1),(i,3),(6+i,2) : (6+i,3),(4+i,2),(2+i,1)), \\ & ((i,1),(1+i,1),(2+i,1) : (3+i,1),(5+i,1),(i,2)), \\ & ((i,1),(2+i,1),(4+i,1) : (i,2),(1+i,2),(2+i,2)), \\ & ((i,1),(i,2),(1+i,1) : (4+i,2),(3+i,1),(i,3)), \\ & ((i,1),(1+i,2),(3+i,1) : (i,3),(5+i,1),(1+i,3)), \\ & ((i,1),(3+i,3),(i,2) : (5+i,3),(1+i,2),(6+i,3)), \\ & ((i,1),(4+i,3),(4+i,2) : (5+i,3),(i,3),(6+i,3)), \\ & ((i,2),(1+i,2),(4+i,2) : (2+i,2),(5+i,2),(5+i,3)), \\ & ((i,2),(2+i,2),(1+i,3) : (4+i,3),(5+i,3),(6+i,3)) \\ & \qquad \mid 0 \le i \le 6\}, \end{aligned}$$

and

$$
\begin{aligned}
C = \{ & ((i,3),(2+i,3),(5+i,2),(6+i,1),(6+i,3),(4+i,1)),\\
& ((5+i,1),(i,3),(6+i,2),(6+i,3),(4+i,2),(2+i,1)),\\
& ((i,1),(1+i,1),(2+i,1),(3+i,1),(5+i,1),(i,2)),\\
& ((i,1),(2+i,1),(4+i,1),(i,2),(1+i,2),(2+i,2)),\\
& ((i,1),(i,2),(1+i,1),(4+i,2),(3+i,1),(i,3)),\\
& ((i,1),(1+i,2),(3+i,1),(i,3),(5+i,1),(1+i,3)),\\
& ((i,1),(3+i,3),(i,2),(5+i,3),(1+i,2),(6+i,3)),\\
& ((i,1),(4+i,3),(4+i,2),(5+i,3),(i,3),(6+i,3)),\\
& ((i,2),(1+i,2),(4+i,2),(2+i,2),(5+i,2),(5+i,3)),\\
& ((i,2),(2+i,2),(1+i,3),(4+i,3),(5+i,3),(6+i,3))\\
& \mid 0 \le i \le 6\} \quad \cup
\end{aligned}
$$

$$
\begin{aligned}
\{ & ((i,1),(5+i,3),(6+i,1),(6+i,3),(1+i,1),(4+i,1)),\\
& ((i,1),(1+i,3),(4+i,3),(2+i,2),(4+i,1),(5+i,3)),\\
& ((i,3),(5+i,3),(1+i,2),(3+i,2),(i,2),(3+i,3)),\\
& ((i,2),(6+i,3),(3+i,2),(5+i,3),(4+i,2),(i,1)),\\
& ((i,1),(4+i,2),(3+i,1),(2+i,2),(6+i,1),(4+i,2)) \mid 0 \le i \le 6\}.
\end{aligned}
$$

□

LEMMA 4.4 *There exists a 3-fold W-design with a metamorphosis into a 3-fold 6-cycle system, (X, K, C), of $K_{21} \setminus K_9$.*

Proof Let $X = \mathbb{Z}_{11} \cup \{\infty\} \cup \{A_i \mid 1 \le i \le 9\}$; the hole of size 9 is $\{A_i \mid 1 \le i \le 9\}$.

$K = K_1 \cup K_2$ where $K_1 = \{(\infty, 5+i, 3+i : i, 4+i, 6+i), (i, 8+i, 3+i : 7+i, 2+i, 1+i) \mid 0 \le i \le 10\}$ and K_2 contains 36 copies of $K_{3,3} = W$, on the following sets of vertices, as given in the 9×4 table below. (For example, $A_1A_2A_3$ and $1\,2\,3$ give rise to one copy of $K_{3,3}$ on the vertex set $\{\{A_1, A_2, A_3\}, \{1, 2, 3\}\}$, and we may decide later which 6-cycle in this copy of W to retain for the metamorphosis, and which 3 edges to use in further 6-cycles.)

