ON SUPER EDGE-MAGIC GRAPHS

R. M. FIGUEROA-CENTENO, R. ICHISHIMA, AND F. A. MUNTANER-BATLE

Dedicated to Farrokh Saba

ABSTRACT. A (p, q) graph G is edge-magic if there exists a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that $f(u) + f(v) + f(uv) = k$ is a constant, called the valence of f, for any edge uv of G. Moreover, G is said to be super edge-magic if $f(V(G)) = \{1, 2, \ldots, p\}$. Every super edge-magic (p, q) graph is cordial, and it is harmonious and sequential whenever it is a tree or $q \geq p$. In this paper, it is shown to be edge-antimagic as well. The super edge-magic properties of several classes of connected and disconnected graphs are studied. Furthermore, we prove that there can be arbitrarily large gaps among the possible valences for certain super edge-magic graphs. We also establish that the disjoint union of multiple copies of a super edge-magic linear forest is super edge-magic if the number of copies is odd.

1. INTRODUCTION

Lately, new life has been injected into the subject of edge-magic labelings of graphs through a paper by Ringel and Lladó [10]; the study of which originated in 1970 in two papers by Kotzig and Rosa [7, 8]. This has led naturally to the definition of a particular type of edge-magic labelings, called super edge-magic labelings, introduced by Enomoto, Lladó, Nakamigawa and Ringel [2]. These are interesting since relationships between super edge-magic labelings and previously well studied labelings have been found by the authors [3], e.g., every super edge-magic (p, q)graph is cordial and whenever it is a tree or $q \geq p$ it is harmonious and sequential as well. Thus, the construction of classes of super edge-magic graphs enlarges the collection of graphs that are, for instance, known to be harmonious. Also, since there are few graphs that were previously shown to be (super) edge-magic (such as caterpillars and cycles), we have decided to initially

---

Date: March 23, 2001.

Key words and phrases. cordial labelling, edge-magic labelling, harmonious labelling, sequential labelling.

The second author would like to thank Pittsburg State University, where part of this research was conducted while he was there as a visiting researcher.

increase our knowledge of the problem by investigating graphs that are somehow related to these. Another motivation is to develop the theoretical tools necessary to deal with questions that pertain to (super) edge-magic labelings of graphs. Indeed, the obvious proof technique to show that a particular class of graphs is (super) edge-magic is to provide a labeling of it; however, proving that a graph that satisfies the elementary necessary conditions is not (super) edge-magic can be far harder. This accumulation of knowledge is, of course, done with the hope that we may eventually assault the central question of the topic, that is, are all trees (super) edge-magic?

In this paper, we present several classes of (super) edge-magic graphs (connected and disconnected); noting that, in the past, it has been difficult to obtain classes of disconnected graphs that are cordial or sequential.

In order to formalize this presentation, we introduce some necessary definitions and refer the reader to Chartrand and Lesniak [1] or Hartsfield and Ringel [6] for all other terms and notation not provided in this paper.

For a (p, q) graph G, a bijective function

$$f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$$

is an *edge-magic labeling* of G if $f(u)+f(v)+f(uv) = k$ is a constant, which is independent on the choice of any edge uv of G. If such a labeling exists, then k is called the *valence* of the labeling and G is said to be an *edge-magic graph.* Furthermore, $f$is a *super edge-magic labeling* if $f(V(G)) = \{1, 2, \ldots, p\}$. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling.

We will find the following basic results useful; see [3].

Lemma 1.1. *A (p, q) graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the set*

$$S = \{f(u) + f(v) : uv \in E(G)\}$$

consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = p + q + s$, where $s = \min(S)$ and

$$\begin{aligned} S &= \{f(u) + f(v) : uv \in E(G)\} \\ &= \{k - (p+1), k - (p+2), \ldots, k - (p+q)\}. \end{aligned}$$

In light of this result, it suffices to exhibit the vertex labeling of a super edge-magic graph. However, we will also provide the valences to increase the clarity of our results.

The next result is particularly useful in showing that a regular graph is *not* super edge-magic; see [3].

Lemma 1.2. *Let G be an r-regular super edge-magic (p, q) graph, where $r \geq 1$, then q is odd and the valence is $(4p + q + 3)/2$.*

In the following lemma, Enomoto, Lladó, Nakamigawa and Ringel [2] provide an upper bound for the size of super edge-magic graphs.

Lemma 1.3. *If a (p,q) graph is super edge-magic, then $q \leq 2p-3$.*

This lemma together with the First Theorem of Graph Theory implies that the minimum degree is at most 3 for every super edge-magic graph.

Now, observe that given an edge-magic labeling f of a (p,q) graph, it is always possible to find a *complementary* edge-magic labeling $\bar{f}$ such that $\bar{f}(x) = p+q+1-f(x)$ for every $x \in V(G) \cup E(G)$; see [7]. Notice that this operation does not preserve super edge-magic labelings unless $G \cong \overline{K}_n$.

