PERFECT BINARY MATROIDS

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Abstract. In this paper a definition of perfect binary matroids is considered and it is shown that, analogous to the Perfect Graph Theorem of Lovász and Fulkerson, the complement of a perfect matroid is also a perfect matroid. In addition, the classes of critically imperfect graphic matroids and critically imperfect graphs are compared.

1. Introduction

The matroid notation and terminology used here will follow Oxley [7], and only simple graphs and matroids will be considered. Since being introduced by Berge [1], perfect graphs have been extensively studied. In this paper, we investigate a definition of perfect binary matroids analogous to the definition of perfect graphs. Recall that a graph G is said to be *perfect* if $\omega(H) = \chi(H)$ for all vertex-induced subgraphs H of G. Therefore, in order to extend the notion of a perfect graph to matroid theory, matroidal analogues for the clique number, $\omega(G)$, and the chromatic number, $\chi(G)$, of a graph G are needed.

Since the clique number of a graph G is the maximum cardinality of a set of vertices that induces a complete subgraph of G, a matroidal analogue can be identified by exploiting an analogy between projective geometries in matroid theory and complete graphs in graph theory. In particular, since every rank-r simple matroid representable over $GF(q)$ can be obtained from the projective geometry $PG(r-1,q)$ by deleting elements, just as every graph on n vertices can be obtained from the complete graph K_n by deleting edges, we make the following definition.

Definition 1.1. Let M be a rank-r matroid representable over $GF(q)$. Then $\omega(M;q)$ is defined as $\max\{r(M|K) : K \cong PG(n-1,q)\}$ where $1 \leq n \leq r$.

Thus, for a $GF(q)$-representable matroid, $\omega(M)$ is the rank of the largest projective geometry $PG(n-1,q)$ that is a restriction of M. In particular, as $PG(1,2)$ is a circuit on 3 elements, $\omega(M;2) \geq 2$ for a binary matroid having a 3-circuit. Moreover, if M is the polygon matroid of a graph, then

$\omega(M;2) \leq 2$ since the rank-3 Fano matroid, $PG(2,2)$, is an excluded minor for the class of graphic matroids.

There are several ways one may attempt to define the chromatic number of a simple matroid. We shall use the *critical exponent* of a matroid, introduced by Crapo and Rota [4], as the matroidal analogue of the chromatic number of a graph. Recall that for a positive integer k and a graph G, a *proper k-coloring* of G is a function f from the vertices of G into $\{1, 2, \ldots, k\}$ such that if uv is an edge of G, then $f(u) \neq f(v)$. It is well-known that, when the number $\chi_G(k)$ of such colorings is viewed as a function of k, it is a polynomial. This polynomial is called the *chromatic polynomial* of G and the chromatic number of G may be defined by $\chi(G) = \min\{k : \chi_G(k) > 0\}$. The *characteristic polynomial* $p(M;\lambda)$ (see, for example, [9, p. 120]) of a matroid M generalizes the chromatic polynomial of a graph.

Definition 1.2. The *critical exponent* of a loopless $GF(q)$-representable matroid M is defined by $c(M;q) = \min\{j \in \mathbb{N} : p(M;q^j) > 0\}$.

Therefore the critical exponent of a simple binary matroid M is the smallest positive integer j such that $p(M;2^j) > 0$, just as the chromatic number of a graph G is the smallest positive integer k such that $\chi_G(k) > 0$. We now list several useful results about the critical exponent of a matroid (see, for example, [3, p. 163] or [9, p. 129]). The first is based on the fact that if $p(M;q^j) > 0$, then $p(M;q^k) > 0$ for all integers $k \geq j$.

Lemma 1.3. $c(M;q) = \min\{j \in \mathbb{N} : p(M;q^k) > 0$ *for all integers* $k \geq j\}$.

We will often use the following interpretation of the critical exponent of a graphic matroid.

Lemma 1.4. *If M is the polygon matroid of a graph G, then $c(M(G);2)$ is the least integer c such that the chromatic number of G does not exceed 2^c.*

For a matroid representable over $GF(q)$, the next result [3, Corollary 6.4.13] provides an alternative characterization of the critical exponent.