$A_1A_2A_3$	123	456	**789**	**10∞0**
$A_1A_2A_3$	123	456	789	10∞0
$A_1A_2A_3$	123	456	789	10∞0
$A_4A_5A_6$	**123**	**456**	789	10∞0
$A_4A_5A_6$	123	456	789	10∞0
$A_4A_5A_6$	123	456	789	10∞0
$A_7A_8A_9$	123	456	789	10∞0
$A_7A_8A_9$	123	456	789	10∞0
$A_7A_8A_9$	123	456	789	10∞0

Then $C = C_0 \cup C_1 \cup C_2 \cup C_3$ where C_0 is the set of 58 6-cycles remaining in the copies of W in the metamorphosis (so we do not explicitly list these), and $C_1 \cup C_2 \cup C_3$ contains the remaining 29 new 6-cycles arriving from the metamorphosis: C_1 comes from the edges removed from the copies of W in K_1, apart from a 1-factor $\{\{0,6\},\{1,3\},\{2,8\},\{4,5\},\{7,9\},\{10,\infty\}\}$, C_2 comes from the four bold entries in the above table, together with the six edges in the 1-factor just listed, and C_3 comes from all the remaining edges from the rest of the copies of W in the above table.

$$\begin{aligned} C_1 &= \{(0,\infty,1,2,4,7), (2,\infty,3,4,6,9), (4,\infty,5,6,8,0),\\ &\quad (6,\infty,7,8,10,2), (8,\infty,9,10,1,4), (1,0,2,3,6,7),\\ &\quad (5,1,6,10,7,2), (9,0,3,7,5,8), (0,5,3,9,4,10), (5,9,1,8,3,10)\}; \end{aligned}$$

$$C_2 = \{(A_1,\infty,10,A_2,7,9), (A_4,1,3,A_5,4,5), (0,6,A_6,2,8,A_3)\};$$

$$\begin{aligned} C_3 &= \{(A_1,5,A_3,4,A_2,6), (A_1,7,A_3,9,A_2,8), (A_1,10,A_3,0,A_2,\infty),\\ &\quad (A_4,\infty,A_6,10,A_5,0), (A_4,1,A_6,3,A_5,2), (A_4,4,A_6,6,A_5,5),\\ &\quad (A_1,1,A_2,5,A_7,4), (A_1,2,A_9,5,A_8,3),\\ &\quad (A_2,2,A_8,6,A_3,3), (A_3,1,A_9,3,A_7,2), (A_7,1,A_8,4,A_9,6),\\ &\quad (A_4,7,A_5,\infty,A_7,10), (A_4,8,A_9,\infty,A_8,9),\\ &\quad (A_5,8,A_8,0,A_6,9), (A_6,7,A_9,9,A_7,8), (A_7,7,A_8,10,A_9,0)\}. \end{aligned}$$

This completes the metamorphosis for this lemma. □

We now have

COROLLARY 4.5 *For all orders* 1 *or* 9 *(mod* 12*), there exists a* 3*-fold* $K_{3,3}$*-design with a metamorphosis into a* 3*-fold* 6*-cycle system.*

5 The case $\lambda = 6$

The order here is $n \equiv 0$ or 1 (mod 3). We begin with three lemmas.

