On Harmonious Graphs of Order 6

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Abstract

In this paper, we determine all harmonious graphs of order 6.

All graphs in this paper are finite, simple and undirected. We shall use the basic notation and terminology of graph theory as in [1].

A graph G of vertex set V(G) and edge set E(G) and of size q = |E(G)| is said to be harmonious [2] if there exists an injective function f, called a harmonious labeling,

$$f : V(G) \longrightarrow Z_q \text{ (= the group of integers modulo q)}$$

such that the induced function

$$f^* : E(G) \longrightarrow Z_q$$

defined by $f^*(xy) = f(x) + f(y)$, for all $xy \in E(G)$ is again injective. The function f is the vertex labeling function and the function f^* is the corresponding edge labeling function. The image of the function f (= $I_m(f)$) is called the corresponding set of vertex labels. This definition extends to the case where G is a tree by allowing exactly two vertices to have the same value under f.

Graham and Sloane [3] determined the harmonious graphs of order ≤ 5. In this paper we extend this result to graphs of order 6.

This paper is divided into two sections. In Section 1 we accumulate the results needed to establish our main theorems. In Section 2 we obtain our main theorems, Theorem 2.1 (resp. Theorem 2.2) which determine all connected (resp. disconnected) harmonious graphs of order 6.

1. Auxiliary results

We shall make frequent reference to the following results :

Result 1 : [2, Theorem 11]

If a harmonious graph G has an even size $q = |E(G)|$ and if $2^k \mid deg(v)$ for all $v \in V(G)$ for some $k \geq 1$, then $2^{k+1} \mid q$.

Result 2 : [3, Theorem 7]

K_n is harmonious if and only if $n \leq 4$.

Result 3 : [3, Theorem 19]

$K_{m,n}$ is harmonious if and only if m or n = 1.

Result 4 : [2, Proof by Aldred/ Mckay**]**

All trees of order ≤ 26 are harmonious.

Result 5 : [3, Table 2]

Let m(p) be the maximum size of a harmonious graph of order p, then we have the following table :

P	2	3	4	5	6	7	8	9	10
m(p)	1	3	6	9	13	17	24	30	.36

Result 6 : [3, Theorem 8, Corollary i]

Any vertex in a harmonious graph can be assigned the label 0 under a suitable harmonious labeling function.

Result 7 : [5, Theorem 5]

The graph obtained from $K_{2,n}$, $n \geq 2$, by adding p and q pendant edges out from the two vertices of degree n is not harmonious ; where $p, q \geq 0$,

Let $C_{m,n}$ be the graph consisting of two cycles of lengths m and n with exactly one vertex in common. The harmonious property of such graphs is investigated in the next two theorems.

Theorem 1.1.

If $C_{m,n}$ is harmonious, then $m+n \equiv 0 \pmod 4$.

Proof

Suppose that $C_{m,n}$ is harmonious and let v_0 be the common vertex of the two cycles of $C_{m,n}$. Then $G = C_{m,n} \cup K_1$ is again harmonious and we have $q = |E(G)| = m+n$. By Result 6, there exists a harmonious labeling function f of G such that $f(v_0) = 0$, so that

$$\frac{q(q-1)}{2} \equiv \sum_{e \in E(G)} f^*(e) \equiv 2\left(\sum_{v \in V(G)} f(v) - f(w)\right)$$

$$\equiv 2\left(\frac{q(q-1)}{2} - f(w)\right) \pmod q$$

where $\{w\} = V(K_1)$. If q is odd, let $q = 2r+1, r \in N$. Then we get $(2r+1)r \equiv 2((2r+1)r - f(w)) \pmod{2r+1}$, i.e., $f(w) \equiv 0 \pmod q$. Now if $q = m+n = 4r+2$, then 2 divides the degree of every vertex of the graph, but 2^2 does not divide q, the result

Theorem 1.2.

$C_{3,n}$ is harmonious if and only if $n \equiv 1 \pmod 4$.

Proof

$\Rightarrow$ This follows from Theorem 1.1.

$\Leftarrow$ Suppose $n \equiv 1 \pmod 4$ and let $V(C_{3,n}) = \{x, y, v_1, v_2, \ldots, v_n\}$, where v_1 is the common vertex of the two cycles of $C_{3,n}$, and x, y are the two other vertices of the 3-cycle in $C_{3,n}$. Define the function

$$f : V(C_{3,n}) \longrightarrow Z_{n+3}$$

by

$$f(v_i) = \begin{cases} \frac{i-1}{2} & i \text{ odd} \le \frac{n+1}{2} \\ \frac{i+1}{2} & \frac{n+1}{2} < i \text{ odd} \le n \\ \frac{n+5}{2} + \frac{i}{2} & i \text{ even} \le n-1 \end{cases}$$

$$f(x) = \frac{n+3}{2}, \quad f(y) = \frac{n+5}{2}.$$

Clearly, f is injective. We have $f^*(v_n v_1) = \frac{n+1}{2}$, $f^*(x v_1) = \frac{n+3}{2}$, $f^*(y v_1) = \frac{n+5}{2}$, $f^*(x y) = 1$

and $\{f^*(v_i v_{i+1}) : 1 \leq i \leq \frac{n+1}{2}\} = \{\frac{n+5}{2} + i : 1 \leq i \leq \frac{n+1}{2}\}$

and $\{f^*(v_i v_{i+1}) : \frac{n+1}{2} < i < n\} = \{\frac{n+5}{2} + i+1 : \frac{n+1}{2} < i < n\}$

$$= \{j : 2 \leq j \leq \frac{n-1}{2}\} \ (\text{mod } (n+3)),$$

hence f^* is onto as desired.

