Some properties of two-color partitions

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ABSTRACT: Let $R(n)$ denote the number of two-color partitions of n. We obtain several identities concerning $R(n)$.

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Introduction: We prove several identities that link $R(n)$, the number of two-color partitions of n, with $p(n)$, the unrestricted partition function, and $q_0(n)$, the number of self-conjugate partitions of n. We also show that $R(n)$ changes parity infinitely often.

Preliminaries: Let $x \in C$, $|x| < 1, n \in N \cup \{0\}$.

Definition 1: Let $R(n)$ denote the number of two-color partitions of n.

Remark: In the literature, $R(n)$ is sometimes denoted $p_{-2}(n)$.

Definition 2: Let $p(n)$ denote the unrestricted partition function.

Definition 3: Let $q_0(n)$ denote the number of self-conjugate partitions of n.

Definition 4: Let $c_2(n)$ denote the number of 2-core partitions of n.

Definition 5: Let $\omega(k) = k(3k-1)/2$ where $k \in Z$. (pentagonal numbers)

$$\sum_{n=0}^{\infty} R(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-2} \tag{1}$$

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-1} \tag{2}$$

$$\sum_{n=0}^{\infty} q_0(n)x^n = \prod_{n=1}^{\infty}(1+x^{2n-1}) \tag{3}$$

$$\sum_{n=0}^{\infty} c_2(n)x^n = \prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{1-x^n} = \prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^{2n-1}} \tag{4}$$

$$c_2(n) = \begin{cases} 1 \text{ if } n = j(j+1)/2 \\ 0 \text{ otherwise} \end{cases} \tag{5}$$

$$\prod(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2} \tag{6}$$

$$\prod(1-x^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{k(k+1)/2} \tag{7}$$

$$\prod_{n=1}^{\infty}\frac{1-x^n}{1+x^n} = \sum_{n=-\infty}^{\infty}(-1)^n x^{n^2} = 1 + 2\sum_{n=1}^{\infty}(-1)^n x^{n^2} \tag{8}$$

$$\prod_{n=1}^{\infty}(1-x^n)^3(1-x^{2n-1})^2 = \sum_{k=-\infty}^{\infty}(1-6k)x^{\omega(k)} \tag{9}$$

$$\prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}(1-x^{2n-1})^{-1} \tag{10}$$

Remarks: Most of these identities are well-known. (9) is due to B. Gordon. (See [2]).

The Main Results

The first several theorems involve $R(n)$ and $p(n)$.

Theorem 1:

$$p(n) = \sum_{j=0}^{[n/2]} c_2(n-2j)R(j)$$

Proof: (4) implies

$$\prod_{n=1}^{\infty} p(n)x^n = (\sum_{n=0}^{\infty} c_2(n)x^n) \prod_{n=1}^{\infty} (1-x^{2n})^{-2}$$

Now (1) and (2) imply

$$\sum_{n=0}^{\infty} p(n)x^n = (\sum_{n=0}^{\infty} c_2(n)x^n)(\sum_{n=0}^{\infty} R(n)x^n)$$

The conclusion now follows by matching coefficients of like powers of x. ■

Corollary 1:

$$p(2m) = \sum \{R(m - \frac{k(k+1)}{4}) : k \equiv 0, 3 \pmod 4\}$$

that is,

$$p(2m) = R(m) + R(m-3) + R(m-5) + R(m-14) + \cdots$$

Proof: Let $n = 2m$ in Theorem 1 and apply (5). ■

Corollary 2:

$$p(2m+1) = \sum \{R(m - \frac{k(k+1)}{4} - \frac{1}{2}) : k \equiv 1, 2 \pmod 4\}$$

that is,

$$p(2m+1) = R(m-1) + R(m-2) + R(m-8) + R(m-11) + \cdots$$

Proof: Let $n = 2m+1$ in Theorem 1 and apply (5). ■

Theorem 2:

$$p(n) = \sum_{k=-\infty}^{\infty} (-1)^k R(n - \omega(k))$$

that is,

$$p(n) = R(n) - R(n-1) - R(n-2) + R(n-5) + R(n-7) + \cdots$$

Proof: (1) implies

$$\left(\prod_{n=1}^{\infty}(1 - x^n)\right)\left(\sum_{n=0}^{\infty} R(n)x^n\right) = \prod_{n=1}^{\infty}(1 - x^n)^{-1}$$

