Remarks on a general model of a random digraph

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Abstract Basic properties of in-degree distribution of a general model of a random digraphs $D(n, \mathcal{P})$ are presented. Then some relations between random digraphs $D(n, \mathcal{P})$ for different probability distributions $\mathcal{P}$'s are studied. In this context, a problem of the existence of a threshold function for every monotone digraph property of $D(n, \mathcal{P})$ is discussed.

1. Introduction

We begin with a definition of a general model of a random digraph that was introduced in [6]. Let $\mathcal{P} = (P_0, P_1, \ldots, P_{n-1})$ be a probability distribution, i.e. an n-tuple of non-negative real numbers which satisfy $\sum_{m=0}^{n-1} P_m = 1$. Denote by $D(n, \mathcal{P})$ a random digraph on a vertex set $V = \{1, 2, \ldots, n\}$ such that (here, and what follows, $N^+(i)$ denotes the set of images of a vertex i):

1) each vertex "chooses" its out-degree and then its images independently of all other vertices,
2) each vertex $i \in V$ chooses its out-degree according to the probability distribution $\mathcal{P}$:

$$\Pr\{|N^+(i)| = k\} = P_k, \qquad k = 0, 1, \ldots, n-1$$

3) for every $S \subseteq V \setminus \{i\}$, with $|S| = k$,

$$p_k = \Pr\{S = N^+(i)\} = P_k / \binom{n-1}{k}, \tag{1.1}$$

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i.e. p_k is the probability that a subset S coincides with the set of images of a vertex i.

In particular, if $\mathcal{P}$ is such that $P_d = 1$ for some d, $1 \leq d \leq n-1$, the model $D(n,\mathcal{P})$ is equivalent to a random d-out-regular digraph $D(n,d)$. Such a digraph can also be defined as an element chosen at random from the family of all $\binom{n-1}{d}^n$ digraphs on n labeled vertices each of out-degree d. (Alternatively, $D(n,d)$ can be thougth as a representation of a sum of d dependent random mappings as illustrated in [4]).

In a case when $\mathcal{P}$ is a binomial distribution $\mathcal{B}(n-1,p)$, i.e.

$$\mathcal{P} = \left(q^{n-1}, \dots, \binom{n-1}{k} p^k q^{n-1-k}, \dots, p^{n-1}\right),$$

the model $D(n,\mathcal{P})$ is equivalent to a random digraph $D(n,\mathcal{B})$ on n labeled vertices in which each of $n(n-1)$ possible arcs appears independently with a given probability $p = 1-q$.

One more step takes us to a classical random graph $G(n,p')$. In order to do this, let us define an *underlying simple graph* G of a digraph D as a graph obtained from D by omission of the orientation of arcs and replacement of eventual double edges by a single one. Now an underlying simple graph of a random digraph $D(n,\mathcal{B})$ is a random graph $G(n,p')$, obtained by independent deletion of edges of a complete graph on n vertices, so that each edge has the same probability $p' = 2p - p^2$ of being present.

Other spaces of random graphs, which can be obtained from the model $D(n,\mathcal{P})$, are discribed with relations between them in [4], [5] and [6].

In [6] some relations between the general model $D(n,\mathcal{P})$ and its special case $D(n,d)$ were considered. Also it was proved that, under some technical restrictions, the vertex connectivity, edge connectivity and minimum vertex degree of a random digraph $D(n,\mathcal{P})$ have asymptotically the same distribution. Here, in Section 2, we present basic properties of in-degree distribution of $D(n,\mathcal{P})$.

Recall that a graph property $\mathcal{A}$ is *monotone increasing* (*decreasing*) if from the fact that a graph G has $\mathcal{A}$, it follows that every spanning supergraph (subgraph) of G has also this property. Erdős and Rényi (see [3]) discovered the important fact that most monotone properties of the classical model of random graph $G(n,p)$ appear rather suddenly. They introduced the notion of a threshold function for a monotone property. Let $\mathcal{A}$ be a given monotone increasing property. Then a function $p^*(n)$ is said to be a *threshold function* for $\mathcal{A}$ if

$$\lim_{n\to\infty} \Pr\{G(n,p) \text{ has } \mathcal{A}) = \begin{cases} 0 & \text{if } \frac{p}{p^*} \to 0 \\ 1 & \text{if } \frac{p}{p^*} \to \infty. \end{cases}$$

One can show that probability that $G(n,p)$ has a monotone property is a monotone function of the parameter p (see [1, p.33]). Bollobás and Thomason (see [2]) observed that every monotone property of subsets of a set has a threshold function. In particular it follows that any monotone property of $G(n,p)$ has a threshold function. It is natural to ask whether this fact is also true in more general setting.