The following question remains open (except for the case covered in Theorem 1.2) :

Question :

What about the harmoniousness of the graph $C_{m,n}$ if $m+n \equiv 0 \pmod 4$?

We shall also need to investigate the harmonious property of several special graphs. This is done in the following theorems.

Theorem 1.3.

Let G be the graph

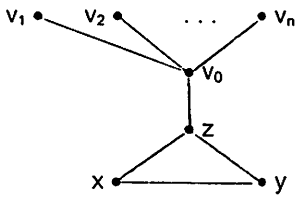


Figure 1.

then G is harmonious if and only if $n \equiv 0 \pmod 4$.

Proof

Let $q = |E(G)| = n+4$

$\Rightarrow$ Suppose that G has a harmonious labeling f with $f(v_0) = 0$, then there must exist natural numbers $0 < x_1, y_1, z_1 < q$ such that $x_1 + y_1 \equiv 0$, $x_1 + z_1 \equiv y_1$, $y_1 + z_1 \equiv x_1 \pmod q$. Hence $2 z_1 \equiv 0 \pmod q$ and $q = 2 z_1$. Therefore x_1 and y_1 are either both odd or both even and z_1 must be even, then $n \equiv 0 \pmod 4$.

$\Leftarrow$ Suppose that $n \equiv 0 \pmod 4$. Define a bijection

$$f : V(G) \longrightarrow Z_q$$

such that $f(v_0) = 0$, $f(x) = \frac{1}{4}q$, $f(y) = \frac{3}{4}q$, $f(z) = \frac{1}{2}q$, then one can easily check that f^* is bijective as well.

Theorem 1.4.

Let G be the graph

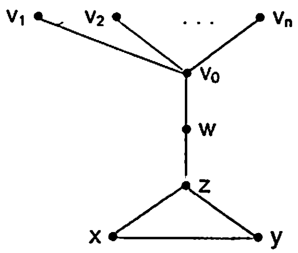


Figure 2.

then G is not harmonious for all $n \geq 0$.

Proof

Let $q = |E(G)| = n+5$.

Suppose that G has a harmonious labeling f such that $f(v_0) = 0$, then there exists distinct natural numbers $0 < x_1, y_1, z_1, w_1 < q$ such that

$$x_1 + y_1 \equiv z_1, \quad x_1 + z_1 \equiv y_1, \quad z_1 + w_1 \equiv 0, \quad z_1 + y_1 \equiv x_1 \pmod q.$$

or $$y_1 + x_1 \equiv z_1, \quad x_1 + z_1 \equiv y_1, \quad z_1 + w_1 \equiv x_1, \quad z_1 + y_1 \equiv 0 \pmod q.$$

The first line gives $2x_1 = 2z_1 \equiv 0 \pmod q$ which is absurd and the second line gives $y_1 + x_1 + z + w_1 \equiv z_1 + x_1 \pmod q$, *i.e.*, $y_1 + w_1 \equiv 0 \pmod q$, and together with $y_1 + z_1 \equiv 0 \pmod q$, we get $w_1 \equiv z_1 \pmod q$ which is absurd as well.

Theorem 1.5.

The graph G :

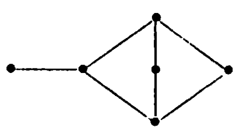


Figure 3.

is not harmonious.

Proof

Suppose that the graph G has a harmonious labeling f where the label assigned to each vertex is as indicated in Figure 4.

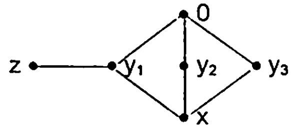


Figure 4.

Observe that $x \in I_m(f^*)$ gives $z + y_1 \equiv x \pmod 7$ and $z \in I_m(f^*)$ gives $x+y_i \equiv z \pmod 7$ for i = 2 or 3 (since otherwise $x+y_1 \equiv z \pmod 7$ gives, with $z + y_1 \equiv x_1 \pmod 7$ that $2y_1 \equiv 0 \pmod 7$ which is absurd). Hence $x+y_1 \equiv 0 \pmod 7$ (since otherwise $y_i \equiv z + y_1 \pmod 7$ for i = 2 or 3 which is absurd). Therefore we may assume that

$z + y_1 \equiv x, \quad x + y_2 \equiv 0, \quad x + y_3 \equiv z \pmod 7$ and so $y_1 + y_3 \equiv 0 \pmod 7$.

Now it follows that $\sum_{e \in E(G)} f^*(e) = 3(x + y_1) + 2(y_2 + y_3) + z \equiv 0 \pmod 7$.

Solving these four congruences gives $x \equiv 0 \pmod 7$ which is absurd.

Theorem 1.6.

The graph G :

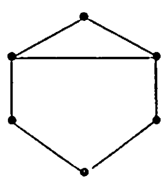


Figure 5.

is not harmonious.

Proof

Suppose that the graph G has a harmonioous labeling f where the label assigned to each vertex is as indicated in Figure 6

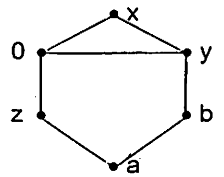


Figure 6.

We have

$$\sum_{e \in E(G)} f^*(e) = 2(x+a+b+z) + 3y \equiv 0 \pmod 7 \; . \; . \; . \; . \; . \; . \; . \qquad (1)$$

There are only three cases to consider :

Case 1 : $z + a \equiv b$, $y + b \equiv a \pmod 7$

It follows that

$$z + y \equiv 0 \pmod 7 \quad \ldots\ldots (2)$$

Now if $a + b \not\equiv 0 \pmod 7$, then we have also

$$x + y \equiv 0 \pmod 7 \ldots\ldots (3)$$

From (2) and (3) it follows that $x \equiv z \pmod 7$ which is impossible.