The following two lemmas provide a simple but often powerful method to find new edge-magic graphs from known edge-magic graphs.

Lemma 1.4. *Let G be an edge-magic graph, f an edge-magic labeling of G, and $u, v \in V(G)$ such that $f(u)+f(v) = k$, where k is the valence of f, then $G+uv$ is edge-magic.*

Proof. Notice that if $f(u)+f(v) = k$, then $uv \notin E(G)$; for otherwise $f(uv) = 0$. Therefore, we immediately obtain an edge-magic labeling g of $G+uv$ by letting $g(x) = f(x)+1$ for every $x \in (V(G) \cup E(G)) - \{uv\}$ and $g(uv) = 1$. ∎

Lemma 1.4 cannot be applied if f is a super edge-magic labeling of a connected graph G. To see why, let G be a super edge-magic graph with a super edge-magic labeling f. By Lemma 1.1, the valence of f is

$$k = p + q + \min\left(\{f(u)+f(v) : uv \in E(G)\}\right).$$

Thus, $k \geq p+q+1+2 \geq 2p+2$ since G is connected so that $q \geq p-1$. Now,

$$\max\left(\{f(u)+f(v) : uv \in E(G)\}\right) \leq p + (p-1) = 2p-1$$

since f is a super edge-magic labeling. Therefore,

$$f(u)+f(v) \leq 2p-1 < 2p+2 \leq k.$$

Whereas Lemma 1.4 concerns the addition of an edge, our next lemma involves the deletion of an edge.

Lemma 1.5. *If G is an edge-magic graph and f is an edge-magic labeling of G for which there exists $e \in E(G)$ such that $f(e) = 1$, then $G-e$ is edge-magic.*

Proof. We immediately obtain an edge-magic labeling g of $G-e$ by letting $g(x) = f(x)-1$ for every $x \in (V(G) \cup E(G)) - \{e\}$. ∎

2. Results on Path-Related Graphs

Paths were shown to be super edge-magic by Ringel and Lladó [10]. Thus, in this section, it is natural to explore some infinite classes of graphs obtained from paths.

The following result is of interest because it shows how taking the k-th power of a super edge-magic graph may or may not imply that the resulting graph is super edge-magic. Observe that the graphs $P_2^2 \cong P_2 \cong K_2$ and $P_3^2 \cong K_3$ are clearly super edge-magic.

Theorem 2.1. *For every integer $n \geq 4$, the graph $G \cong P_n^k$ is super edge-magic if and only if $k = 1$ or 2.*

Proof. The order of P_n^3 is n and its size is $3n - 6$ whenever $n \geq 4$; so, by Lemma 1.3, P_n^k is not super edge-magic for $k \geq 3$.

For the converse, let G be the graph defined as follows:

$$V(G) = \{v_i : i = 1, 2, \ldots, n\}$$

and

$$E(G) = \{v_i v_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_i v_{i+2} : i = 1, 2, \ldots, n-2\}.$$

Then consider the vertex labeling $f : V(G) \to \{1, 2, \ldots, n\}$suc h that $f(v_i) = i$. Consequently, $f(v_i)+f(v_{i+1}) = 2i+1$ for each i with $1 \leq i \leq n-1$ and $f(v_i) + f(v_{i+2}) = 2i + 2$ for every i with $1 \leq i \leq n - 2$. Thus,

$$\{f(u) + f(v) : uv \in E(G)\} = \{3, 4, \ldots, 2n - 1\}$$

is a set of $2n - 3$ consecutive integers.

Therefore, it follows that f is a super edge-magic labeling of G with valence $3n$ by Lemma 1.1, obtaining the desired result. ∎

The following result is due to Kotzig and Rosa [7].

Theorem 2.2. *The graph nP_2 is super edge-magic if n is odd. Conversely, if nP_2 is edge-magic, then n is odd.*

The next corollary provides a necessary condition for a given labeling of the vertices and edges of a graph to be edge-magic.

Corollary 2.3. *Let G be a graph, and assume that H is a 1-regular subgraph of G of even size. Then a labeling of G, where the vertices and edges of H are labeled with consecutive integers, is never an edge-magic labeling.*

Proof. Assume, to the contrary, that there exists an edge-magic labeling of G and a 1-regular subgraph H of G of even size whose vertices and edges are labeled with consecutive integers. Now, let s be the smallest integer assigned to any vertex or edge of H. Subtract $s - 1$ from each of the labels of H. Then this produces an edge-magic labeling of nP_2, where n is even, contradicting the previous theorem. ∎

The next theorem partially extends Theorem 2.2 in terms of linear forests (forests whose components are paths).