The main aim of our paper is to study some relations between random digraphs $D(n,\mathcal{P})$ with respect to different probability distributions $\mathcal{P}$'s. In particular, in Section 3, we show the heuristically obvious fact that a monotone increasing property of a random digraph $D(n,\mathcal{P})$ is the more likely to occur the more arcs we are likely to have.

2. Basic properties of $D(n,\mathcal{P})$

Let X^+ be a discrete random variable having a probability distribution $\mathcal{P} = (P_0, P_1, \dots, P_{n-1})$:

$$\Pr\{X^+ = k\} = P_k, \qquad k = 0, 1, \dots, n-1.$$

Due to the homogeneous structure of the random digraph $D(n,\mathcal{P})$, the random variable X^+ defines the out-degree of a given vertex of $D(n,\mathcal{P})$. Actually, we have n independent, identically distributed random variables $X^+(1), X^+(2), \dots, X^+(n)$. Therefore we can write shortly X^+ instead of $X^+(i)$.

The first property expresses, by appropriate factorial moment of X^+, the probability that a given subset of vertices is contained in the set of images of a vertex $i \in V = \{1, 2, \dots n\}$. As usuall, $(n)_k = n(n-1)\dots(n-k+1)$ and $E_k(X)$ stands for the k-th factorial moment of a random variable X.

Property 2.1. *For a given i, $1 \le i \le n$, let $U \subseteq V \setminus \{i\}$ and $|U| = t \ge 1$. Then*

$$\Pr\{U \subseteq N^+(i)\} = \frac{1}{(n-1)_t} E_t(X^+).$$

Proof. Assume that we are given a vertex $i \in V$ and t-element subset $U \subseteq V \setminus \{i\}$. Let W be an l-element subset of $V \setminus (U \cup \{i\})$. Then

$$\begin{aligned}\Pr\{U \subseteq N^+(i)\} &= \sum_{W \subseteq V \setminus U \setminus \{i\}} \Pr\{U \cup W = N^+(i)\} \\ &= \sum_{l=0}^{n-t-1} \sum_{\substack{W \subseteq V \setminus U \setminus \{i\} \\ |W| = l}} \Pr\{U \cup W = N^+(i)\}.\end{aligned}$$

So, by (1.1) we have

$$\begin{aligned}\Pr\{U \subseteq N^+(i)\} &= \sum_{l=0}^{n-t-1} \binom{n-t-1}{l} p_{t+l} \\ &= \sum_{k=t}^{n-1} \binom{n-t-1}{k-t} p_k \\ &= \frac{1}{(n-1)_t} \sum_{k=t}^{n-1} (k)_t P_k \\ &= \frac{1}{(n-1)_t} E_t(X^+). \quad \square\end{aligned}$$

In particular, if $t = 1$ the above property defines an arc occurence probability in digraph $D(n, \mathcal{P})$. Denoting this probability by p^* we have

$$p^* = \frac{1}{n-1} E(X^+). \tag{2.1}$$

Now let $X^- = X^-(i)$ be the in-degree of a given vertex $i \in \{1, 2, \ldots, n\}$ of $D(n, \mathcal{P})$. Clearly, the probability distribution of X^- depends on $\mathcal{P}$. We have the following result.

Property 2.2. *For $i = 1, 2, \ldots, n$ the random variable $X^-(i)$ has binomial distribution $\mathcal{B}(n-1, p^*)$.*

Proof. For a given vertex $i \in \{1, 2, \ldots, n\}$, let $N^-(i)$ be the set of vertices that "have chosen" vertex i. Without loosing generality we may take $i = 1$. Then, again due to the homogeneous structure of $D(n, \mathcal{P})$, we obtain

$$\begin{aligned}\Pr\{X^-(1) = k\} &= \Pr\{|N^-(1)| = k\} \\ &= \binom{n-1}{k} \Pr\{1 \in N^+(2), 1 \in N^+(3), \ldots, 1 \in N^+(k+1), \\ &\qquad 1 \notin N^+(k+2), \ldots, 1 \notin N^+(n)\} \\ &= \binom{n-1}{k} [\Pr\{1 \in N^+(2)\}]^k [\Pr\{1 \notin N^+(n)\}]^{n-1-k}\end{aligned}$$

and the result follows from Property 2.1. $\square$

In contrast with out-degrees of vertices of $D(n, \mathcal{P})$, the random variables $X^-(i)$, $i = 1, 2, \ldots, n$, are not, in general, independent. The only case when these variables are independent is when X^+ is binomially distributed. As a matter of fact we have the following equivalence.