Lemma 1.5. *If M is isomorphic to the restriction of $PG(n-1;q)$ to the set E, then*

$$\begin{aligned} c(M;q) &= \min\{j \in \mathbb{N} : PG(n-1,q) \text{ has hyperplanes } H_1, H_2, \ldots, H_j \\ &\qquad \text{such that } (\cap_{i=1}^{j} H_i) \cap E = \emptyset\} \\ &= \min\{j \in \mathbb{N} : PG(n-1,q) \text{ has a flat of rank } n-j \text{ having} \\ &\qquad \text{empty intersection with } E\}. \end{aligned}$$

Thus a $GF(q)$-representable rank-r matroid M with critical exponent one can be embedded in the complement of a hyperplane of $PG(r-1,q)$; that is, M is affine. This useful geometric interpretation of the critical exponent is part of the next result (see, for example, [9, Corollary 7.6.3] or [3, Exercise 6.50]).

Lemma 1.6. *The following are equivalent for a simple binary matroid M.*

(i) *Every circuit of M has even cardinality.*
(ii) *M is a binary affine matroid.*
(iii) $c(M) = 1$.

From Lemma 1.5 it is evident that if M is simple and T is a subset of $E(M)$, then $c(M|T;q) \leq c(M;q)$. On combining this with the fact that both the rank and critical exponent of $PG(n-1,;q)$ equal n, we have the following lemma.

Lemma 1.7. *If M is a simple matroid representable over $GF(q)$, then $\omega(M;q) \leq c(M;q)$.*

The next lemma is a special case of [7, Propostion 9.3.4].

Lemma 1.8. *Let C be a circuit of a simple binary matroid M and let e be an element of $cl(C) - C$. Then there is a partition of C into non-empty subsets X_1 and X_2 so that $X_1 \cup e$ and $X_2 \cup e$ are circuits of M, and M has no other circuits that contain e and are contained in $C \cup e$.*

2. Perfect Binary Matroids

The availability of matroidal analogues for the chromatic number and clique number of a graph naturally leads to the following definition.

Definition 2.1. A simple $GF(q)$-representable matroid M is *perfect* if $\omega(M|F;q) = c(M|F;q)$ for each flat F of M.

Example 2.2. Since $\omega(PG(n-1,2)) = c(PG(n-1,2)) = n$ and each flat of a projective geometry is also a projective geometry, it follows that $PG(n-1,2)$ is a perfect binary matroid.

We shall abbreviate $\omega(M;q)$ and $c(M;q)$ to $\omega(M)$ and $c(M)$, respectively when considering only $GF(q)$-representable matroids. In addition, we will often shorten the terms perfect $GF(q)$-representable matroid and imperfect $GF(q)$-representable matroid to perfect matroid and imperfect matroid, respectively.

Example 2.3. $M(K_4)$ is a perfect binary matroid. Since $\chi(K_4) = 4$, it follows from Lemma 1.4 that $c(M(K_4)) = 2$. Moreover, as K_4 contains a 3-cycle as a restriction, $\omega(M(K_4)) = 2$. Furthermore, for each proper flat F of $M(K_4)$, we have

$$\omega(M(K_4)|F) = c(M(K_4)|F) = \begin{cases} 2, & \text{if } F \text{ contains a 3-circuit} \\ 1, & \text{otherwise.} \end{cases} \tag{2.1}$$

Thus K_4 is a perfect graph and $M(K_4)$ is a perfect binary matroid. However, not all perfect graphs yield perfect binary matroids. For instance, the perfect graph K_5 yields an imperfect binary matroid since $c(M(K_5)) = 3$, while $\omega(M(K_5)) = 2$. In fact, any graph G such that $\chi(G) \geq 5$ will yield an imperfect binary matroid $M(G)$.

An elementary result in graph theory characterizes bipartite graphs as those having no odd cycles. Since the bipartite graphs are an example of a class of perfect graphs, an attractive part of the next result is that the matroids having no odd circuits are perfect binary matroids. Let C be a circuit of a matroid M. An element e of the matroid is a *chord* of the circuit C if $e \in cl_M(C) - C$.