Theorem 2.4. *Let $F \cong \bigcup_{i=1}^{l} P_{n_i}$, where n_i is a positive integer for all values of i, be a super edge-magic linear forest. Then mF is super edge-magic if m is odd.*

Proof. For $m = 1$, the result is trivial, so we assume that $m \geq 3$.
Let

$$V(F) = \bigcup_{i=1}^{l} \{v_{i,j} : 1 \leq j \leq n_i\}$$

and

$$E(F) = \bigcup_{i=1}^{l} \{v_{i,j}v_{i,j+1} : 1 \leq j \leq n_i - 1\},$$

and suppose then that $f : V(F) \to \left\{1, 2, \ldots, \sum_{i=1}^{l} n_i\right\}$ is a vertex labeling that extends to a super edge-magic labeling of F with valence k.
Now, let mF be the linear forest with

$$V(mF) = \bigcup_{t=1}^{m} \bigcup_{i=1}^{l} \{v_{i,j}^t : 1 \leq j \leq n_i\}$$

and

$$E(mF) = \bigcup_{t=1}^{m} \bigcup_{i=1}^{l} \{v_{i,j}^t v_{i,j+1}^t : 1 \leq j \leq n_i - 1\}.$$

Then consider the vertex labeling $g : V(mF) \to \left\{1, 2, \ldots, m\sum_{i=1}^{l} n_i\right\}$ such that

$$g\left(v_{i,j}^t\right) = \begin{cases} mf(v_{i,j}) - m + t, & \text{if } j \text{ is even and } 1 \leq t \leq m; \\ mf(v_{i,j}) + \frac{1-m}{2} + t, & \text{if } j \text{ is odd and } 1 \leq t \leq \frac{m-1}{2}; \\ mf(v_{i,j}) + \frac{1-3m}{2} + t, & \text{if } j \text{ is odd and } \frac{m+1}{2} \leq t \leq m. \end{cases}$$

Finally, notice that g extends to a super edge-magic labeling of mF with valence $mk + 3(1 - m)/2$. ∎

The converse of the previous theorem is not true. For example, the linear forest $2P_4$ is super edge-magic (label consecutively the vertices of one path 1, 7, 2 and 5, and the ones of the other 3, 8, 4 and 6 to obtain a super edge-magic labeling of $2P_4$wit h valence 21).

The previous theorem makes it worthwhile to investigate classes of super edge-magic linear forests.

It is interesting, in light of Kotzig and Rosa's result for nP_2, to point out that the next result holds for all positive integers n.

Theorem 2.5. *The linear forest $F \cong P_3 \cup nP_2$ is super edge-magic for every positive integer n.*

Proof. Let F be the linear forest with

$$V(F) = \{x, y, z\} \cup \{u_i, v_i : 1 \le i \le n\}$$

and

$$E(F) = \{xy, yz\} \cup \{u_i v_i : 1 \le i \le n\}.$$

We now consider three possible cases.

Case 1: Suppose that $n = 1$, and let $f : V(F) \to \{1, 2, \ldots, 5\}$ be the vertex labeling of F such that $f(x) = 2$, $f(y) = 3$, $f(z) = 4$, $f(u_1) = 1$ and $f(v_1) = 5$. Then f extends to a super edge-magic labeling of F with valence 13.

Case 2: Suppose that $n = 2m + 1$, where m is a positive integer, and let $f : V(F) \to \{1, 2, \ldots, 2n + 3\}$ be the vertex labeling of F defined as follows:

$$f(w) = \begin{cases} 3m+3, & \text{if } w = x; \\ 2m+3, & \text{if } w = y; \\ m+3, & \text{if } w = z; \\ i, & \text{if } w = u_i \text{ and } 1 \le i \le m+2; \\ i+3m+3, & \text{if } w = v_i \text{ and } 1 \le i \le m+2; \\ i+1, & \text{if } w = u_i \text{ and } m+3 \le i \le 2m+1; \\ i+m+1, & \text{if } w = v_i \text{ and } m+3 \le i \le 2m+1. \end{cases}$$

Then f extends to a super edge-magic labeling of F whose valence is $(9n + 17)/2$.

Case 3: Suppose that $n = 2m$, where m is a positive integer, and let $f : V(F) \to \{1, 2, \ldots, 2n + 3\}$ be the vertex labeling of F obtained as follows:

$$f(w) = \begin{cases} 2m+2, & \text{if } w = x; \\ m+1, & \text{if } w = y; \\ 2m+3, & \text{if } w = z; \\ i, & \text{if } w = u_i \text{ and } 1 \le i \le m; \\ i+3m+3, & \text{if } w = v_i \text{ and } 1 \le i \le m; \\ i+1, & \text{if } w = u_i \text{ and } m+1 \le i \le 2m; \\ i+m+3, & \text{if } w = v_i \text{ and } m+1 \le i \le 2m. \end{cases}$$

Then f extends to a super edge-magic labeling of F whose valence is $(9n + 16)/2$.