It can be shown that $a+b \equiv 0 \pmod 7$ leads to a contradiction.

Case 2 : $z+a \equiv b$, $x+y \equiv a \pmod 7$

Here it follows that $a+b \equiv 0 \pmod 7$, otherwise $y+b \equiv 0 \pmod 7$. Then $a+b \equiv x+y+b \equiv x \pmod 7$, which is absurd. Now from equation (1) it follows that $2(x+z)+3y \equiv 0 \pmod 7$, hence $2x+2z+2y+y \equiv 0 \pmod 7$, *i.e.*, $2a+2z+y \equiv 0 \pmod 7$, so $y+b \equiv a \pmod 7$, which is impossible.

Case 3 : $y+b \equiv a$, $x+y \equiv b \pmod 7$

In this case we must have $a+b \equiv 0 \pmod 7$, since otherwise $z+a \equiv 0 \pmod 7$, *i.e.*, $z+y+b \equiv 0 \pmod 7$. Hence $\{z,y,b\} \equiv \{1,2,4\}$ *or* $\{3,5,6\} \pmod 7$. Equation (1) further gives $2x+2a+2b+2z+2y+y \equiv 2x+2a+y \equiv 2x+2y+2b+y \equiv 0 \pmod 7$. Hence $4b+y \equiv 0 \pmod 7$, which cannot hold for any choice of the set {z, y, b}.

Theorem 1.7.

The graph G :

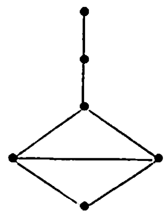


Figure 7.

is not harmonious.

Proof

Suppose that the graph G has a harmonious labeling f where the label assigned to each vertex is as indicated in Figure 8.

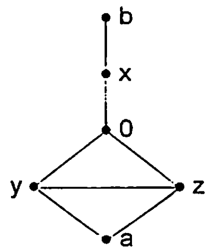


Figure 8.

We have only three cases to consider :

Case 1 : y+a ≡ b , x+b ≡ a (mod 7)

In this case we must have z+a ≡ 0 (mod 7) , otherwise y+z ≡ 0 (mod 7). But $y+a \equiv b \pmod 7$ and $x+b \equiv a \pmod 7$ give $y+x \equiv 0 \pmod 7$, so $x \equiv z \pmod 7$, which is absurd.

Now $(b+x)+x+y+z+(y+z)+(y+a)+(z+a) \equiv 0 \pmod 7$. But since $a+z \equiv 0 \pmod 7$ and $x+y \equiv 0 \pmod 7$, it follows that $y+z+b \equiv 0 \pmod 7$. So $0 \equiv y+z+(y+a) \equiv 2y+z+a \equiv 2y \pmod 7$, hence $y \equiv 0 \pmod 7$, which is absurd.

Case 2 : y+z ≡ b , x+b ≡ a (mod 7)

In this case we may assume that z+a ≡ 0 (mod 7), then it follows that a + x + y + z + (y+z) + (y + a) + (z + a) ≡ 0 (mod 7), hence

$x + y + 2y \equiv 0 \pmod 7$ ....(1)

But $y + a \equiv y + x + b \equiv y + x + y + z \pmod 7$, so $a \equiv x + y + z \pmod 7$, since $z + a \equiv 0 \pmod 7$, we have

$x + y + 2z \equiv 0 \pmod 7$ ....(2)

From (1) and (2) we get $y \equiv z \pmod 7$, which is impossible.

Case 3 : **y+a ≡ b , y+z ≡ a (mod 7)**

In this case we have a+z ≡ 0 (mod 7), otherwise y+a+x ≡ b + x ≡ 0 (mod 7), and since b + x + x + y + z + a+ (y + a) + (z + a) ≡ 0 (mod 7), we have

$0 \equiv b + z + a + z \equiv y + a + z + a + z \equiv 3a + z \pmod 7$ .... (1)

Now {a, x, y} ≡ {1, 2, 4} or {3, 5, 6} (mod 7). Considering all possible values of *a, y* and hence of *z* and substituting in (1) , we always obtain a contradiction. Hence

$a + z \equiv 0 \pmod 7$ ....(2)

From (2) it follows that

$a + b + y + z + x + b + x \equiv 0 \pmod 7$ ....(3)

But we also have : $y + 2z \equiv y - 2a \equiv y - 2y - 2z \equiv -(y + 2z) \pmod 7$, and so

$y + 2z \equiv 0 \pmod 7$ ....(4)

From (3) and (4), and since $a \equiv y + z \pmod 7$, we have

$y + 2(b + x) \equiv 0 \pmod 7$ ....(5)

From (4) and (5) we obtain $z \equiv b + x \pmod 7$, which is absurd.

Theorem 1.8.

The graph G :

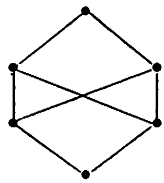


Figure 9

is not harmonious.

Proof

Suppose that the graph G has a harmonious labeling f, where the label assigned to each vertex is as indicated in Figure 10.

