Inducing regularization of graphs, multigraphs and pseudographs

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Abstract

For a given structure (graph, multigraph, or pseudograph) G and an integer $r \geq \Delta(G)$, a smallest inducing r-regularization of G (which is an r-regular superstructure of the smallest possible order, with bounded edge multiplicities, and containing G as an induced substructure) is constructed.

1 Introduction

Graphs are simple, multigraphs without loops, and pseudographs may contain both loops and multiple edges. For terminology and notation we refer to Chartrand and Lesniak [2].

For a given structure (graph, multigraph, or pseudograph) G and an integer $r \geq \Delta(G)$, an r-regular superstructure containing G as an induced substructure is called an *inducing r-regularization/regularization* of G. The problem of inducing r-regularization is originated in 1916 by König [5, 6]. Therein an r-regular multigraph containing a given multigraph G and its copy G' (if G is not r-regular), both as disjoint induced submultigraphs, is constructed so that additional (possibly multiple) edges join a vertex x of G to its copy x' in G', r being any cardinal number, $r \geq \Delta(G)$. Optimal (i.e. the smallest) inducing regularization of a simple graph G within simple graphs with the smallest possible degree $\Delta(G)$ is published in two papers by Erdős and Kelly [3, 4]. Inducing r-regularization of a simple graph G within simple r-regular graphs with $r \geq \Delta(G)$ is achieved in Chartrand and Lesniak [2] by splitting König's method into $r - \delta(G)$ steps. Then only simple xx' edges are added at each step and the order of the resulting r-regular graph is $2^{r-\delta(G)}$ times as large as that of G.

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We extend Erdős and Kelly's result both to pseudographs and multigraphs with any multiplicity bound p, maintaining optimality for any upper bound r on the maximum degree Δ. We present examples to show that the upper bound we give on the number of necessary new vertices is sharp. If G is an n-vertex graph and regularization of G is also a graph then in Erdős and Kelly's case (with $r = \Delta(G)$) the optimal inducing r-regularization of G may require up to n new vertices, which is the case if $n \geq 4$ and $G = K_n - e$, see [3, 4]; whereas, if r is odd and $r > n$ then exactly $r+1$ new vertices are necessary precisely for all graphs G with $\delta(G) = 0$ and even n, otherwise up to $\max\{r, n\}$ new vertices only.

2 Main result

For $X \in \{P, M\}$, let X-graph stand for P-graph (pseudograph) if $X = P$, otherwise for M-graph (multigraph) if $X = M$. Let G be an X-graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let δ and Δ stand for the minimum and maximum degrees among vertices of G. Let r be any integer such that $r \geq \Delta$. Call $a_i := r - \deg_G(v_i)$ to be the *r-deficiency* of the vertex v_i. Then $r - \delta$ is the *maximum r-deficiency* among vertices in G. Let $\sigma = \sum_i a_i$ be the sum of the r-deficiencies. The number of edges joining two vertices v, u is called the *multiplicity* of the vertex-pair v, u. The end of a claim's proof will be marked by the symbol □.

Theorem *Given an X-graph G of order n, with minimum and maximum degrees δ and Δ, let p and r be integers such that $r \geq \Delta$ and p is an upper bound on the maximum multiplicity in G. Let F denote any r-regular X-graph with maximum multiplicity at most p and containing an induced sub-X-graph isomorphic to G. The necessary and sufficient condition that $n+t$ be the minimal order possible for F is that t is the least nonnegative integer such that*

(*i*) $tr \geq \sigma$;
(*ii*) $pt \geq r - \delta$;
(*iii*) $(t+n)r$ *is an even integer;*
(*iv*) *either* (*P*) $pt^2 - (r-p)t + \sigma \geq 0$ *if* $X = P$ *or*
(*M*) $pt^2 - (r+p)t + \sigma \geq 0$ *if* $X = M$.