Therefore, we conclude that F is super edge-magic. ∎

Another infinite class of linear forests is shown to be super edge-magic in the following theorem.

Theorem 2.6. *The linear forest $F \cong P_2 \cup P_n$ is super edge-magic for every integer $n \geq 3$.*

Proof. Let $F \cong P_2 \cup P_n$ be the linear forest with

$$V(F) = \{u_1, u_2\} \cup \{v_i : 1 \leq i \leq n\}$$

and

$$E(F) = \{u_1u_2\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\},$$

and then consider a vertex labeling $f : V(F) \to \{1, 2, \ldots, n+2\}$.

We now proceed by cases.

Case 1: Assume that $n \equiv 0 \pmod 4$, and let $f(u_1) = 1$; $f(u_2) = \frac{1}{2}n + 3$; and

$$f(v_j) = \begin{cases} \frac{n}{2} + 2, & \text{if } j = 1; \\ \frac{n}{2} + 4, & \text{if } j = 3; \\ 2i, & \text{if } j = 4i \text{ and } 1 \leq i \leq \frac{n}{4}; \\ \frac{n}{2} + 2i + 4, & \text{if } j = 4i+1 \text{ and } 1 \leq i \leq \frac{n-4}{4}; \\ 2i + 3, & \text{if } j = 4i+2 \text{ and } 0 \leq i \leq \frac{n-4}{4}; \\ \frac{n}{2} + 2i + 3, & \text{if } j = 4i+3 \text{ and } 1 \leq i \leq \frac{n-4}{4}. \end{cases}$$

Then f extends to a super edge-magic labeling of F with valence $(5n + 12)/2$.

Case 2: Assume that $n \equiv 1 \pmod 4$, and let $f(u_1) = 1$; $f(u_2) = n + 2$; and

$$f(v_j) = \begin{cases} \frac{2j+n+5}{4}, & \text{if } j \text{ is odd}; \\ \frac{2j+3n+5}{4}, & \text{if } j \text{ is even and } 2 \leq j \leq \frac{n-1}{2}; \\ \frac{4j-n+5}{4}, & \text{if } j \text{ is even and } \frac{n+3}{2} \leq j \leq n-1. \end{cases}$$

Then f extends to a super edge-magic labeling of F with valence $(5n + 11)/2$.

Case 3: Assume that $n \equiv 2 \pmod 4$, and let $f(u_1) = 1$; $f(u_2) = \frac{1}{2}n + 2$; and

$$f(v_j) = \begin{cases} n + 2, & \text{if } j = 1; \\ n, & \text{if } j = 3; \\ n + 1, & \text{if } j = n; \\ \frac{n}{2} - 2i + 2, & \text{if } j = 4i \text{ and } 1 \leq i \leq \frac{n-2}{4}; \\ n - 2i, & \text{if } j = 4i+1 \text{ and } 1 \leq i \leq \frac{n-2}{4}; \\ \frac{n}{2} - 2i - 1, & \text{if } j = 4i+2 \text{ and } 0 \leq i \leq \frac{n-6}{4}; \\ n - 2i + 1, & \text{if } j = 4i+3 \text{ and } 1 \leq i \leq \frac{n-6}{4}. \end{cases}$$

Then f extends to a super edge-magic labeling of F with valence $(5n + 10)/2$.

Case 4: Assume that $n \equiv 3 \pmod 4$, and let $f(u_1) = 1$; $f(u_2) = n + 2$; and

$$f(v_j) = \begin{cases} \frac{j+3}{2}, & \text{if } j \text{ is odd and } 1 \leq j \leq \frac{n-1}{2}; \\ \frac{j+n+2}{2}, & \text{if } j \text{ is odd and } \frac{n+3}{2} \leq j \leq n; \\ \frac{j+n+3}{2}, & \text{if } j \text{ is even and } 2 \leq j \leq \frac{n-3}{2}; \\ \frac{j+4}{2}, & \text{if } j \text{ is even and } \frac{n+1}{2} \leq j \leq n-1. \end{cases}$$

Then f extends to a super edge-magic labeling of F with valence $(5n + 11)/2$.

Therefore, we conclude that F is super edge-magic. ∎

3. Results on Star-Related Graphs

In [2], it was shown that the complete bipartite graph $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$. The next theorem partially extends their result by determining all edge-magic and super edge-magic labelings of the star $K_{1,n}$.