LEMMA 5.1 *There exists a 6-fold W-design of order 6 with a metamorphosis into a 6-fold 6-cycle system, (X, K, C).*

Proof Let $X = \{\infty\} \cup \mathbb{Z}_5$, $K = \{(\infty, i, 1+i : 3+i, 2+i, 4+i), (\infty, i, 1+i : 2+i, 3+i, 4+i) \mid 0 \leq i \leq 4\}$, and

$$C = \{(\infty, i, 1+i, 3+i, 2+i, 4+i), (\infty, i, 1+i, 2+i, 3+i, 4+i), \\ (\infty, i, 2+i, 4+i, 1+i, 3+i) \mid 0 \leq i \leq 4\}.$$

□

LEMMA 5.2 *There exists a 6-fold W-design of order 7 with a metamorphosis into a 6-fold 6-cycle system, (X, K, C).*

Proof Let $X = \mathbb{Z}_7$, $K = \{(i, 1+i, 2+i : 4+i, 6+i, 3+i), (i, 1+i, 2+i : 4+i, 6+i, 3+i) \mid 0 \leq i \leq 6\}$, and

$$C = \{(i, 1+i, 2+i, 4+i, 6+i, 3+i), (i, 1+i, 2+i, 4+i, 6+i, 3+i), \\ (i, 6+i, 3+i, 1+i, 2+i, 5+i) \mid 0 \leq i \leq 6\}.$$

□

LEMMA 5.3 *There exists a 6-fold W-design of order 9 with a metamorphosis into a 6-fold 6-cycle system, (X, K, C).*

Proof Let $X = \{\infty\} \cup \mathbb{Z}_8$,

$$K = \{(\infty, i, 3+i : 1+i, 4+i, 2+i),\ (\infty, 1+i, 5+i : 2+i, 3+i, 6+i), \\ (i, 1+i, 3+i : 6+i, 5+i, 2+i) \mid 0 \leq i \leq 7\},$$

and

$$X = \{(\infty, i, 3+i, 1+i, 4+i, 2+i),\ (\infty, 1+i, 5+i, 2+i, 3+i, 6+i), \\ (i, 1+i, 3+i, 6+i, 5+i, 2+i) \mid 0 \leq i \leq 7\} \\ \cup \{(\infty, 4+i, i, 1+i, 2+i, 6+i) \mid 0 \leq i \leq 7\} \\ \cup \{(0,2,4,5,7,1),\ (1,3,5,6,0,2),\ (2,4,6,7,1,3),\ (3,5,7,0,6,4)\}.$$

□

COROLLARY 5.4 *For all orders 0 or 1 (mod 3), there exists a 6-fold $K_{3,3}$-design with a metamorphosis into a 6-fold 6-cycle system.*

Proof First, when n is 0 or 1 (mod 6), we use the Construction with $h = 0$ or 1, $t = 3$, $a' = a = 2$, $s = 2b$ and $n = h + 6b$, together with three copies of Lemma 3.2 and Lemma 5.1 or 5.2.

Secondly, when n is 3 or 4 (mod 6), we use the Construction with $h = 0$ or 1, $t = 3$, $a' = 3$ and $a = 2$, together with three copies of Lemma 3.2, and Lemmas 5.1 and 5.3 (when $h = 0$), or Lemma 5.2 and three copies of Lemma 3.1 (when $h = 1$). □

6 The case $\lambda = 9$

Here the order n must be 1 (mod 4). We start with two lemmas.

LEMMA 6.1 *There exists a 9-fold W-design of order 17 with a metamorphosis into a 9-fold 6-cycle system, (X, K, C).*

Proof Let $X = \mathbb{Z}_{17}$,

$$\begin{aligned} K = \{&(i, 1+i, 2+i : 7+i, 5+i, 8+i), (i, 14+i, 12+i : 11+i, 6+i, 10+i),\\ &(i, 7+i, 3+i : 8+i, 9+i, 6+i), (i, 4+i, 8+i : 9+i, 2+i, 14+i),\\ &(i, 12+i, 7+i : 8+i, 1+i, 3+i), (i, 6+i, 12+i : 10+i, 9+i, 4+i),\\ &(i, 7+i, 14+i : 6+i, 2+i, 16+i), (i, 8+i, 16+i : 15+i, 12+i, 5+i)\\ &\mid 0 \le i \le 16\}, \end{aligned}$$