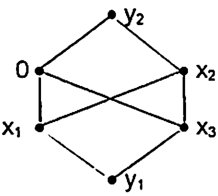


Figure 10

We have

$$\sum_{e \in E(G)} f^*(e) = 3(x_1 + x_2 + x_3) + 2(y_1 + y_2) \equiv 0 + 1 + \ldots + 7 \equiv 4 \pmod 8 \ldots\ldots(1)$$

hence $x_1 + x_2 + x_3$ is even and exactly one x_i is even for some $1 \leq i \leq 3$ (since otherwise x_i is even for $1 \leq i \leq 3$ and $x_2 \notin I_m(f^*)$). Therefore we have two cases to consider :

Case 1 : **x_2 is even**

In this case x_1 and x_3 are odd, hence we must have y_1 odd and y_2 even. Summing over the odd labeled edges we get $x_1 + x_3 + (x_1 + x_2) + (x_2 + x_3) \equiv 0$ (mod 8) hence $x_1 + x_2 + x_3 \equiv 0$ (mod 8). Substituting in equation (1) we obtain $y_1 + y_2$ is even which is absurd.

Case 2 : **x_2 is odd**

In this case we may assume that x_1 is even and that x_3 is odd. We have four subcases to consider :

Subcase 2.1. **y_1 and y_2 are both even :**

In this subcase we must have $x_2 + x_3 \equiv y_1$ and $x_3 + y_1 \equiv x_2$ (mod 8) hence $2x_3 \equiv 0$ (mod 8) which is absurd.

Subcase 2.2. **y_1 and y_2 are both odd :**

Since the odd edge labels are 1,3,5 and 7 (mod 8), we must have $x_2 + x_3 \equiv 0 \pmod 8$, otherwise $x_2 + y_2 \equiv 0 \pmod 8$ and $y_1 + x_3 \equiv 0 \pmod 8$, which is absurd. Hence $x_1 + x_2 \equiv y_1$, $x_1 + y_1 \equiv x_2 \pmod 8$, consequently $2x_1 \equiv 0 \pmod 8$, i.e., $x_1 \equiv 4 \pmod 8$. Since $0 \equiv x_2 + x_3 \equiv x_1 + y_1 + x_3 \pmod 8$, and $x_1 \equiv 4 \pmod 8$, it follows that $y_1 + x_3 \equiv 4 \pmod 8$, which is absurd.

Subcase 2.3. **y_1 is even and y_2 is odd :**

In this subcase we must have $y_1 + x_3 \equiv x_2$ (mod 8), and $x_2 + y_2 \equiv y_1$ (mod 8) , since otherwise $x_2 + x_3 \equiv y_1$ (mod 8), which together with $y_1 + x_3 \equiv x_2 \pmod 8$ give $2x_3 \equiv 0$ (mod 8), i.e., $x_3 \equiv 4 \pmod 8$, which is absurd. It follows that $x_2 + y_1 \equiv y_1 + x_3 + x_2 + y_2 \pmod 8$, hence

$$x_3 + y_2 \equiv 0 \pmod 8 \quad \ldots\ldots(1)$$

Now summing over the odd (resp. even) labeled edges and using (1) , we get:

$$y_2 + (y_1 + x_3) + x_3 + (x_1 + x_2) \equiv x_1 + 2x_2 \equiv 0 \pmod 8 \quad \ldots\ldots(2)$$

(resp.) $x_1 + (x_1 + y_1) + (x_2 + x_3) + (x_2 + y_2) \equiv 2x_1 + 2x_2 + y_1 \equiv 4$ (mod 8)..........(3)

From (2) and (3) , we get $x_1 + y_1 \equiv 4$ (mod 8) which is absurd.

Subcase 2.4. **y_1 is odd and y_2 is even :**

In this subcase we must have $x_1 + y_1 \equiv x_2$ (mod 8). Summing over the odd (resp. even) labeled edges, we get

$$(x_1 + y_1) + (x_2 + y_2) + (x_1 + x_2) + x_3 \equiv 0 \pmod 8 \quad \ldots \quad (1)$$

(resp.) $x_1 + y_2 + (x_2 + x_3) + (x_3 + y_1) \equiv 4$ (mod 8) . . . . (2)

Hence $x_3 \equiv x_1 + x_2 + 4$ (mod 8) . . . . (3)

We must have $x_2 + y_2 \equiv y_1$ (mod 8), otherwise $y_1 \equiv x_1 + x_2 \equiv 2x_1 + y_1 \pmod 8$, and hence $x_1 \equiv 4 \pmod 8$, from (3) we get $x_3 \equiv x_2$ (mod 8) which is absurd.

Since $x_2 \equiv x_1 + y_1 \pmod 8$, it follows that

$$x_1 + y_2 \equiv 0 \pmod 8) \dots\dots\dots (4)$$

Also $x_2 + x_3 \equiv 0 \pmod 8$, otherwise $y_1 + x_3 \equiv 0 \pmod 8$, and consequently from (3) we get $0 \equiv y_1 + x_1 + x_2 + 4 \equiv 2x_2 + 4 \pmod 8$, then $2x_2 \equiv 4 \pmod 8$, i.e., x_2 is even, which is absurd. Consequently, it follows, using (2), that

$4 \equiv x_1 + y_2 + x_3 + y_1 \equiv x_1 + y_2 + x_3 + x_2 - x_1 \pmod 8$, and since $x_2 + x_3 \equiv 0 \pmod 8$, we get $y_2 \equiv 4 \pmod 8$. But (4) now gives that $y_2 \equiv x_1 \pmod 8$ which is absurd.

Theorem 1.9.

The graph G :

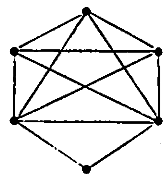


Figure 11.

is not harmonious.

Proof

Suppose that the graph G has a harmonious labeling f where the label assigned to each vertex is as indicated in Figure 12.

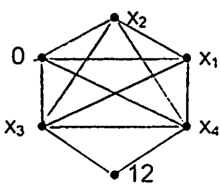


Figure 12.