Theorem 3.1. *Every star $K_{1,n}$ is super edge-magic. Moreover, there are exactly $3 \cdot 2^{\,n}$ distinct edge-magic labelings of $K_{1,n}$ of which only two are super edge-magic labelings up to isomorphisms.*

Proof. First, notice that the order of $K_{1,n}$ is $n + 1$ and its size is n. Next, define the star $G \cong K_{1,n}$ as follows: $V(G) = \{u\} \cup \{v_i : 1 \leq i \leq n\}$ and $E(G) = \{e_i = uv_i : 1 \leq i \leq n\}$. Assume that there exists an edge-magic labeling f of G, and let k be its valence. Then

$$\left(\sum_{i=1}^{n} (f(v_i) + f(e_i)) \right) + nf(u) = nk.$$

Thus, n divides $\sum_{i=1}^{n} (f(v_i) + f(e_i))$.

Now,

$$\begin{aligned} \left(\sum_{i=1}^{n} (f(v_i) + f(e_i)) \right) + f(u) &= 1 + \cdots + (2n + 1) \\ &= 2n^2 + 3n + 1, \end{aligned}$$

so

$$\sum_{i=1}^{n} (f(v_i) + f(e_i)) = 2n^2 + 3n + (1 - f(u)).$$

Hence, n divides $f(u) - 1$, but $1 \leq f(u) \leq 2n + 1$, which implies that $f(u)$ is 1, $n + 1$ or $2n + 1$. Since

$$nk = 2n^2 + 3n + 1 + (n - 1)g(u),$$

it follows that $k = 2n + 4$, $3n + 3$ or $4n + 2$, which correspond to $f(u) = 1$, $n + 1$, $2n + 1$, respectively.

It suffices now to exhibit labelings with each of the three possible valences, and then describe how to obtain all of the other labelings from them. Let f_1, f_2, and f_3 be edge-magic labelings of G defined as follows:

$$\begin{array}{lll} f_1(u) = 1, & f_1(v_i) = i + 1, & f_1(uv_i) = 2n + 2 - i, \\ f_2(u) = n + 1, & f_2(v_i) = i, & f_2(uv_i) = 2n + 2 - i, \\ f_3(u) = 2n + 1, & f_3(v_i) = i, \text{ and } & f_3(uv_i) = 2n + 1 - i, \end{array}$$

where $1 \leq i \leq n$. Then the valences of f_1,f_2,an d f_3 are $2n+4$, $3n+3$, and $4n+2$, respectively. Notice that all other edge-magic labelings of G can be obtained by permuting the labels of uv_i and v_i for any i with $1 \leq i \leq n$, and that of these $3 \cdot 2^n$ possible permutations, only f_1 and f_2 are super edge-magic labelings of G. ∎

The following corollary is an immediate consequence of the proof of the preceding theorem. It is interesting since Godbold and Slater [5] have conjectured that for sufficiently large cycles, there are no *gaps* between the possible valences.

Corollary 3.2. *Forevery integer $n \geq 2$, there exists a super edge-magic graph G such that $|k_1 - k_2| \geq n - 1$, where k_1 and k_2 are two possible distinct valences of G.*

The next corollary describes how new super edge-magic graphs can be found from known super edge-magic graphs.

Corollary 3.3. *For every positive integer n, the graph $K_2 + \overline{K}_n$ is super edge-magic.*

Proof. Let $G \cong K_{1,n}$ be defined as in the proof of the previous theorem, and consider the following edge-magic labeling g of G: $g(u) = n+1$, $g(v_i) = 2(n + 1) - i$ and $g(uv_i) = i$ for $1 \leq i \leq n$. Now, notice that the valence k is $3n + 3$ and $g(v_1) + g(v_i) = 4n + 3 - i$ for $2 \leq i \leq n$.

Then define the graph $H \cong K_2 + \overline{K}_n$ as follows: $V(H) = V(G)$ and $E(H) = E(G) \cup \{v_1v_i : 2 \leq i \leq n\}$; and consider the following edge-magic labeling of H with valence $6n$: $f(v) = g(v) + n - 1$ for any vertex v of H, $f(uv_i) = g(uv_i) + n - 1$ for $1 \leq i \leq n$ and $f(v_1v_i) = i - 1$ for $2 \leq i \leq n$.

Finally, observe that f is a super edge-magic labeling of this graph since $f(v) > f(e)$ for any vertex v and edge e of H. ∎

Notice that the above corollary establishes the sharpness of Lemma 1.3.

We remark that from the preceding proof, we can obtain a sequence of super edge-magic graphs as follows. Take the labeling f employed for $K_2 + \overline{K}_n$in the proof and then remove the edge labeled 1 from it and decrease all labels by 1. Continue in this fashion until arriving to $K_{1,n}$.

Every such labeling of each graph is edge-magic by Lemma 1.5 and its complementary labeling is super edge-magic.

The above corollary also allows us to characterize all the super edge-magic complete m-partite graphs.