Property 2.3. *Random variables $X^-(1), X^-(2), \dots, X^-(n)$ are independent if and only if the out-degree of each vertex has a binomial distribution.*

Proof. Let $X^-(1), \dots, X^-(n)$ be independent random variables. By Property 2.1 each of these variables has the binomial distribution $\mathcal{B}(n-1, p^*)$ with $p^* = \frac{1}{n-1}E(X^+)$. Now reverse direction of each arc (out-degrees become in-degrees and vice versa). Then in a new digraph, by Property 2.2, the in-degree of a given vertex (but the out-degree in the original digraph) has binomial distribution with parameters $n-1$ and p^*.

Conversely, let $\mathcal{P}$ be a binomial distribution $\mathcal{B}(n-1, p)$. Then, for each vertex, appearances of arcs going from the vertex can be considered as independent events. Therefore the definition of $D(n, \mathcal{P})$ implies that all arcs appear independently and with probability p in such a digraph. This insures the independence of each in-degree in the random digraph $D(n, \mathcal{P})$. □

3. Main results

Using a standard terminology of stochastic ordering, we say that a random variable Y is smaller than a random variable X *in the usual stochastic order* (and we write $Y \leq X$) if

$$\Pr\{Y > u\} \leq \Pr\{X > u\} \tag{3.1}$$

for all $u \in (-\infty, \infty)$. In the particular case, when X and Y are discrete random variables such that

$$\Pr\{X = k\} = P_k$$

$$\Pr\{Y = k\} = Q_k$$

for $k = 0, 1, 2 \dots, n-1$ the condition (3.1) is equivalent to

$$\sum_{k=0}^{u} P_k \leq \sum_{k=0}^{u} Q_k \tag{3.2}$$

for all $u = 0, 1, 2, \dots, n-1$. In such a case we simply write $\mathcal{Q} \leq \mathcal{P}$, where clearly $\mathcal{P} = (P_0, P_1, \dots, P_{n-1})$ and $\mathcal{Q} = (Q_0, Q_1, \dots, Q_{n-1})$.

Now we are ready to state our main result and its consequences. It is a generalization of a well-known result about monotone properties in a classical random graph model $G(n, p)$ (see [1, p.33]).

Theorem 3.1. *If $\mathcal{Q} \leq \mathcal{P}$ then for any digraph monotone increasing property $\mathcal{A}$*

$$\Pr\{D(n,\mathcal{Q}) \text{ has } \mathcal{A}\} \leq \Pr\{D(n,\mathcal{P}) \text{ has } \mathcal{A}\}. \tag{3.3}$$

Moreover, if $\mathcal{A}$ is any monotone decreasing digraph property, then the opposite inequality holds.

The proof is based on the following lemma which shows, that even a very small "distruction" (known as a *Dalton transfer*) of the probability distribution of out-degree of a random digraph $D(n,\mathcal{P})$ has a significant influence on the probability of this graph having a monotone property $\mathcal{A}$.

Lemma 3.1. *Let $\mathcal{P} = (P_0, \dots, P_k, \dots, P_{n-1})$ be a probability distribution. Let $\{\mathcal{P}_k(x)\}$, $k = 1, \dots, n-1$ be a distribution defined as follows*

$$\mathcal{P}_k(x) = (P_0, \dots, P_{k-1} + x, P_k - x, \dots, P_{n-1})$$

where x is any real number such that $0 \leq x \leq P_k$. Then for any monotone increasing digraph property $\mathcal{A}$ and for any integer k, $1 \leq k \leq n-1$,

$$\Pr\{D(n,\mathcal{P}) \text{ has } \mathcal{A}\} \geq \Pr\{D(n,\mathcal{P}_k(x)) \text{ has } \mathcal{A}\}. \tag{3.4}$$

Moreover, if $\mathcal{A}$ is any monotone decreasing digraph property, then the opposite inequality holds.

Proof. We show the lemma in a case of monotone increasing property $\mathcal{A}$ (if a property $\mathcal{A}$ is monotone decreasing then „not $\mathcal{A}$" is monotone increasing). Let k, $1 \leq k \leq n-1$, be fixed and put

$$f_k(x) = \Pr\{D(n,\mathcal{P}_k(x)) \text{ has } \mathcal{A}\}.$$

To prove (3.4) it is enough to show that

$$f_k'(x) = \frac{df_k(x)}{dx} \leq 0. \tag{3.5}$$

For, if $f_k(x)$ is a decreasing function on the interval $[0, P_k]$, then for any $x \in [0, P_k]$ and any k we have

$$\Pr\{D(n,\mathcal{P}) \text{ has } \mathcal{A}\} = f_k(0) \geq f_k(x) = \Pr\{D(n,\mathcal{P}_k(x)) \text{ has } \mathcal{A}\}.$$