Moreover, $t \leq t_0$ *where*

$$t_0 := \begin{cases} \lceil r/p \rceil + 1 & \text{if both } r \text{ and } \lceil r/p \rceil + n \text{ are odd,} \\ & \lceil r/p \rceil > n, \text{ and } \delta < r + p - p\lceil r/p \rceil, \\ \max\{\lceil r/p \rceil, n\} & \text{otherwise.} \end{cases} \tag{1}$$

Proof. Necessity. Suppose the order $n + t$ of the super-X-graph F is minimal. Let G' be the sub-X-graph of F isomorphic to G and let H

be the sub-X-graph induced by the t vertices of F not in G'. Then in F there are σ edges between the sub-X-graphs G' and H. Since each of the t vertices of H is incident with at most r of these edges, (i) follows. The vertex of G' whose r-deficiency is $r-\delta$ has at least $\lceil (r-\delta)/p \rceil$ neighbors in H, whence (ii) holds. Clearly $(t+n)r$ is an even number so (iii) holds. The sum of the degrees in H of the vertices of H is $tr-\sigma$ which cannot exceed the following obvious upper bound on this sum: $t(t-1)p$ if $X=M$ and $t(t+1)p$ if $X=P$. This gives (P) and (M). So all four conditions (i)–(iv) are necessary.

To establish sufficiency, let t be the least nonnegative integer satisfying conditions (i)–(iv). If $\sigma=0$ then G is r-regular whence $t=0$, which agrees with conditions (i)–(iv). Assume that $\sigma>0$. Then $r\geq\Delta\geq 0$ and $r>0$. Moreover, $t>0$ by (i). Recall that v_i denotes the ith vertex of the X-graph G and $V=V(G)$. Let U be a set of t extra vertices $u_1,u_2,\ldots,u_t$. Let B be the bipartite V–U multigraph comprising a_i edges joining v_i to U for $i=1,2,\ldots,n$. Thus σ is the number of all V–U edges. Define an X-graph F to be the edge-disjoint union of X-graphs G, B and H where H is induced by the set U. Assume that V–U edges make up a sequence $\mathcal{A}$ such that, for each i, edges incident with v_{i+1} follow all those incident with v_i. In order to establish incidence of V–U edges with vertices in U, consider the sequence of vertices $(\tilde{u}_1,\tilde{u}_2,\ldots,\tilde{u}_t,\ldots,\tilde{u}_\sigma)$, where $\tilde{u}_j=u_i$ if $j\equiv i \pmod t$ for $j=1,\ldots,\sigma$ and $i=1,\ldots,t$. Assume that the jth vertex $\tilde{u}_j$ is made incident to the jth edge of $\mathcal{A}$. Then multiplicity of the pair v_i, u_j equals at most $\lceil a_i/t \rceil$ which is at most p by (ii). Hence $F=G\cup B$ if equality holds in condition (i) (which is the case if both $X=M$ and $t=1$).

Otherwise strong inequality holds in (i) and additionally $t>1$ if $X=M$. We are going to show that adding some U–U edges can complete the X-graph F. Notice that $nr+\sigma$ is even as the sum of degrees of vertices in the X-graph $G\cup B$. Hence and by (iii) $tr-\sigma$ is an even number. Then for $t=1$ and $X=P$, it is enough to add $(tr-\sigma)/2$ ($\leq p$ by (P)) loops to the vertex u_1 in order to get a required F. For $t\geq 2$ and $X\in\{P,M\}$, define nonnegative integers h and s so that $\sigma/t=h+s/t$ where $s<t$. Hence $h<r$ because strong inequality in (i) is assumed. Then $\deg_B(u_j)=h+1$ for $j=1,2,\ldots,s$, otherwise $\deg_B(u_j)=h$. Thus the degrees in B of vertices of U differ mutually by one at most. Moreover, the greatest remaining deficiency $r-h$ can be covered up in H if either $r-h\leq(t+1)p$ for $X=P$ or $r-h\leq(t-1)p$ for $X=M$. But the required conditions (on integers) hold because, by definition of h and s,

$$r-h-s/t=r-\sigma/t\leq\begin{cases}(t+1)p & \text{due to } (P) \text{ if } X=P,\\ (t-1)p & \text{due to } (M) \text{ if } X=M,\end{cases}$$

where $0\leq s/t<1$. A required X-graph H can be extracted from the complete X-graph on t vertices by using edge-disjoint complete subgraphs

K_t and possibly an edge-decomposition of one K_t into Hamiltonian cycles and, additionally for even t, a perfect matching (see Berge [1] for the well-known solution to Kirkman's problem on packing Hamilton cycles into K_t. The solution presented in Berge appears already in Lucas [7] of 1883 wherein no reference to Kirkman is made).