Theorem 2.4. *Let M be a simple binary matroid such that $c(M) \leq 2$.*

(i) *If $c(M) = 1$, then M is perfect.*
(ii) *M is perfect if and only if every odd circuit C of M such that $|C| \geq 5$ has a chord.*

Proof. If $c(M) = 1$, then M is affine and Lemma 1.6 implies that M has no odd circuits. Hence $\omega(M) = 1$. It follows that M is perfect and (i) holds.

We now prove statement (ii). Suppose M is perfect and $c(M) \leq 2$. If $c(M) = 1$, then M has no odd circuits and the result holds. Now assume $c(M) = 2$ and C is an odd circuit of M such that $|C| \geq 5$. If C has no chord, then $c(cl(C)) = 2$, since the flat $cl(C)$ is an odd circuit. However, as $cl(C)$ has no 3-circuit as a restriction, $\omega(cl(C)) = 1$. This contradicts the assumption that M is perfect and we conclude C has a chord.

Now suppose every odd circuit of M has a chord and $c(M) \leq 2$. By (i), the matroid M is perfect if $c(M) = 1$, so we may assume $c(M) = 2$. Let F be a non-empty flat of M. If $c(M|F) = 1$, then F contains no odd circuits, and $\omega(M|F) = 1$. If $c(M|F) = 2$, then F contains an odd circuit. Since each odd circuit of F has a chord, it follows from Lemma 1.8 that F has a 3-circuit. Then $\omega(M|F) = 2$. Hence M is a perfect matroid. □

Corollary 2.5. *A graphic matroid M is a perfect binary matroid if and only if $\chi(G) \leq 4$ and every odd cycle of G of length at least 5 has a chord.*

The *complement* $\overline{G}$ of a simple graph G is the graph with vertex set $V(G)$ such that two distinct vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. The analogy between projective geometries in matroid theory and complete graphs in graph theory allows one to consider complements for simple matroids that are uniquely representable over $GF(q)$. If M is a simple uniquely $GF(q)$-representable matroid such that $M \cong PG(k-1,q)|T$, then the *$(GF(q), k)$-complement* of M is the matroid $PG(k-1,q)\backslash T$. For example, $U_{2,3}$ is the $(GF(2), 3)$-complement of $U_{3,4}$.

The well-known Perfect Graph Theorem of Lovász [6] and Fulkerson [5] states that G is perfect if and only if $\overline{G}$ is perfect. An analogous theorem for perfect matroids is proved next.

Theorem 2.6. *A simple rank-r binary matroid M is perfect if and only if its $(GF(2), r)$-complement is perfect.*

Proof. Let M be a simple rank-r perfect binary matroid. Then M can be embedded in a projective geometry $PG(r-1, 2)$ and has a $(GF(2), r)$-complement which we shall denote by M^c. Let W_1 and W_2 be largest rank binary projective geometries that are restrictions of M and M^c, respectively. Then Lemma 1.5 implies that $c(M) + r(W_2) = r$ and $c(M^c) + r(W_1) = r$. Hence $c(M)+\omega(M^c) = r$ and $c(M^c)+\omega(M) = r$. Now, as M is perfect, $\omega(M) = c(M)$. Thus the fact that $c(M)+\omega(M^c) = c(M^c)+\omega(M)$ implies $\omega(M^c) = c(M^c)$.

Now let F_1 be a non-empty flat of M^c. Then $F_1 = F - E(M)$ for some flat F of $PG(r-1, 2)$. Since $F \cong PG(k-1, 2)$ for some k, the flat F_1 has a $(GF(2), k)$-complement F_2 that is a subset of $E(M)$. Moreover, as M is perfect, $\omega(M|F_2) = c(M|F_2)$. Since F_1 is the $(GF(2), k)$-complement of F_2, it follows from the above argument that $\omega(M^c|F_1) = c(M^c|F_1)$. We conclude that M^c is perfect. □

The next lemma lists two useful properties of the characteristic polynomial. The first follows from the fact that the characteristic polynomial is a Tutte–Grothendieck invariant [3, Proposition 6.2.5] and the second was proven by Brylawski [2, Theorem 7.8].