and

$$\begin{aligned} C = \{&(i, 1+i, 2+i, 7+i, 5+i, 8+i), (i, 14+i, 12+i, 11+i, 6+i, 10+i),\\ &(i, 7+i, 3+i, 8+i, 9+i, 6+i), (i, 4+i, 8+i, 9+i, 2+i, 14+i),\\ &(i, 12+i, 7+i, 8+i, 1+i, 3+i), (i, 6+i, 12+i, 10+i, 9+i, 4+i),\\ &(i, 7+i, 14+i, 6+i, 2+i, 16+i), (i, 8+i, 16+i, 15+i, 12+i, 5+i)\\ &\mid 0 \le i \le 16\} \quad \cup\\ \{&(i, 8+i, 2+i, 6+i, 4+i, 7+i), (i, 8+i, 2+i, 6+i, 4+i, 7+i),\\ &(i, 8+i, 3+i, 5+i, 13+i, 2+i), (i, 11+i, 3+i, 9+i, 15+i, 13+i)\\ &\mid 0 \le i \le 16\}. \end{aligned}$$

□

LEMMA 6.2 *There exists a 9-fold W-design of $9(K_{17} \setminus K_5)$ with a metamorphosis into a 9-fold 6-cycle system, (X, K, C).*

Proof Let $X = \{\infty\} \cup \mathbb{Z}_{11} \cup \{A, B, C, D, E\}$, where the hole of size 5 is $\{A, B, C, D, E\}$.

$K = K_1 \cup K_2$ where

$$K_1 = \{(\infty, 5+i, 3+i : i, 4+i, 6+i), (i, 8+i, 3+i : 7+i, 2+i, 1+i) \mid 0 \leq i \leq 10\},$$

each taken *three* times, and K_2 is given by 60 copies of $K_{3,3}$ on the following sets of vertices as given in the 10×6 table below. (For example, ABC and $0\,1\,2$ give rise to one copy of $K_{3,3}$, on vertex set $\{\{A, B, C\}, \{0, 1, 2\}\}$, and we may decide later which 6-cycle to retain for the metamorphosis.)

ABC	0 1 2	3 4 5	6 7 8	9 10 ∞	0 1 2	6 7 8
ABD	0 1 2	3 4 5	6 7 8	9 10 ∞	0 1 2	6 7 8
ABE	0 1 2	3 4 5	6 7 8	9 10 ∞	0 1 2	6 7 8
ACD	0 1 2	3 4 5	6 7 8	9 10 ∞	3 4 5	9 10 ∞
ACE	0 1 2	3 4 5	6 7 8	9 10 ∞	3 4 5	9 10 ∞
ADE	0 1 2	3 4 5	6 7 8	9 10 ∞	3 4 5	9 10 ∞
BCD	0 1 2	3 4 5	6 7 8	9 10 ∞	3 4 5	9 10 ∞
BCE	0 1 2	3 4 5	6 7 8	9 10 ∞	3 4 5	9 10 ∞
BDE	0 1 2	3 4 5	6 7 8	9 10 ∞	3 4 5	9 10 ∞
CDE	0 1 2	6 7 8	6 7 8	6 7 8	0 1 2	0 1 2

Then $C = C_1 \cup C_2 \cup C_3 \cup C_4$ where C_1 comes from K_1, C_2 and C_3 come from the edges removed from the copies of W in K_1, apart from a 1-factor $\{\{0, 6\}, \{1, 3\}, \{2, 8\}, \{4, 5\}, \{7, 9\}, \{10, \infty\}\}$. This 1-factor is then used, with edges removed from the copies of W in K_2, to make a further collection of 31 6-cycles, C_4.