Now denote by $A(n_0, n_1, \dots, n_{n-1})$ the number of digraphs (without loops and multiple arcs) with n_i vertices of out-degree i, $i = 0, 1, \dots, n-1$, having an increasing property $\mathcal{A}$. Then

$$\Pr\{D(n,\mathcal{P}) \text{ has } \mathcal{A}\} = \sum_{(n_0,\dots,n_{n-1})} A(n_0, n_1, \dots, n_{n-1}) \prod_{i=0}^{n-1} p_i^{n_i},$$

where p_i is given by (1.1) and the summation is over all n-tuples $(n_0, n_1, \dots, n_{n-1})$ such that each $n_i \geq 0$ and

$$\sum_{i=0}^{n-1} n_i = n.$$

Therefore

$$f_k(x) = \sum_{(n_0,\dots,n_{n-1})} A(n_0, n_1, \dots, n_{n-1}) \left[\frac{P_{k-1}+x}{\binom{n-1}{k-1}}\right]^{n_{k-1}} \left[\frac{P_k - x}{\binom{n-1}{k}}\right]^{n_k} \cdot \prod_{\substack{i=0\\ i\neq k-1,k}}^{n-1} \left[\frac{P_i}{\binom{n-1}{i}}\right]^{n_i}.$$

Calculating the derivative $f_k'(x)$ and then using the inequality (see [4])

$$\begin{aligned}&(n-k)n_{k-1}\, A(n_0, n_1, \dots, n_{n-1})\\ &\leq (n_k+1)\, k\, A(n_0, \dots, n_{k-1}-1, n_k+1, \dots, n_{n-1})\end{aligned}$$

which holds for a monotone increasing property $\mathcal{A}$ and any k, $1 \leq k \leq n-1$, one can confirm that (3.5) is true. □

Now we are ready to prove the main result.

Proof of Theorem 3.1. By (3.2) it is clear that $Q_{n-1} \leq P_{n-1}$. Let

$$x_{n-1} = P_{n-1} - Q_{n-1}$$

and let us consider the distribution

$$\begin{aligned}\mathcal{P}^* &= (P_0, P_1, \dots, P_{n-2} + x_{n-1}, P_{n-1} - x_{n-1})\\ &= (P_0, P_1, \dots, P_{n-2} + x_{n-1}, Q_{n-1}).\end{aligned}$$

By Lemma 3.1, for a monotone increasing digraph property $\mathcal{A}$,

$$\Pr\{D(n,\mathcal{P}) \text{ has } \mathcal{A}\} \geq \Pr\{D(n,\mathcal{P}^*) \text{ has } \mathcal{A}\}.$$

Now let

$$\begin{aligned}x_{n-2} &= P_{n-2} + x_{n-1} - Q_{n-2}\\ &= (P_{n-2} + P_{n-1}) - (Q_{n-2} + Q_{n-1}).\end{aligned}$$

Again, by (3.2), $x_{n-2} \geq 0$. Taking

$$\begin{aligned}\mathcal{P}^{**} &= (P_0, P_1, \dots, P_{n-3} + x_{n-2}, P_{n-2} + x_{n-1} - x_{n-2}, Q_{n-1}) \\ &= (P_0, P_1, \dots, P_{n-3} + x_{n-2}, Q_{n-2}, Q_{n-1})\end{aligned}$$

we have, again by Lemma 3.1,

$$\Pr\{D(n, \mathcal{P}^*) \text{ has } \mathcal{A}\} \geq \Pr\{D(n, \mathcal{P}^{**}) \text{ has } \mathcal{A}\}.$$

Analogously, applying succesively this lemma for

$$x_k = (P_k + \dots + P_{n-1}) - (Q_k + \dots + Q_{n-1}),$$

with $k = n-3, \dots, 2, 1$ we finally obtain the inequality (3.3). □

An interesting problem is to answer the question: what assumptions should be made about the class of stochasticaly ordered probability distributions to quarantee the existence of a threshold function (threshold probability distribution) for every monotone property of $D(n, \mathcal{P})$. At least one can expect that this is the case when $\mathcal{P}$ depends only on two parameters, including n. Below we present corollaries of our theorem for some distributions of this type.

The first special case is the following result on monotone properties of the classical random graph $G(n,p)$ which was already mentioned in the Introduction.