Lemma 2.7. *Let M_1, M_2, and M be matroids.*

(i) *If $M = M_1 \oplus M_2$, then $p(M;\lambda) = p(M_1;\lambda)p(M_2;\lambda)$.*

(ii) *If M is the generalized parallel connection of the matroids M_1 and M_2 across the modular flat X, then $p(M;\lambda) = \frac{p(M_1;\lambda)p(M_2;\lambda)}{p(M|X;\lambda)}$.*

The next two results concern ways of combining perfect matroids to form larger perfect matroids.

Theorem 2.8. *If M_1 and M_2 are perfect $GF(q)$-representable matroids, then the direct sum of M_1 and M_2 is also perfect.*

Proof. Suppose M_1 and M_2 are perfect $GF(q)$-representable matroids and let $N = M_1 \oplus M_2$ be the direct sum of M_1 and M_2. Then

$$\begin{aligned} c(N) &= \min\{j \in \mathbb{N} : p(N, q^j) > 0\} \\ &= \min\{j \in \mathbb{N} : p(M_1, q^j)p(M_2, q^j) > 0\} \\ &= \min\{j \in \mathbb{N} : p(M_1, q^j) > 0 \text{ and } p(M_2, q^j) > 0\} \\ &= \max\{c(M_1), c(M_2)\} \end{aligned} \tag{2.2}$$

where the second equality follows from Lemma 2.7(i) and the last equality follows from Lemma 1.3. Moreover, as M_1 and M_2 are perfect, we have

$\omega(M_1) = c(M_1)$ and $\omega(M_2) = c(M_2)$. On combining this with Lemma 1.7, we have that

$$\begin{aligned}\max\{\omega(M_1), \omega(M_2)\} &= \max\{c(M_1), c(M_2)\} \\ &= c(N) \\ &\geq \omega(N) \geq \max\{\omega(M_1), \omega(M_2)\}.\end{aligned}$$

Hence $c(N) = \omega(N)$.

Now let F be a non-empty proper flat of N. Then $F = F_1 \cup F_2$ where F_1 is a flat of M_1 and F_2 is a flat of M_2. Thus $\omega(M_1|F_1) = c(M_1|F_1)$ and $\omega(M_2|F_2) = c(M_2|F_2)$. Moreover, as $N|F = (M_1|F_1) \oplus (M_2|F_2)$, it follows from (2.2) that $\max\{c(M_1|F_1), c(M_2|F_2)\} = c(N|F)$. On combining this with Lemma 1.7 we obtain

$$\begin{aligned}\max\{\omega(M_1|F_1), \omega(M_2|F_2)\} &= \max\{c(M_1|F_1), c(M_2|F_2)\} \\ &= c(N|F) \\ &\geq \omega(N|F) \geq \max\{\omega(M_1|F_1), \omega(M_2|F_2)\}.\end{aligned}$$

Hence $c(N|F) = \omega(N|F)$ for all flats F of N, and we conclude that $N = M_1 \oplus M_2$ is a perfect matroid. □

Theorem 2.9. *If M_1 and M_2 are perfect $GF(q)$-representable matroids, then the parallel connection of M_1 and M_2 with basepoint p is also perfect.*

Proof. Suppose M_1 and M_2 are perfect $GF(q)$-representable matroids and let $N = P(M_1, M_2; p)$ be the parallel connection of M_1 and M_2 with basepoint p. Then

$$\begin{aligned}c(N) &= \min\{j \in \mathbb{N} : p(N; q^j) > 0\} \\ &= \min\{j \in \mathbb{N} : \tfrac{p(M_1;q^j)p(M_2;q^j)}{q^j-1} > 0\} \\ &= \min\{j \in \mathbb{N} : p(M_1, q^j)p(M_2, q^j) > 0\} \\ &= \max\{c(M_1), c(M_2)\}\end{aligned} \tag{2.3}$$

where the second equality follows from Lemma 2.7(ii) and the final equality follows from Lemma 1.3. On combining this with Lemma 1.7 we have that

$$\begin{aligned}\max\{\omega(M_1), \omega(M_2)\} &= \max\{c(M_1), c(M_2)\} \\ &= c(N) \\ &\geq \omega(N) \geq \max\{\omega(M_1), \omega(M_2)\}.\end{aligned}$$

Now let F be a non-empty proper flat of N. Define F_1 to be the flat $F \cap E(M_1)$ of M_1 and F_2 to be the flat $F \cap E(M_2)$ of M_2. We now consider two cases.