$C_1 = \{(\infty, 5+i, 3+i, i, 4+i, 6+i), (i, 8+i, 3+i, 7+i, 2+i, 1+i) \mid 0 \leq i \leq 10\}$, *three* times each, $C_2 = \{(\infty, i, 5+i, 1+i, 4+i, 2+i), (i, 1+i, 2+i, 4+i, 8+i, 5+i) \mid 0 \leq i \leq 10\}$,

$$\begin{aligned} C_3 \;=\; & \{(0, \infty, 1, 2, 4, 7), (2, \infty, 3, 4, 6, 9), (4, \infty, 5, 6, 8, 0), \\ & (6, \infty, 7, 8, 10, 2), (8, \infty, 9, 10, 1, 4), (1, 0, 2, 3, 6, 7), \\ & (5, 1, 6, 10, 7, 2), (9, 0, 3, 7, 5, 8), (0, 5, 3, 9, 4, 10), (5, 9, 1, 8, 3, 10)\}, \end{aligned}$$

$$\begin{aligned} C_4 = \{&(A,0,B,1,C,2),\ (A,6,B,7,C,8),\ (A,0,B,1,D,2),\ (A,6,B,7,D,8),\\ &(A,0,B,1,E,2),\ (A,6,B,7,E,8),\ (A,3,C,4,D,5),\ (A,9,C,10,D,\infty),\\ &(A,3,C,4,E,5),\ (A,9,C,10,E,\infty),\ (A,3,D,4,E,5),\ (A,9,D,10,E,\infty),\\ &(B,3,C,4,D,5),\ (B,9,C,10,D,\infty),\ (B,3,C,4,E,5),\ (B,9,C,10,E,\infty),\\ &(B,3,D,4,E,5),\ (B,9,D,10,E,\infty),\ (C,6,D,7,E,8),\ (C,0,D,1,E,2),\\ &(A,\infty,10,B,4,5),\ (C,2,8,D,0,6),\ (C,3,1,E,7,9),\ (A,10,B,1,D,0),\\ &(A,4,B,7,E,3),\ (A,2,C,0,D,6),\ (A,8,C,1,E,9),\ (A,7,D,2,E,1),\\ &(B,2,C,6,D,\infty),\ (B,8,C,7,E,0),\ (B,5,D,8,E,6)\}. \end{aligned}$$

This completes the lemma. □

Since the order n is 1 (mod 4), we consider $n \equiv 1$, 5 or 9 (mod 12). Note that orders 1 and 9 (mod 12) come from taking three copies of a 3-fold design (see Section 4 above). For order 5 (mod 12), the Construction with $h = 5$, $t = 12$, $a' = a = 1$, and Lemmas 6.1 and 6.2 above, together with nine copies of Lemma 2.2, yield the result. So we have the following corollary.

COROLLARY 6.3 *For all orders* 1 *(mod* 4*) there exists a* 9*-fold* $K_{3,3}$*-design with a metamorphosis into a* 9*-fold* 6*-cycle system.*

7 The case $\lambda = 18$

Here we consider *any* order $n \geq 6$. Three copies of a 6-fold system deal with orders 0 or 1 (mod 3), so consider $n \equiv 2$ (mod 3).

We need the following three lemmas.

LEMMA 7.1 *There exists an* 18*-fold* W*-design of order* 8 *with a metamorphosis into an* 18*-fold* 6*-cycle system,* (X, K, C)*.*

Proof Let $X = \mathbb{Z}_8$,

$$\begin{aligned} K = \{&(i,7+i,5+i:3+i,1+i,4+i),\ (i,3+i,4+i:7+i,5+i,1+i),\\ &(i,1+i,2+i:6+i,3+i,7+i),\ (i,1+i,5+i:4+i,6+i,3+i),\\ &(i,7+i,5+i:6+i,2+i,3+i),\ (i,3+i,2+i:5+i,7+i,1+i),\\ &(i,4+i,7+i:5+i,6+i,1+i) \mid 0 \leq i \leq 7\}, \end{aligned}$$