First we notice that the vertex label 12 cannot be given to a vertex adjacent to the vertex of label 0, otherwise we obtain two edge labels 1. We have

$$\sum_{e \in E(G)} f^*(e) = 4x_1 + 5(x_2 + x_3 + x_4) + 36 \equiv 0 \pmod{13} \dots\dots\dots (1)$$

All the congruence relations used in this proof are mod 13.

There are only two cases :

Case 1 : $x_1 + x_i \equiv 12$ **for some i** $\in$ **{2, 3, 4}.**

Since the edge label 0 should exist, we must have either $x_i \equiv 1$, for some

$i \in \{2, 3, 4\}$, or $x_3 + x_4 \equiv 0$, since otherwise $x_i + x_j$ for some $i \in \{1, 2\}$ and some $j \in \{3, 4\}$ would give always in the four possible cases two equal edge labels, which is absurd.

If $x_3 + x_4 \equiv 0$, then equation (1) with $x_1 + x_2 \equiv 12$ give $x_1 = 5$, $x_2 = 7$ and consequently $\{x_3, x_4\} \equiv \{2, 11\}, \{3, 10\}$ or $\{4, 9\}$ which give $|f^{*-1}(7)|$, $|f^{*-1}(10)|$ or $|f^{*-1}(9)| > 1$ which is absurd.

If $x_2 \equiv 1$, then $x_1 \equiv 11$ and equation (1) gives $x_3 + x_4 \equiv 9$ and consequently $\{x_3, x_4\} \equiv \{2, 7\}, \{3, 6\}$ or $\{4, 5\}$ which gives $|f^{*-1}(1)|$, $|f^{*-1}(4)|$ or $|f^{*-1}(3)| > 1$ which is absurd.

Therefore we may assume that $x_3 \equiv 1$, then equation (1) with $x_1 + x_2 \equiv 12$ give $x_2 + 5x_4 \equiv 2$ and $\{x_1, x_2\} \equiv \{2, 10\}, \{3, 9\}, \{4, 8\}$ or $\{5, 7\}$ which gives $x_4 \in \{0, 1\}, \{5, 9\}, \{4, 10\}$ or $\{2, 12\}$, but in each case we obtain either two equal vertex labels or two equal edge labels, which is absurd.

Case 2 : $x_2 + x_3$, $x_2 + x_4$ **or** $x_3 + x_4 \equiv 12$

We may assume that $x_2 + x_3 \equiv 12$, then we have the following subcases :

Subcase 2.1. $x_1 + x_4 \equiv 0$

In this subcase, equation (1) gives $x_1 \equiv 5$, $x_4 \equiv 8$ and $\{x_2, x_3\} \equiv \{1, 11\}$ or $\{2, 10\}$ or $\{3, 9\}$ which gives $|f^{*-1}(6)|$ or $|f^{*-1}(7)|$ or $|f^{*-1}(8)| > 1$, which is absurd.

Subcase 2.2. $x_4 \equiv 1$

In this subcase, equation (1) gives $x_1 \equiv 4$ and $\{x_2, x_3\} \equiv \{2, 10\}$ or $\{3, 9\}$ which gives $|f^{*-1}(1)|$ or $|f^{*-1}(4)| > 1$, which is absurd.

Subcase 2.3. $x_2 + x_4$ **or** $x_3 + x_4 \equiv 0$

We must have $x_3 \equiv 12 + x_4$ or $x_2 \equiv 12 + x_4$ which is absurd, since we would have, in both cases, the edge label $12 + x_4$ repeated twice.

Subcase 2.4. x_2 **or** $x_3 \equiv 1$

We may assume that $x_2 \equiv 1$, hence $x_3 \equiv 11$ and equation (1) gives $4x_1 + 5x_4 \equiv 8$, the solutions of this congruence for $x_4 \neq 0$ or 12 are listed in the following table :

x_4	1	2	3	4	5	6	7	8	9	10	11
x_1	4	6	8	10	12	1	3	5	7	9	11
$x_1 + x_4$	5	8	11	1	4	7	10	0	3	6	9

Plotting the graph, in case there are six different vertex labels, we find always two equal edge labels, so this subcase is impossible

Subcase 2.5. $\quad x_1 + x_2$ **or** $x_1 + x_3 \equiv 0$

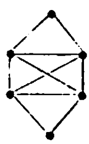
We may assume that $x_1 + x_2 \equiv 0$, hence equation (1) gives $4x_1 + 5x_4 \equiv 8$. Exactly as in subcase 2.4, we find that this subcase is impossible.

2. Main Theorems :

Now we can establish our main theorems :

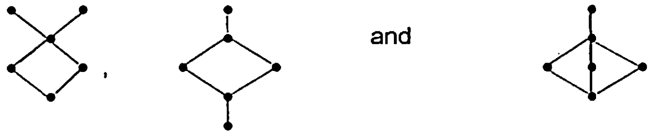
Theorem 2.1.

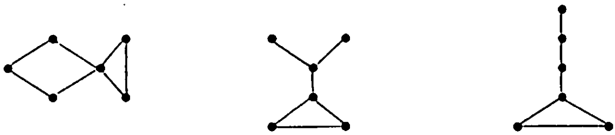
All connected graphs of order 6 are harmonious except for the following 18 special graphs which are not harmonious.