Corollary 3.1. *Suppose $\mathcal{A}$ is any graph monotone increasing property and let $0 \leq p_1 < p_2 \leq 1$. Then*

$$\Pr\{G(n, p_1) \text{ has } \mathcal{A}\} \leq \Pr\{G(n, p_2) \text{ has } \mathcal{A}\}.$$

Proof. Let $\mathcal{Q} = \mathcal{B}(n-1, p_1)$ and $\mathcal{P} = \mathcal{B}(n-1, p_2)$. Since the first derivative

$$\frac{d}{dp} \sum_{k=0}^{u} \binom{n}{k} p^k (1-p)^{n-k} = -n \binom{n-1}{u-2} p^{u-2} (1-p)^{n-u+1}$$

is less or equal to zero for any $u = 0, 1, \dots, n-1$, so $\mathcal{Q} \leq \mathcal{P}$ and by Theorem 3.1 we obtain our result. □

Clearly for two degenerate distributions $\mathcal{Q}$ and $\mathcal{P}$, where $Q_{d_1} = 1$ and $P_{d_2} = 1$, $d_1 < d_2$, we have by (3.2) that $\mathcal{Q} \leq \mathcal{P}$. Hence by Theorem 3.1 we obtain

Corollary 3.2. *Suppose $\mathcal{A}$ is any graph monotone increasing property and let $1 \le d_1 < d_2 \le n-1$. Then*

$$\Pr\{D(n,d_1) \text{ has } \mathcal{A}\} \le \Pr\{D(n,d_2) \text{ has } \mathcal{A}\}. \qquad \square$$

Note that possessing an increasing graph property by a match graph $M(n,d)$ is an increasing digraph property of $D(n,d)$. Therefore we immediately obtain

Corollary 3.3. *Suppose $\mathcal{A}$ is any graph monotone increasing property and let $1 \le d_1 < d_2 \le n-1$. Then*

$$\Pr\{M(n,d_1) \text{ has } \mathcal{A}\} \le \Pr\{M(n,d_2) \text{ has } \mathcal{A}\}. \qquad \square$$

For many graph (digraph) properties $\mathcal{A}$ we know (or at least we believe) that there exists d such that

$$\Pr\{D(n,d) \text{ has } \mathcal{A}\} = 0 \quad \text{and} \quad \Pr\{D(n,d+1) \text{ has } \mathcal{A}\} = 1\,.$$

It is natural to search for the "threshold" two-point distribution on $\{d, d+1\}$ such that $P_d = 1-p$, $P_{d+1} = p$. Denote by $D(n,d;\, p)$ the corresponding digraph. Then the following direct consequence of Lemma 3.1 can be useful.

Corollary 3.4. *Suppose $\mathcal{A}$ is any graph monotone increasing property and let d be fixed and $0 \le p_1 < p_2 \le 1$. Then*

$$\Pr\{D(n,d;\, p_1) \text{ has } \mathcal{A}\} \le \Pr\{D(n,d;\, p_2) \text{ has } \mathcal{A}\}. \qquad \square$$

Finally let us consider digraphs "generated" by geometrical distribution, more precisely by distribution

$$\mathcal{P}(n,q) = \left(\frac{q}{1-(1-q)^n}, \frac{q(1-q)}{1-(1-q)^n}, \frac{q(1-q)^2}{1-(1-q)^n}, \dots, \frac{q(1-q)^{n-1}}{1-(1-q)^n}\right).$$

Denote the corresponding random digraph by $D\big(n, \mathcal{P}(n,q)\big)$.

Corollary 3.5. *Suppose $\mathcal{A}$ is any graph monotone increasing property let $0 \le q_1 < q_2 \le 1$. Then*

$$\Pr\big\{D\big(n, \mathcal{P}(n,q_2)\big) \text{ has } \mathcal{A}\big\} \le \Pr\big\{D\big(n, \mathcal{P}(n,q_1)\big) \text{ has } \mathcal{A}\big\}.$$

Proof. Let $\mathcal{Q} = \mathcal{P}(n, q_2)$ and $\mathcal{P} = \mathcal{P}(n, q_1)$. One can easily check that the first derivative

$$\frac{d}{dq}\left(\sum_{k=0}^{u}\frac{q(1-q)^k}{1-(1-q)^n}\right) = \frac{d}{dq}\left(\frac{1-(1-q)^{u+1}}{1-(1-q)^n}\right)$$
$$= \frac{(1-q)^u}{[1-(1-q)^n]^2}\Big((n-u-1)(1-q)^n - n(1-q)^{n-u-1} + (u+1)\Big)$$

is greater or equal to zero for any $u = 0, 1, \ldots, n-1$ and $q > 0$, so $\mathcal{Q} \leq \mathcal{P}$ and by Theorem 3.1 we obtain our corollary. □

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