If F is a flat of N not containing the basepoint p, then $F = (M_1|F_1) \oplus (M_2|F_2)$. Moreover, as M_1 and M_2 are perfect matroids, $M_1|F_1$ and $M_2|F_2$

are perfect. Then Theorem 2.8 implies that $N|F$ is perfect and hence $c(N|F) = \omega(N|F)$.

Now suppose F is a flat of N containing the basepoint p. Then F is the parallel connection of $M_1|F_1$ and $M_2|F_2$ over the basepoint p. Moreover, as M_1 and M_2 are perfect, $\omega(M_1|F_1) = c(M_1|F_1)$ and $\omega(M_2|F_2) = c(M_2|F_2)$. On combining this with (2.3) and Lemma 1.7 we have that

$$\begin{aligned} \max\{\omega(M_1|F_1), \omega(M_2|F_2)\} &= \max\{c(M_1|F_1), c(M_2|F_2)\} \\ &= c(N|F) \\ &\geq \omega(N|F) \geq \max\{\omega(M_1|F_1), \omega(M_2|F_2)\}. \end{aligned}$$

Hence $c(N|F) = \omega(N|F)$. Therefore $c(N|F) = \omega(N|F)$ for all flats F of N and we conclude that N is a perfect matroid. □

3. Critically Imperfect Graphs and Matroids

Recall that an imperfect graph G is said to be *critically imperfect* if each of its proper induced subgraphs is perfect. We now extend this concept to binary matroids.

Definition 3.1. A simple binary matroid M is *critically imperfect* if M is imperfect and $M|F$ is perfect for each proper flat F of M.

Example 3.2. Let C_n denote a cycle on n vertices. If n is odd and exceeds three, then, as C_n is not a two-colorable graph, $c(M(C_n)) = 2$. Moreover, as C_n contains no 3-cycle as a restriction, $\omega(M(C_n)) = 1$. Since $c(M(C_n)|F) = \omega(M(C_n)|F) = 1$ for all proper flats F of $M(C_n)$, we see that the matroids derived from odd cycles are critically imperfect.

The matroid $M(K_5)$ is another example of a critically imperfect matroid since $c(M(K_5)) = 3$ and $\omega(M(K_5)) = 2$, while, for each proper flat F,

$$\omega(M(K_5)|F) = c(M(K_5)|F) = \begin{cases} 2, & \text{if } F \text{ contains a 3-circuit} \\ 1, & \text{otherwise.} \end{cases} \tag{3.1}$$

Therefore the matroid $M(C_5)$ and its $(GF(2),4)$-complement, $M(K_5)$, are an example of the next result which follows from Theorem 2.6.

Theorem 3.3. *A simple rank-r binary matroid M is critically imperfect if and only if its $(GF(2), r)$-complement is critically imperfect.*

The next result characterizes graphic critically imperfect matroids. A graph G is said to be *k-vertex-critical* if $\chi(G) = k$ and $\chi(G - v) < k$ for each vertex v of G.

Theorem 3.4. *Let M be a simple binary matroid.*

(i) *M is critically imperfect and $c(M) = 2$ if and only if $M \cong U_{n-1,n}$ for an odd integer n such that $n \geq 5$.*

(ii) *M is graphic, critically imperfect, and $c(M) = 3$ if and only if $M \cong M(G)$ where G is 5-vertex-critical and every odd cycle of length exceeding 3 has a chord.*