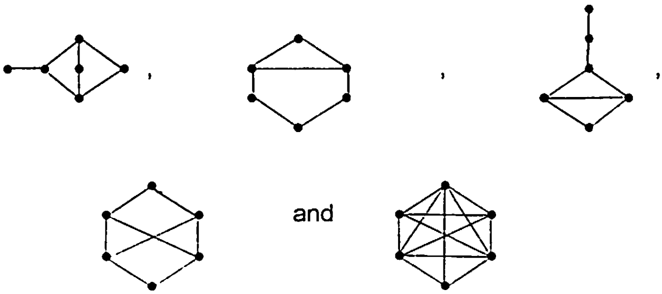
(i) C_6, C_6^2 and 

(ii) $K_{2,4}$ and $K_{3,3}$.

(iii) K_6 and $K_6 - e$.

(iv) 

(v) 

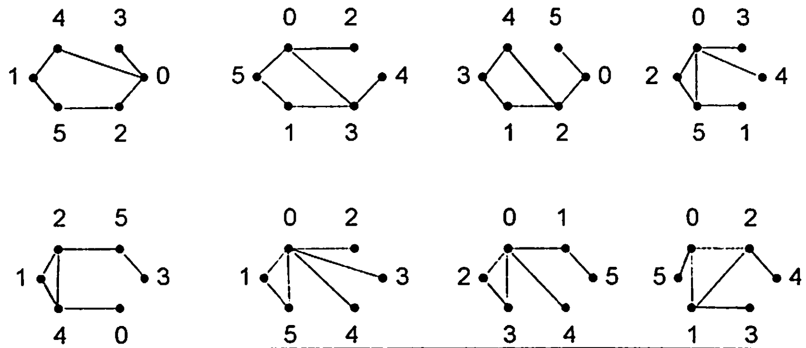
(vi) 

Proof

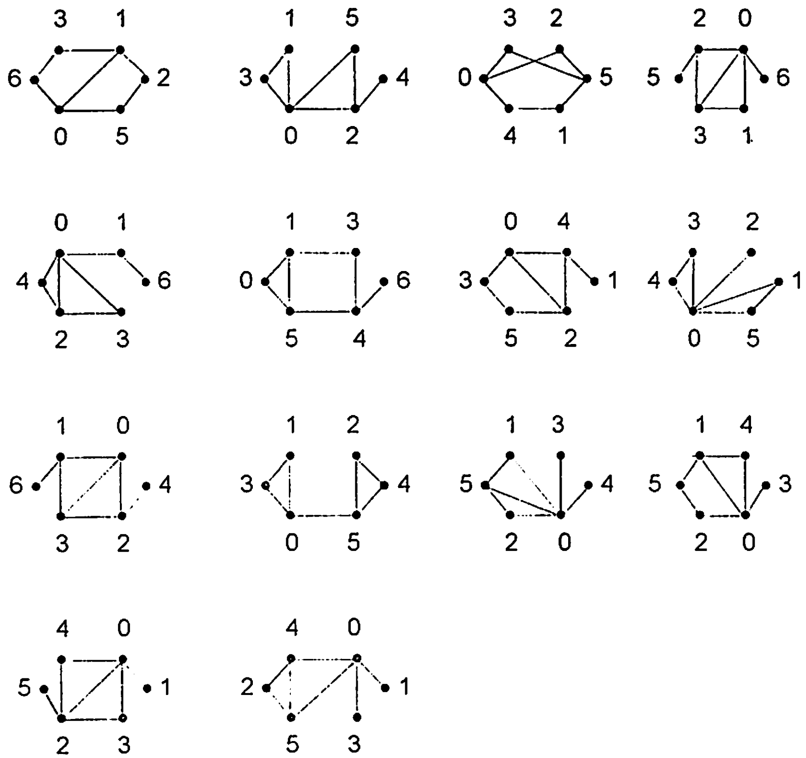
The graphs in case (i) are not harmonious by Result 1. Those in case (ii) are not harmonious by Result 3. The graphs in case (iii) are not harmonious by Result 5. Those in case (iv) are not harmonious by Result 7. The graphs in case (v) are not harmonious by Theorems 1.2, 1.3 and 1.4 respectively. The graphs in case (vi) are not harmonious by Theorems 1.5, 1.6, 1.7, 1.8 and 1.9 respectively.

According to Harary [4] we show that the remaining 94 connected graphs of order 6 are harmonious. Note that there are exactly 6 non-isomorphic trees of order 6 and these are harmonious by Result 4. The remaining 88 conncected graphs of order 6 are shown to be harmonious by giving a specific harmonious labeling assigned to the vertices of each such graph, these graphs are classified according to their size q and are drawn each with a specific harmonious labeling in the following table :