The conclusion now follows from (2) and (6), matching coefficients of like powers of x. ■

Theorem 3:

$$R(n) = \sum_{j=0}^{n} p(n-j)p(j)$$

Proof: This follows easily from (1) and (2). ■

Corollary 3:

$$R(n) \equiv \begin{cases} p(n/2) & (\text{mod } 2) \text{ if } n \text{ is even} \\ 0 & (\text{mod } 2) \text{ if } n \text{ is odd} \end{cases}$$

Proof: Theorem 3 implies

$$R(2m) = 2\sum_{j=0}^{m-1} p(2m-j)p(j) + p(m)^2$$

$$R(2m-1) = 2\sum_{j=0}^{m-1} p(2m-1-j)p(j)$$

The conclusion now follows. ■

Corollary 4: $R(n)$ changes parity infinitely often.

Proof: This follows from Corollary 3, since it is known that $p(n)$ changes parity infinitely often. (See [3].) ■

The next theorem is a recurrence involving only $R(n)$.

Theorem 4:

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)R(n-\frac{k(k+1)}{2}) = \begin{cases} (-1)^j \text{ if } n = j(3j\pm 1)/2 \\ 0 \text{otherwise} \end{cases}$$

Proof: (1) implies

$$(\prod_{n=1}^{\infty}(1-x^n)^3)(\sum_{n=0}^{\infty}R(n)x^n = \prod_{n=1}^{\infty}(1-x^n)$$

The conclusion now follows from (6) and (7), matching coefficients of like powers of x. ∎

The final theorem involves $R(n)$ and $q_0(n)$.

Theorem 5:

$$\sum_{k=-\infty}^{\infty}(1-6k)R(n-\omega(k)) = (-1)^n(q_0(n)+2\sum_{j=1}^{\infty}q(n-j^2))$$

Proof: (9) implies

$$\prod_{n=1}^{\infty}(1-x^n)(1-x^{2n-1})^2 = \prod_{n=1}^{\infty}(1-x^n)^{-2}(\sum_{n=-\infty}^{\infty}(1-6n)x^{\omega(n)})$$

(10) implies

$$\prod_{n=1}^{\infty}\frac{1-x^n}{1+x^n}\prod_{n=1}^{\infty}(1-x^{2n-1}) = \prod_{n=1}^{\infty}(1-x^n)^{-2}(\sum_{n=-\infty}^{\infty}(1-6n)x^{\omega(n)})$$

(8), (3), and (1) imply

$$(1+2\sum_{n=1}^{\infty}(-1)^n x^{n^2})(\sum_{n=0}^{\infty}(-1)^n q_0(n)x^n) = (\sum_{n=0}^{\infty}R(n)x^n)(\sum_{n=-\infty}^{\infty}(1-6n)x^{\omega(n)})$$

The conclusion now follows by matching coefficients of like powers of x. ∎

The table below lists $R(n)$ for $0 \leq n \leq 26$.

n	$R(n)$	n	$R(n)$	n	$R(n)$
0	1	9	300	18	12230
1	2	10	481	19	17490
2	5	11	752	20	24842
3	10	12	1165	21	35002
4	20	13	1770	22	49030
5	36	14	2665	23	68150
6	65	15	3956	24	94235
7	110	16	5822	25	129512
8	185	17	8470	26	177087

Concluding remarks: The reader may note that apparently

$$R(n) \equiv 0 \pmod 5 \quad if \quad n \equiv 2,3,4 \pmod 5$$

This is a known result. (See [1], (4.9), p. 271.)

References

1. J. M. Gandhi Congruences for $p_r(m)$ and Ramanujan's τ function *Amer. Math. Monthly* 70 (1963) 265-274

2. B. Gordon Some identities in combinatorial analysis *Quart. J. Math. Oxford* Ser. (2) 12 (1961) 285-290

3. O. Kolberg Note on the parity of the partition function *Math. Scand.* 7 (1959) 377-378