Note
On Directed Odd or Even Minimum (s, t)-Cut Problem and Generalizations

Ottilia Fülöp*

Institute of Mathematics, Technical University, Budapest

June 18, 2002

Abstract

We show that if $M(n, m)$ denotes the time of a (u, v)-minimum cut computation in a directed graph with $n \geq 2$ nodes, m edges, and s and t are two distinct given nodes, then there exists an algorithm with $O(n^2 m + n \cdot M(n, m))$ running time for the directed minimum odd (or even) (s, t)-cut problem and for its certain generalizations.

Let $\vec{G} = (V, \vec{E})$ be a directed graph with $n \geq 2$ nodes and m edges, s and t two distinct given nodes of $\vec{G}$. The *cut* of the graph is a subset C of the nodes. An (s, t)-*cut* of the graph is a subset $C \subset V$ with $s \in C$ and $t \notin C$. The *value* of the cut $f(C)$ is the number or the total capacity of the edges leaving C. A function f over all subsets of a ground set V is called *submodular* if all $X, Y \subseteq V$ satisfy $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$. An example of a submodular function is the cut value function. The notion of minimum cut and its generalization, the minimum of a submodular function, plays an important role in combinatorial optimization. See [5] for a survey of the application of submodular functions and [6] for that of minimum cuts.

Grötschel et al. ([4]) generalize the notion of an odd (cardinality) set and define a *triple family* as follows. A family of subsets of a ground set V forms a *triple family over* V if for all $X \subseteq V$ and $Y \subseteq V$ whenever three of the four sets X, Y, $X \cap Y$ and $X \cup Y$ are not in the triple family, then so is the fourth.

* *E-mail address:* otti@math.bme.hu
Supported from OTKA grant T-29772 and T-037547

We consider the following optimization problems related to (s, t)-minimum cuts in directed graphs:

- The odd (even) minimum (s, t)-cut problem asks for a cut C, such that $s \in C$, $t \notin C$, $|C|$ is odd (even) with $f(C)$ minimum.
- For a prescribed node subset T, the T-odd (T-even) minimum (s, t)-cut problem asks for a cut C, such that $s \in C$, $t \notin C$, $|C \cap T|$ is odd (even) with $f(C)$ minimum.
- The problem of minimum (s, t)-cut with cardinality not divisible by a given integer p (or for a given node subset T the problem of minimum (s, t)-cut C with $|C \cap T|$ not divisible by p).
- The problem of minimum (s, t)-Steiner cut asks for a cut C such that $s \in C$, $t \notin C$, C subdivides a given subset T of V (i.e. $\emptyset \neq C \cap T \neq T$) with $f(C)$ minimum.
- The problem of minimum (s, t)-generalized Steiner cut asks for a cut C such that $s \in C$, $t \notin C$, C subdivides at least one of the given subsets $T_1, \ldots, T_k \subseteq V$ with $f(C)$ minimum.

If we leave out the condition $s \in C$, $t \notin C$ everywhere, all families of sets from each example are triple families over V (see [1], Section 1.3), but the above mentioned problems do not ask for the minimum value cut in these triple families like problems of [1]. Here each example asks for the minimum value (s, t)-cut in these triple families. Notice that if $\mathcal{G}$ is a triple family over V, then for two arbitrarily fixed distinct nodes s and t $\mathcal{G} \cap \{X \subset V : s \in X, t \notin X\}$ is not a triple family over V.

Lemma 1 *Let $\mathcal{G} \subseteq 2^V$ be a triple family over V, s, $t \in V$ two distinct given nodes from V. Then $\mathcal{G}^* := \{X - \{s\} : X \in \mathcal{G}, s \in X, t \notin X\}$ forms a triple family over $V - \{s, t\}$.*

Proof. Note that a subset A of $V - \{s, t\}$ is not in $\mathcal{G}^*$ iff $A \cup \{s\}$ is not in $\mathcal{G}$. Let A and B two arbitrarily fixed subsets of $V - \{s, t\}$. Suppose that three of the four sets A, B, $A \cap B$ and $A \cup B$ are not in $\mathcal{G}^*$, this means that three of the four sets $A \cup \{s\}$, $B \cup \{s\}$, $(A \cap B) \cup \{s\}$ and $(A \cup B) \cup \{s\}$ are not in the triple family $\mathcal{G}$, hence so is the fourth. If we leave out s from the fourth set we obtain the fourth set from A, B, $A \cap B$ and $A \cup B$, this is a subset of $V - \{s, t\}$, which is not in $\mathcal{G}^*$. □

Theorem 2. *Let $\mathcal{G} \subseteq 2^V$ be a triple family over V, let s, $t \in V$ be two distinct given nodes of V and let $V_{s,t} := \{X \subset V : s \in X, t \notin X\}$. Then there exists an algorithm with $O(n^2 m + n \cdot M(n, m))$ running time for finding an*

f-minimizer set C over $\mathcal{G}$ such that $s \in C$, $t \notin C$, where $M(n, m)$ denotes the time of a submodular function minimization over $V_{u,v}$.

Proof. Let $\mathcal{G}^*$ be the triple family over $V - \{s, t\}$ from Lemma 1. We use for $\mathcal{G}^*$ our algorithm for minimizing submodular functions over triple families from [1], which may return $\emptyset$ or $V - \{s, t\}$, with $O(n^2 m + n \cdot M(n, m))$ running time, where $M(n, m)$ denotes the time of a (u, v)-minimum cut computation ([1], Section 4.2). A (u, v)-minimum cut computation in $\mathcal{G}^*$ corresponds to computing a minimum cut with source s and u contracted and sink t and v contracted. Our algorithm from [1] uses two major tools: the Cheng-Hu flow-equivalent tree and a special uncrossing step, and means a factor $O(n)$ improvement over the running time of the previous most efficient algorithm of Goemans and Ramakrishnan for triple families [3]. If Y_0 is the output of the algorithm, i.e. $Y_0 \subseteq V - \{s, t\}$, $Y_0 \in \mathcal{G}^*$ with $f(Y_0)$ minimum, then $C := Y_0 \cup \{s\}$ is an f-minimizer over $\mathcal{G}$ such that $s \in C$, $t \notin C$. □

Acknowledgement The author would like to thank András Frank and András A. Benczúr for fruitful discussions.

References

[1] A.A. Benczúr and O. Fülöp, Fast algorithms for even/odd minimum cuts and generalizations, Lecture Notes in Computer Science 1879, Algorithms-ESA 2000, Proceedings of the 8th Annual European Symposium, Saarbrücken, Germany, September 2000, Springer-Verlag, Berlin, pp. 88-99.

[2] C.K. Cheng and T.C. Hu, Ancestor trees for arbitrary multi-terminal cut functions, Ann. Op. Res. 33, 1991, pp. 199-213.

[3] M.X. Goemans and V.S. Ramakrishnan, Minimizing submodular functions over families of sets, Combinatorica 15(4), 1995, pp. 499-513

[4] M. Grötschel, L. Lovász and A. Schrijver, Corrigendum to our paper The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 4, 1984, pp. 291-295.

[5] L. Lovász, Submodular functions and convexity, in: A. Bachem, M. Grötschel and B. Korte (eds.), Mathematical Programming: The State of the Art, Bonn, 1982, Springer, Berlin, 1983, pp. 235-257.

[6] J.C. Picard and M. Queyranne, Selected applications of minimum cuts in networks, I.N.F.O.R.: Canadian Journal of Operations Research and Information Processing 20, 1982, pp. 394-422.