Now we are going to prove the statement on t_0, t_0 defined in (1).

Claim 1 *If $r/p \geq n$ and $\delta < r + p - p\lceil r/p \rceil$ then conditions (ii) and (iii) imply that $t \geq t_0$.*

Proof. Note that (*ii*) implies $t \geq t_1 := \lceil r/p \rceil$ if and only if $r - \delta > (\lceil r/p \rceil - 1)p$ or (equivalently) $\delta < r + p - p\lceil r/p \rceil$. However, if r and $n + t_1$ are odd integers and $t_1 > n$ then $t \neq t_1$ by condition (*iii*). □

Claim 2 *Conditions (i) and (ii) hold for $t \geq \max\{\lceil r/p \rceil, n\}$.*

Proof. Since $nr \geq \sigma$, condition (*i*) holds for $t \geq n$. Moreover, (*ii*) holds for $t \geq \lceil r/p \rceil$. □

Assume that $\sigma > 0$ in the following part of the proof. Hence it is easily seen that (P) holds for $t \geq r/p - 1$. This and Claim 2 imply that conditions (*i*),(*ii*) and (P) hold for $t \geq \max\{\lceil r/p \rceil, n\}$. Hence by Claim 1 conditions (*i*)–(*iv*) hold for $t \geq t_0$ if $X = P$.

Claim 3 *Condition (M) holds for $t \geq n$ if $r \leq 2pn$, otherwise for $t \geq \lceil r/p \rceil + 1 - n$.*

Proof. Let $L(t)$ stand for the left-hand side of the inequality (M). Then $L(n) = (\sigma - n(r - \Delta)) + n((n-1)p - \Delta) \geq 0$ as the sum of two nonnegative summands. Hence (M) holds for $t = n$. Assume that $r \leq 2pn$ and note that $L(t)$ is the quadratic trinomial in t which attains its minimum at $\tau := (r/p+1)/2 \leq n+1/2$. Therefore for any integer $t \geq n+1$, $L(t) \geq L(n) \geq 0$.

Otherwise $r > 2pn$ whence $\tau > n+1/2$. Therefore $L(t) \geq 0$ for $t \leq n$. Hence, since $L(t)$ is symmetrical with respect to $t = \tau > n$, $L(t) \geq 0$ for each $t \geq n + 2(\tau - n) = r/p + 1 - n$. □

Claim 3 implies that, for $n \geq 2$ and $r/p > 2n$, condition (M) holds for $t \geq \lceil r/p \rceil - 1$. Hence and by Claims 1, 2 and 3, conditions (*i*)–(*iv*) hold for $t \geq t_0$ if $X = M$. ■

Remark 1. A specification of Theorem in case G and F are simple graphs is obtainable by straightforwardly substituting therein $X = M$ and $p = 1$. Additionally, for $r = \Delta$ this way we get main body of the classical Erdős and Kelly's Theorem [3, 4].

Remark 2. For $X \in \{M, P\}$ the following n-vertex X-graphs G show that the upper bound t_0 on the smallest t in Theorem is attainable. Assume that E_m denotes a set of m edges of the complete graph K_n. For $X = M$, $p = 1$ and $r = \Delta(G) > 0$ (which is the Erdős and Kelly's case): $t = n$ if $n \geq 4$ and $G = K_n - E_m$ where $1 \leq m \leq \lfloor (n-2)/2 \rfloor$ (the Erdős and Kelly example $K_n - e$ beeing included). For any X, any p, and $r/p \leq n$: $t = n$ if G is any X-graph with $e(G) \leq (r-1)/2$ (whence $\Delta < r$); whereas for $r/p > n$: $t = t_0 > n$ if $\delta(G) < r + p - p\lceil r/p \rceil$ (i.e., if $\delta(G) = 0$ for $p = 1$).

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