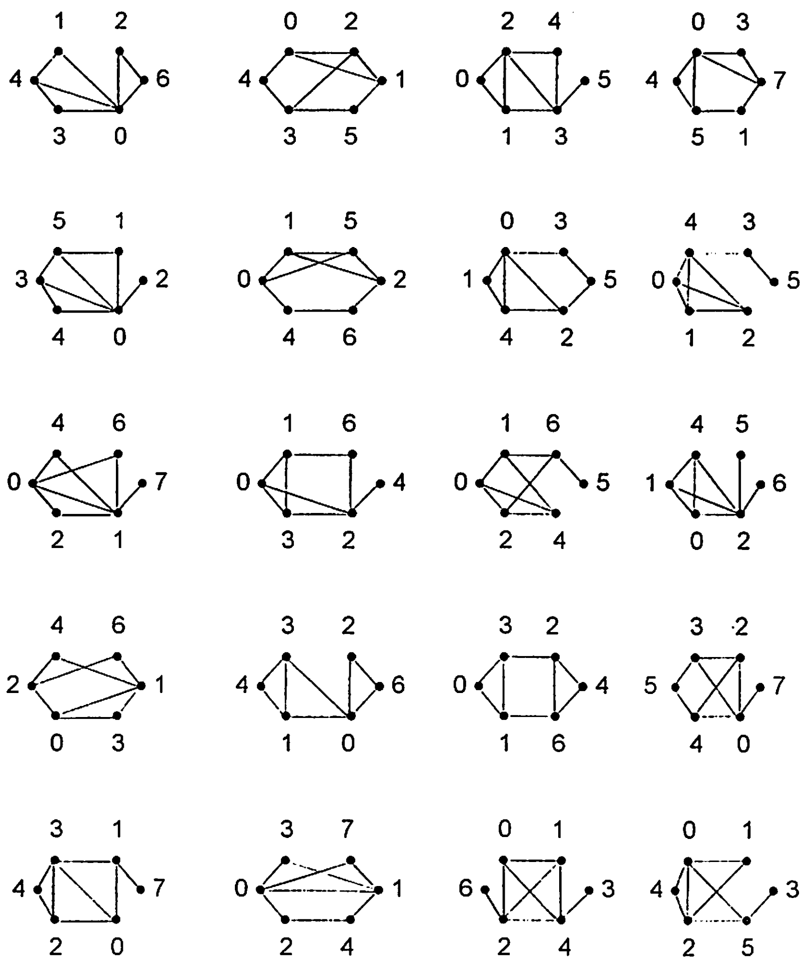
q = 6


4 3 1 0 5 2
0 2 5 4 1 3
4 5 3 0 1 2
0 3 2 4 5 1
2 5 1 3 4 0
0 2 1 3 5 4
0 1 2 5 3 4
0 2 5 4 1 3

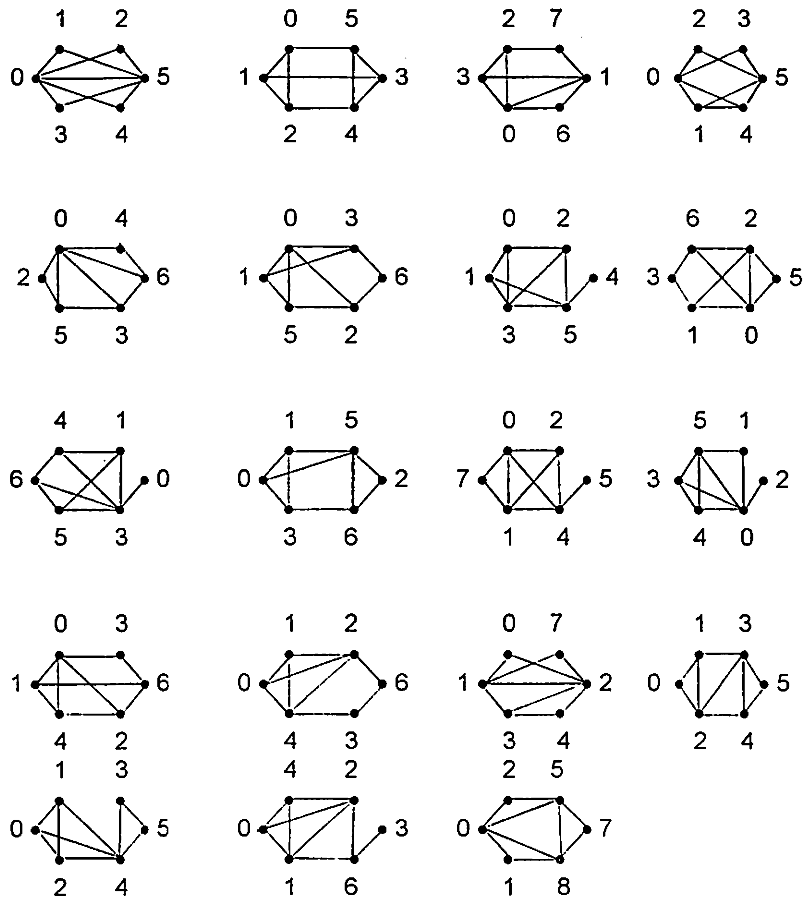
q = 7


3 1 6 2 0 5
1 5 3 4 0 2
3 2 0 5 4 1
2 0 5 6 3 1
0 1 4 6 2 3
1 3 0 6 5 4
0 4 3 1 5 2
3 2 4 1 0 5
1 0 6 4 3 2
1 2 3 4 0 5
1 3 5 4 2 0
1 4 5 3 2 0
4 0 5 1 2 3
4 0 2 1 5 3