Theorem 3.4. *The only super edge-magic complete m-partite graphs are $K_{1,n}$ and $K_{1,1,n}$, where $n \geq 1$.*

Proof. Recall that Enomoto, Lladó, Nakamigawa and Ringel [2] have already shown that the star $K_{1,n}$, $n \geq 1$, is the only super edge-magic complete bipartite graph. Furthermore, the complete 3-partite graph $K_{1,1,n} \cong K_2 + \overline{K}_n$ is super edge-magic by Corollary 3.3.

In order to see that $K_{1,n}$ and $K_{1,1,n}$ are the unique complete bipartite and 3-partite graphs, respectively, notice first that $m \leq 4$; for otherwise the minimum degree would be greater than 3, which is impossible. Thus, it remains to be shown that $K_{1,1,n}$ is the unique super edge-magic complete 3-partite graph and that there are no complete 4-partite graphs with this property.

For the uniqueness of $K_{1,1,n}$, let $G \cong K_{n_1,n_2,n_3}$ be a complete 3-partite graph with $n_1 \geq n_2 \geq n_3 \geq 1$. Then, assume, to the contrary, that $n_2 \geq 2$ and G is super edge-magic. Now, the order of G is $n_1 + n_2 + n_3$ and its size is $n_1n_2 + n_1n_3 + n_2n_3$; so, by Lemma 1.3,

$$n_1n_2 + n_1n_3 + n_2n_3 \leq 2n_1 + 2n_2 + 2n_3 - 3,$$

which, in turn, implies that $n_1n_3 \leq 2n_2 - 3$ since $n_2 \geq 2$ and $n_3 \geq 2$. Hence, $n_2n_3 \leq 2n_2 - 3$, so $2 - n_3 > 0$ from which we conclude that $n_3 = 1$. Therefore, if we apply Lemma 1.3 again, we get that $n_1 \leq 1$, producing a contradiction.

To show that there are no super edge-magic complete 4-partite graphs, observe that $K_{1,1,1,n}$ is not super edge-magic by Lemma 1.3 and all remaining graphs have minimum degrees greater than 3, completing the proof. ∎

The next two results show that some classes of *galaxies* (forests whose components are stars) are super edge-magic.

Theorem 3.5. *The galaxy $G \cong K_{1,n} \cup K_{1,n+1}$, $n \geq 1$, is super edge-magic.*

Proof. Define the galaxy G as follows:

$$V(G) = \{u, v\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n+1\}$$

and

$$E(G) = \{ux_i : 1 \leq i \leq n\} \cup \{vy_i : 1 \leq i \leq n+1\},$$

and then consider the vertex labeling $f : V(G) \to \{1, 2, \ldots, 2n+3\}$ such that $f(u) = 1$, $f(v) = 3$, $f(x_i) = 2i + 3$ $(1 \leq i \leq n)$, and $f(y_i) = 2i$ $(1 \leq i \leq n-1)$.

In order to show that f extends to a super edge-magic labeling of G, it suffices to verify the following by Lemma 1.1:

(a) $f(V(G)) = \{1, 2, \ldots, 2n+3\}$;

(b) $S = \{f(x) + f(y) : xy \in E(G)\}$ is a set of $2n+1$ consecutive integers.

For (a), observe that $f(u) = 1$ and $f(x_n) = 2n+3$. Also, if $i < n$, then $f(x_i) < f(x_n) = 2n+3$; and if $i < n+1$,th en $f(y_i) < f(y_{n+1}) = 2n+2 < 2n+3 = f(x_n)$. Hence, the maximum possible integer that can be used for a vertex label is $2n+3$. Now, $f(x_i) \neq f(x_j)$ if and only if $i \neq j$; and $f(y_i) \neq f(y_j)$ if and only if $i \neq j$. Further, notice that $f(x_i)$ is odd for every i with $1 \leq i \leq n$ and $f(y_i)$ is even for every i with $1 \leq i \leq n+1$. Thus, the set of vertex labels is $\{1, 2, \ldots, 2n+3\}$.

For (b), observe first that the minimum element in S is $3 + f(y_1) = 5$ and the maximum element is $3 + f(y_{n+1}) = 2n+5$. Now, $f(u) + f(x_i) \neq f(u) + f(x_j)$ if and only if $i \neq j$; and $f(u) + f(x_i)$ is even. In addition, $f(v) + f(y_i) \neq f(v) + f(y_j)$ if and only if $i \neq j$; and $f(v) + f(y_i)$ is odd. Therefore, all elements of S are distinct and $|S|$ is the size of G.

Finally, notice that the valence of the labeling f is $4n+9$, which completes the proof. ∎

The following theorem is a partial generalization of Theorem 3.1.