and

$$\begin{aligned}
C &= \{(i,7+i,5+i,3+i,1+i,4+i),\ (i,3+i,4+i,7+i,5+i,1+i),\\
&\quad (i,1+i,2+i,6+i,3+i,7+i),\ (i,1+i,5+i,4+i,6+i,3+i),\\
&\quad (i,7+i,5+i,6+i,2+i,3+i),\ (i,3+i,2+i,5+i,7+i,1+i),\\
&\quad (i,4+i,7+i,5+i,6+i,1+i) \mid 0 \le i \le 7\} \quad \cup\\
&\quad \{(i,2+i,4+i,6+i,3+i,1+i),\ (i,2+i,5+i,7+i,4+i,1+i)\\
&\qquad \mid 0 \le i \le 7\} \quad \cup\\
&\quad \{(0,2,5,1,7,4),\ (1,3,6,2,0,5),\ (0,1,3,6,2,4),\ (1,2,3,5,7,4),\\
&\qquad (3,5,7,2,6,4),\ (0,3,7,1,6,4),\ (0,6,1,4,7,3),\ (0,5,7,2,4,6),\\
&\qquad (2,5,6,7,3,0),\ (1,5,2,4,0,6),\ (7,0,5,4,1,3),\ (2,6,3,5,1,7)\}.
\end{aligned}$$

□

LEMMA 7.2 *There exists an 18-fold W-design of $18(K_8 \setminus K_2)$ with a metamorphosis into an 18-fold 6-cycle system, (X, K, C).*

Proof Let $X = \mathbb{Z}_6 \cup \{A, B\}$, where $\{A, B\}$ is the hole of size 2,

$$\begin{aligned}
K &= \{(A,i,1+i:3+i,2+i,5+i),\ (A,i,1+i:3+i,2+i,5+i),\\
&\quad (A,i,1+i:2+i,3+i,5+i),\ (A,i,1+i:3+i,B,4+i),\\
&\quad (A,i,1+i:3+i,B,4+i),\ (A,i,1+i:3+i,B,4+i),\\
&\quad (i,1+i,3+i:2+i,5+i,B),\ (i,1+i,2+i:3+i,5+i,B),\\
&\quad (i,1+i,2+i:3+i,5+i,B)\},
\end{aligned}$$

and

$$\begin{aligned}
C &= \{(A,i,1+i,3+i,2+i,5+i),\ (A,i,1+i,3+i,2+i,5+i),\\
&\quad (A,i,1+i,2+i,3+i,5+i),\ (A,i,1+i,3+i,B,4+i),\\
&\quad (A,i,1+i,3+i,B,4+i),\ (A,i,1+i,3+i,B,4+i),\\
&\quad (i,1+i,3+i,2+i,5+i,B),\ (i,1+i,2+i,3+i,5+i,B),\\
&\quad (i,1+i,2+i,3+i,5+i,B)\} \quad \cup\\
&\quad \{(A,5+i,1+i,4+i,2+i,i),\ (A,1+i,4+i,B,2+i,i),\\
&\qquad (A,i,3+i,B,4+i,1+i),\ (i,B,3+i,1+i,4+i,2+i)\} \quad \cup\\
&\quad \{(0,2,5,3,1,4),\ (1,3,0,4,2,5),\ (2,4,1,5,3,0)\}.
\end{aligned}$$

LEMMA 7.3 *There exists an 18-fold W-design of $18K_{11}$ with a metamorphosis into an 18-fold 6-cycle system, (X, K, C).*

Proof Note that although a W-design of $9K_{11}$ exists, this is not an admissible order for a 9-fold 6-cycle system; nevertheless, we may take two copies of such a 9-fold W-design of order 11.
Let $X = \mathbb{Z}_{11}$, let K be two copies of

$$\{(i, 1+i, 3+i : 6+i, 2+i, 5+i),\ (i, 4+i, 8+i : 6+i, 3+i, 5+i),\\ (i, 1+i, 6+i : 4+i, 5+i, 3+i),\ (i, 1+i, 4+i : 9+i, 5+i, 6+i),\\ (i, 5+i, 6+i : 3+i, 9+i, 7+i) \mid 0 \le i \le 10\},$$