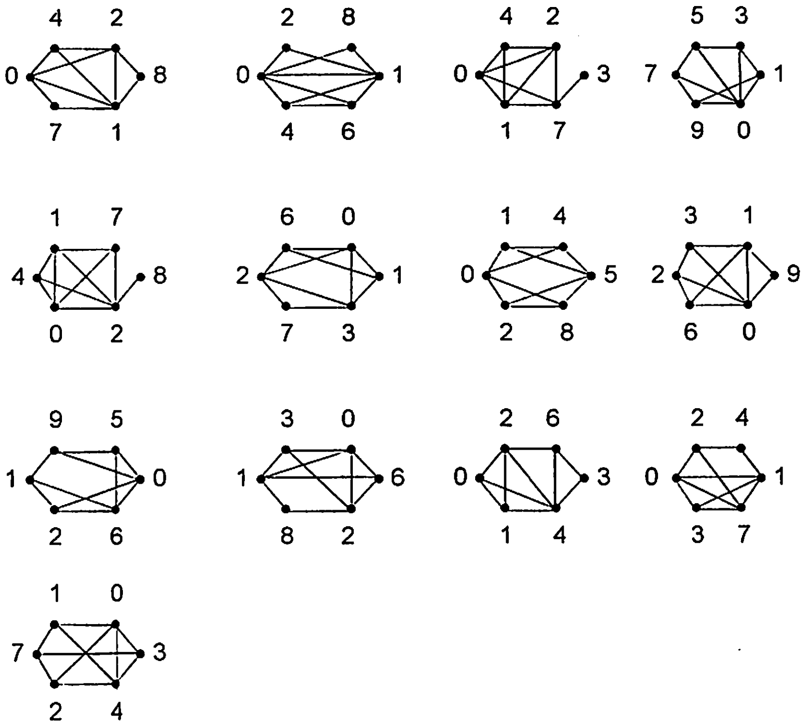
q = 8



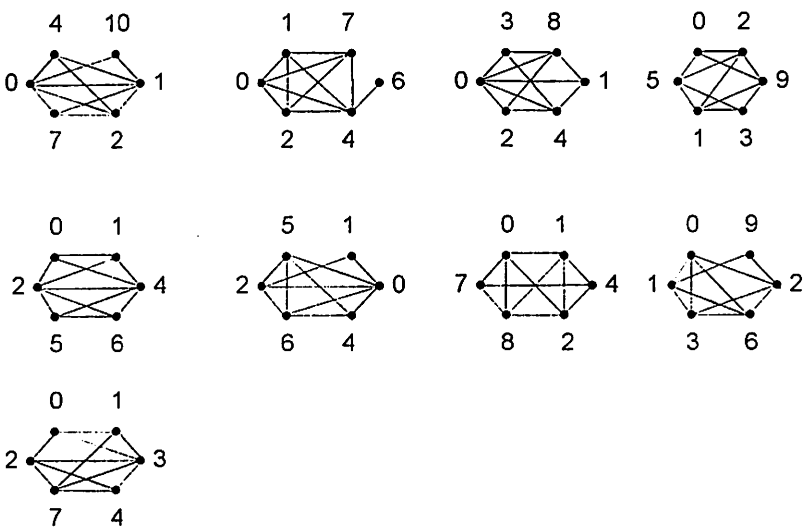
q = 9



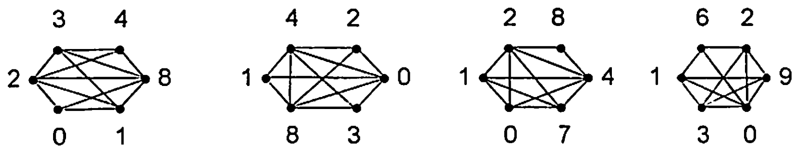
q = 10



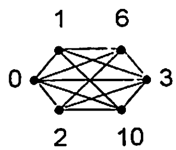
q = 11



q = 12



q = 13



Theorem 2.2.

The following table gives the number of harmonious (resp. not harmonious) disconnected (6,q) graphs = $H_6(q)$ (resp. $NH_6(q)$) for all possible q.

Q	$H_6(q)$	$NH_6(q)$
≤ 5	0	27
6	5	3
7	4	1
8	2	0
9	1	0
10	0	1

Proof

We make the following straightforward remarks :

(1) Let G be a (p, q) graph, $q < p$ and G is not a tree, then G is not harmonious.

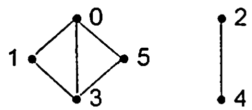
(2) Let G be a (p, q) graph, $q > p$, then the following statements are equivalent :

(i) G is harmonious

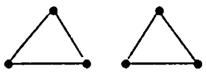
(ii) $G \cup \overline{K}_m$ is harmonious for some m, $1 \leq m \leq q-p$

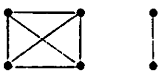
(iii) $G \cup \overline{K}_m$ is harmonious for all m, $1 \leq m \leq q-p$.

According to the list in [3, p. 218] there are 44 disconnected graphs of order 6. The above two remarks determine the harmoniousness condition of all these graphs except for the following three graphs :

(1) 

harmonious with the specific harmonious labeling indicated at the vertices of the graph.

(2) 
not harmonious by Result 1.

(3) 

not harmonious, since otherwise there would exist a harmonious labeling f where the label assigned to each vertex is as indicated in Figure 13.

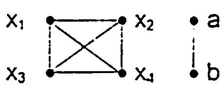


Figure 13.

We may assume that $0 \in I_m(f)$, so that

$$\sum_{i=1}^{4} x_i + (a+b) \equiv 0 \pmod 7$$

and $$\sum_{e \in E(G)} f^*(e) = 3\sum_{i=1}^{4} x_i + (a+b) \equiv 0 \pmod 7$$

Hence $a+b \equiv 0 \pmod 7$ and $x_i + x_j \equiv 0 \pmod 7$ for some $1 \leq i, j \leq 4$, $i = j$ which is absurd. This shows that we have the table in the theorem.

References

[1] **M. Behzad, G. Chartrand and L. Lesniak-Foster**; Graphs and Digraphs, Wadsworth International Group, California 1996.

[2] **J. A. Gallian,** A dynamic survey of graph labeling, The Electronic Journal of Combinatorics (2000).

[3] **R.L. Graham and N.J.A. Sloane**, On addiative bases and harmonious graphs. SIAM J. Alg. Discrete Meth. 1 (1980) 382-404.

[4] **F. Harary**, Graph Theory, Addison-Wesley, Reading, MA 1969.

[5] **M.A. Seoud and M.Z. Youssef**, Families of harmonious and non-harmonious graphs, J. Egypt. Math. Soc., Vol. 7(1) (1999) 117-125.