Theorem 3.6. *For positive integers m and n, where m is odd, the galaxy $G \cong mK_{1,n}$ is super edge-magic.*

Proof. Let G be the galaxy with

$$V(G) = \{u_i : 1 \leq i \leq m\} \cup \{x_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

and

$$E(G) = \{u_i x_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n\}.$$

Then consider the vertex labeling $f : V(G) \to \{1, 2, \ldots, m(n+1)\}$ such that

$$f(w) = \begin{cases} i, & \text{if } w = u_i \text{ for } 1 \leq i \leq m; \\ i + \frac{3m+1}{2}, & \text{if } w = x_{i,1} \text{ for } 1 \leq i < \frac{m+1}{2}; \\ i + \frac{m+1}{2}, & \text{if } w = x_{i,1} \text{ for } \frac{m+1}{2} \leq i \leq m; \\ f(x_{i,1}) + m(j-1), & \text{if } w = x_{i,j} \text{ for } 1 \leq i \leq m \text{ and } 2 \leq j \leq n; \end{cases}$$

hence, f extends to a super edge-magic labeling of G with valence $2mn + 2m + 3$. ∎

4. Results on 2-Regular Graphs

In this section, we consider 2-regular graphs. These are of interest to us since, in their seminal paper, Kotzig and Rosa [7] pondered on whether one may find necessary and sufficient conditions to determine if 2-regular graphs are edge-magic.

Consider the following result by Kotzig and Rosa [7].

Theorem 4.1. *Every n-cycle C_n is edge-magic.*

Also, recall the analogous result for super edge-magic graphs by Enomoto, Lladó, Nakamigawa and Ringel [2].

Theorem 4.2. *The n-cycle C_n is super edge-magic if and only if n is odd.*

The following is a natural generalization of Theorems 4.1 and 4.2.

Theorem 4.3. *The 2-regular graph $G \cong mC_n$ is super edge-magic if and only if $m \geq 1$ and $n \geq 3$ are odd.*

Proof. The case where $m = 1$ has already been handled by Kotzig and Rosa [7], so we assume that $m \geq 3$. Let G be the 2-regular graph with

$$V(G) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and

$$E(G) = \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\} \cup \{v_{i,n}v_{i,1} : 1 \leq i \leq m\}.$$

Then consider a vertex labeling $f : V(G) \to \{1, 2, \ldots, mn\}$ such that

$$f(v_{i,j}) = \begin{cases} i, & \text{if } 1 \leq i \leq m \text{ and } j = 1; \\ m\left(\left\lceil \frac{n}{2} \right\rceil + \frac{j-2}{2}\right) + \frac{2i+1+m}{2}, & \text{if } 1 \leq i \leq \frac{m-1}{2} \text{ and } j \text{ is even}; \\ m\left(\left\lceil \frac{n}{2} \right\rceil + \frac{j-2}{2}\right) + \frac{2i+1-m}{2}, & \text{if } \frac{m+1}{2} \leq i \leq m \text{ and } j \text{ is even}; \\ m\left(\frac{j-1}{2} + 1\right) + 1 - 2i, & \text{if } 1 \leq i \leq \frac{m-1}{2} \text{ and } j \neq 1 \text{ is odd}; \\ m\left(\frac{j-1}{2} + 2\right) + 1 - 2i, & \text{if } \frac{m+1}{2} \leq i \leq m \text{ and } j \neq 1 \text{ is odd}. \end{cases}$$

Therefore, f extends to a super edge-magic labeling of G with valence $\frac{(5n+2)m+3}{2}$.

The converse follows immediately from Lemma 1.2. ∎

The 2-regular graph $2C_4$ is edge-magic (label the vertices of one 4-cycle clockwise 1, 14, 9 and 13, and the ones of the other 4, 6, 12 and 5, and let the valence be 25). Therefore, the previous theorem cannot be strengthened by considering edge-magic graphs instead of super edge-magic graphs. The problem of determining whether mC_n is edge-magic or not when either m or n is even is open.

5. Results on Edge-Antimagic Graphs

Ringel [9] has also provided the definition for edge-antimagic graphs.

For a (p, q) graph G, a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ is an *edge-antimagic labeling* of G if

$$|\{f(u) + f(v) : uv \in E(G)\}| = q.$$

If such a labeling exists, then G is called an *edge-antimagic graph.*

In this section, we present some relationships between super edge-magic graphs and edge-antimagic graphs.

The following is an immediate consequence of Lemma 1.1.

Theorem 5.1. *Every super edge-magic graph is edge-antimagic.*

We then note that Lemma 1.3 follows from Theorem 5.1 and a comment by Ringel [9] to the effect that the inequality $q \leq 2p - 3$ holds for edge-antimagic (p, q) graphs.

Ringel [9] also mentioned that if a graph G of order p is edge-antimagic with an edge-antimagic labeling f, then

$$\{f(u) + f(v) : uv \in E(G)\} \subseteq \{3, 4, \ldots, 2p - 1\}.$$

This remark implies the following partial converse of Theorem 5.1.