and let $C = C_1 \cup C_2$ where C_1 is two copies of

$$\{(i, 1+i, 3+i, 6+i, 2+i, 5+i),\ (i, 4+i, 8+i, 6+i, 3+i, 5+i),\\ (i, 1+i, 6+i, 4+i, 5+i, 3+i),\ (i, 1+i, 4+i, 9+i, 5+i, 6+i),\\ (i, 5+i, 6+i, 3+i, 9+i, 7+i) \mid 0 \le i \le 10\},$$

and

$$C_2 = \{(i, 4+i, 5+i, 7+i, 10+i, 6+i),\ (i, 4+i, 5+i, 7+i, 10+i, 6+i),\\ (i, 10+i, 6+i, 4+i, 7+i, 3+i),\ (i, 5+i, 1+i, 4+i, 2+i, 3+i),\\ (i, 1+i, 6+i, 10+i, 8+i, 9+i) \mid 0 \le i \le 10\}.$$

□

COROLLARY 7.4 *For all orders $n \ge 6$ there exists an 18-fold $K_{3,3}$-design with a metamorphosis into an 18-fold 6-cycle system.*

Proof Three copies of a 6-fold system deal with orders 0 or 1 (mod 3), so consider $n \equiv 2 \pmod 3$. We use the construction with $h = 2$, $t = 3$, and (for $n \equiv 2 \pmod 6$) $a' = a = 2$, or with $h = 2$, $t = 3$, and (for $n \equiv 4 \pmod 6$) $a' = 3$, $a = 2$. Then Lemmas 7.1, 7.2 and 7.3 above, together with nine copies of $2K_{3,3}$ (Lemma 3.2) complete the construction in this case for all orders 2 (mod 3). □

8 Concluding remarks

For any value of $\lambda = 18x + y$ where $0 \le y < 18$, we may now combine x copies of an 18-fold design with one y-fold design (which itself may possibly be a multiple number of copies; see Table 2 in Section 1, reproduced in the theorem below). Hence we obtain a λ-fold $K_{3,3}$-design of any admissible order as given in the table below, which has a metamorphosis into a λ-fold 6-cycle system of the same order.

We record this as follows.

THEOREM 8.1 *There exists a λ-fold $K_{3,3}$-design of order n which has a metamorphosis into a λ-fold 6-cycle system of the same order n, if and only if n is as given below:*

λ (mod 18)	order n
$1, 5, 7, 11, 13, 17$	1 (mod 36)
$2, 4, 8, 10, 14, 16$	1 (mod 9)
$3, 15$	$1, 9$ (mod 12)
$6, 12$	$0, 1$ (mod 3), $n \geq 6$
9	1 (mod 4), $n \geq 6$
0	any $n \geq 6$

Acknowledgements

Most of the work on this paper was done during a visit by the first author (EJB) to the University of Catania in January 2000. EJB thanks both the Department of Mathematics at the University of Catania and also the second author GQ, for wonderful hospitality during her visit.

References

[1] E.J. Billington, *The metamorphosis of lambda-fold 4-wheel systems into lambda-fold bowtie systems*, Australasian Journal of Combinatorics **22** (2000), 265–286.

[2] E.J. Billington and C.C. Lindner, *The metamorphosis of lambda-fold 4-wheel systems into lambda-fold 4-cycle systems*, Utilitas Mathematica **59** (2001), 215–235.

[3] C.C. Lindner and A. Rosa, *The metamorphosis of λ-fold block designs with block size four into λ-fold triple systems*, Journal of Statistical Planning and Inference (to appear).

[4] C.C. Lindner and A.P. Street, *The metamorphosis of λ-fold block designs with block size four into λ-fold 4-cycle systems*, Bulletin of the Institute of Combinatorics and its Applications **28** (2000), 7–18; Corrigendum *ibid.* **29** (2000), 88.

[5] R.C. Mullin and H.-D.O.F. Gronau, *PBDs and GDDs: the basics*, in CRC Handbook of Combinatorial Designs (eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton 1996, pp. 185–193.