Proof. It was shown in Example 3.2 that if $M \cong U_{n-1,n}$ for an odd integer $n \geq 5$, then M is critically imperfect and $c(M) = 2$. Suppose M is a critically imperfect matroid such that $c(M) = 2$. Now, as M is not affine, it has an odd circuit C. Since M is critically imperfect and $c(cl_M(C)) = 2$, the flat $cl_M(C)$ must equal M. Thus C is a spanning set of M. If $x \in E(M)\backslash C$, then Lemma 1.8 implies that there is a partition of C into nonempty subsets X_1 and X_2 such that $X_1 \cup x$ and $X_2 \cup x$ are circuits of M, and M has no other circuits that contain x and are contained in $C \cup x$. Moreover, as C is an odd circuit, we may assume that $|X_1|$ is even and $|X_2|$ is odd. Thus $X_1 \cup x$ is an odd circuit and a proper flat of M contrary to the fact that $c(M|F) = \omega(M|F) \leq 1$ for each proper flat F of M. Therefore $E(M)\backslash C = \emptyset$ and we conclude that $M \cong U_{n-1,n}$ for an odd integer $n \geq 5$.

We now prove (ii). Suppose $M(G)$ is critically imperfect and $c(M(G)) = 3$. Since $M(G - v)$ is a proper flat of $M(G)$, we have $c(M(G - v)) = \omega(M(G-v)) \leq 2$. Thus, although G is not 4-colorable, $G - v$ is 4-colorable for each vertex v of G. Therefore G is 5-vertex-critical.

Now suppose C is an odd cycle of G having length at least 5 and no chords. Then C is a flat of $M(G)$. However, $\omega(M(G)|C) = 1$ and $c(M(G)|C) = 2$, contrary to the fact that $M(G)$ is critically imperfect. Thus every odd cycle of length at least 5 has a chord.

Now assume G is a 5-vertex-critical graph and every odd cycle of length at least 5 has a chord. Then $c(M(G)) = 3$ and $c(M(G)|F) < 3$ for each proper flat F of $M(G)$. Moreover, $\omega(M(G)) \leq 2$, as $M(G)$ is graphic. Let F be a proper flat of $M(G)$. If F has no odd circuits, then $c(M(G)|F) = 1$ and $\omega(M(G)|F) = 1$. If F has an odd circuit, then $c(M(G)|F) = 2$. As every odd cycle of G of length at least five has a chord, it follows that F has a 3-circuit. Hence $\omega(M(G)|F) = 2$ and we conclude that $M(G)$ is a critically imperfect matroid. □

The following lemma, due to Tucker [8], characterizes the critically imperfect graphs having no K_4-restriction.

Lemma 3.5. *The only critically imperfect graphs having no K_4-restriction are the odd circuits of length at least 5 and their complements.*

The next two theorems describe the relationship between the set of critically imperfect graphs and the set of graphs G such that $M(G)$ is a critically imperfect matroid. The graph $\overline{C_7}$ mentioned in the following results is shown in Figure 1.

Theorem 3.6. *If G is a critically imperfect graph such that $M(G)$ is not critically imperfect as a matroid, then $\chi(G) \geq 6$ or $G \cong \overline{C_7}$.*

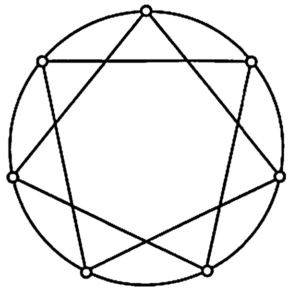

Figure 1. The graph $\overline{C_7}$.

Proof. Suppose G is a critically imperfect graph, but $M(G)$ is not critically imperfect as a matroid. Since G is critically imperfect, $\chi(G) \geq 3$. Moreover, as the only critically imperfect graphs with chromatic number 3 are the odd cycles C_n for $n \geq 5$, Theorem 3.4(i) implies that $\chi(G) \geq 4$.

Suppose that $\chi(G) = 4$. Then, as G is critically imperfect, $\omega(G) < \chi(G) = 4$. Thus G has no K_4-restriction, and Lemma 3.5 implies that G is an odd cycle or the complement of an odd cycle. It follows that $G \cong \overline{C_7}$.