Theorem 5.2. *If G is an edge-antimagic (p, q) graph with $q = 2p - 3$, then G is super edge-magic.*

Proof. Let G be an edge-antimagic (p, q) graph such that $q = 2p - 3$ with an edge-antimagic labeling f. Then

$$\{f(u) + f(v) : uv \in E(G)\} = \{3, 4, \ldots, 2p - 1\},$$

so the result follows from Lemma 1.2. ∎

Ringel [9] presented the following theorem as well.

Theorem 5.3. *If G is a maximal outerplanar graph of order p with exactly two vertices a, b of degree 2 and whose distance $d_H(a, b)$ on the Hamilton cycle H in G is*

$$\left\lfloor \frac{p}{2} \right\rfloor \text{ or } \left\lfloor \frac{p}{2} \right\rfloor - 1,$$

then G is edge-antimagic.

Since all maximal outerplanar (p, q) graphs satisfy $q = 2p - 3$, we have the following result from Theorems 5.2 and 5.3.

Corollary 5.4. *If G is a maximal outerplanar graph of order p with exactly two vertices a, b of degree 2 and whose distance $d_H(a, b)$ on the Hamilton cycle H in G is*

$$\left\lfloor \frac{p}{2} \right\rfloor \text{ or } \left\lfloor \frac{p}{2} \right\rfloor - 1,$$

then G is super edge-magic.

The previous corollary implies that the upper bound in Lemma 1.3 is also sharp for maximal outerplanar graphs.

6. Conclusions

Using the survey of graph labelings by Gallian [4] together with relationships between super edge-magic labelings and other more familiar labeling problems found by the authors [3], we conclude that the work presented in this paper has enlarged the classes of graphs known to be cordial, edge-antimagic, edge-magic, harmonious and sequential. For instance, the forests $P_3 \cup mP_2$, $P_2 \cup P_n$, $K_{1,n} \cup K_{1,n+1}$, and $(2m+1)K_{1,n}$ are now proven to be cordial, edge-antimagic and edge-magic. Additionally, the graphs P_n^2 and $(2m+1)C_{2n+1}$ are now shown to be cordial, harmonious, edge-antimagic, edge-magic and sequential.

Through this work, the authors hope to have convinced the reader that the study of the super edge-magic labelings of graphs may lead to successful approaches to other labeling problems. Indeed, since super edge-magicness is a more restrictive concept, there are more necessary conditions that may be applied when considering cordial, harmonious, edge-magic, edge-antimagic and sequential labelings.

7. Acknowledgments

The authors are greatly indebted to the referee of this paper who suggested numerous and significant improvements. The most important of which are the statement of Theorem 2.4 and the statement and part of the proof of Theorem 2.6, which were not previously known to us. The authors would also like to acknowledge Gary Chartrand, Clifton Ealy, Hikoe Enomoto and Anna S. Lladó without whose assistance, encouragement and advice, this project would not have been possible.

References

[1] G. Chartrand and L. Lesniak, Graphs and Digraphs, Wadsworth & Brooks/Cole Advanced Books and Software, Monterey, Calif. 1986.

[2] H. Enomoto, A. Lladó, T. Nakamigawa and G. Ringel: Super edge-magic graphs, *SUT. J. Math.*, **34** (1998) 105-109.

[3] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.*, **231** (2001) 153-168.

[4] J. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, **5** (2000), #DS6.

[5] R. D. Godbold and P. Slater, All cycles are edge-magic, *Bull. Inst. Combin. Appl.*, **22** (1998) 93-97.

[6] N. Hartsfield and G. Ringel, *Pearls in Graph Theory*: A Comprehensive Introduction, Academic Press, San Diego, (1994).

[7] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970) 451-461.
[8] A. Kotzig and A. Rosa, Magic valuations of complete graphs, *Publications du Centre de Recherches Mathemátiques Université de Montréal*, **175** (1972).
[9] G. Ringel, Labeling problems., Proceedings of the Eight International Conference on Graph Theory, Combinatorics, Algorithms and Applications.
[10] G. Ringel and A. Lladó, Another tree conjecture, *Bull. Inst. Combin. Appl.*, **18** (1996) 83-85.

MATHEMATICS DEPARTMENT, UNIVERSITY OF HAWAI'I-HILO, 200 W. KAWILI ST., HILO, HI 96720, USA.
E-mail address: RamonF@Hawaii.edu

COLLEGE OF HUMANITIES AND SCIENCES, NIHON UNIVERSITY, 3-25-40 SAKURAJOSUI SETAGAYA-KU, TOKYO 156-8550, JAPAN.
E-mail address: ichishim@chs.nihon-u.ac.jp

DEPARTAMENT DE MATEMÀTICA APLICADA I TELEMÀTICA, UNIVERSITAT POLITÈCNICA DE CATULUNYA, 08071 BARCELONA, SPAIN.
E-mail address: muntaner@mat.upc.es