Now suppose that $\chi(G) = 5$. Since G is critically imperfect, each of its odd cycles of length 5 or more has a chord. It follows that $\omega(M(G)) = 2$. Then, as $c(M(G)) = 3$, the matroid $M(G)$ is imperfect. Therefore $M(G)$ contains a proper flat $F = M(G_1)$ that is critically imperfect as a matroid. Now, as G_1 is a proper vertex-induced subgraph of G, we have $\chi(G_1) \leq 4$. The fact that $M(G_1)$ is critically imperfect implies $c(M(G_1)) > 1$. Hence $\chi(G_1) \geq 3$. Then $c(M(G_1)) = 2$ and it follows from Theorem 3.4(i) that G_1 is an odd cycle of length at least 5. However, this contradicts the fact that each vertex-induced proper subgraph of G is perfect. As a result of this contradiction, we conclude that the theorem holds. □

Theorem 3.7. *If $M(G)$ is a critically imperfect matroid and G is not critically imperfect as a graph, then either $G \cong K_5$ or G is 5-vertex-critical and has $\overline{C_7}$ as a proper induced subgraph.*

Proof. Suppose $M(G)$ is a critically imperfect matroid, but G is not a critically imperfect graph. Theorem 3.4(i) implies that $c(M(G)) \geq 3$. Moreover, as $PG(2,2)$ is an excluded minor for graphic matroids, $\omega(M(G)) \leq 2$. Thus $c(M(G)) = 3$ and $\omega(M(G)) = 2$. Furthermore, G is 5-vertex-critical and every odd cycle of length at least 5 has a chord. If $|V(G)| = 5$, then clearly $G \cong K_5$. Now suppose $|V(G)| > 5$. As G is a 5-vertex-critical graph, it has no K_5-restriction. Hence G is an imperfect graph and it follows that G has a proper vertex-induced critically imperfect subgraph G' such that $\chi(G') \leq 4$. If $\chi(G') = 3$, then Lemma 3.5 implies that G' is an odd cycle or the complement of an odd cycle. Now, as $\chi(\overline{C_n}) \geq 4$ for odd integers $n \geq 7$ and $\overline{C_5} = C_5$, we deduce that G' is an odd cycle of length at least 5. However, this contradicts the fact that G has no chordless odd cycles of length at least 5 as induced subgraphs. Hence we may assume that $\chi(G') = 4$. Since G' is critically imperfect it has no K_4-restriction. Then Lemma 3.5 implies that G' is an odd cycle or the complement of an odd cycle, and the fact that $\chi(G') = 4$ implies that G' is $\overline{C_7}$. Thus G is a 5-vertex-critical graph that has $\overline{C_7}$ as a proper induced subgraph and is not critically imperfect. □

Although $M(\overline{C_7})$ is perfect, the graph $\overline{C_7}$ plays an important role in determining critically imperfect matroids. Theorems 3.4 and 3.7 imply that critically imperfect matroids can be obtained from 5-vertex-critical graphs having $\overline{C_7}$ as an induced subgraph, but no induced cycle subgraphs C_n for odd $n \geq 5$. Three examples of such graphs are shown in Figure 2. The graph G_1 is obtained from $\overline{C_7}$ by adding a new vertex that is adjacent to every vertex of $\overline{C_7}$. Thus G_1 is the join of $\overline{C_7}$ and K_1. The graphs G_2 and G_3 are obtained from $\overline{C_7}$ by adding two new vertices each having six "spokes". Several other examples of graphs that yield critically imperfect matroids can be constructed in a similar manner.

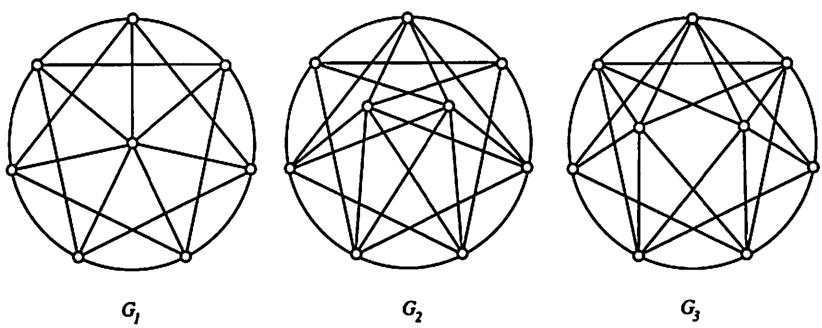

Figure 2.

ACKNOWLEDGEMENT

This work was partially supported by a T.T.U. Faculty Research Grant.

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