On Harmonious Labelings of Some Cycle Related Graphs

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Abstract: It is proved that following graphs are harmonious: (1) shell graphs (2) cycles with maximum possible number of concurrent alternate chords (3) Some families of multiple shells.

1 INTRODUCTION

Let G be a finite simple graph on p vertices and m edges. An edge with end vertices x and y is written as xy.

Definition: A **harmonious labeling** of a graph G is an injective function ϕ from the vertex set $V(G)$ of G to the set $\{1, \cdots, m-1\}$ such that the induced labeling on the edges given by $\phi(x\ y) = [\phi(x) + \phi(y)] \bmod m$ for each edge $x\ y \in E(G)$, is also injective.

This concept was introduced by R.L.Graham and N.J.A.Sloane [6]. They proved that while odd cycles C_{4m+1}, C_{4m+3}, wheels $W_n, n \geq 3$ and

Petersen graph are harmonious, most graphs including even cycles are not harmonious. L.Bolian and Z. Xiankun [1] proved that the graph C_n' obtained by joining a path to a vertex of C_n is harmonious if and only if it has even number of edges, the helm H_n is harmonious when n is odd. P.K.Deb and N. B.Limaye[4] gave harmonious labeling of helm H_n when n is even. L.Bolian and Z.Xiankun [1] gave many families of non-harmonious graphs. They are (a) $C^{n(t)}$ the graph obtained by joining t copies of C_n at a vertex, where $n \equiv 1 \bmod 2, t \equiv 2 \bmod 4$ or $n \equiv 2 \bmod 4, t \equiv 1 \bmod 2$ (b) $C_n + \overline{K_2}$ where $n \equiv 2, 4, 6 \bmod 8$. In [9] S.C.Shee gave harmonious labeling of graph obtained by identifying center of the star S_m with a vertex of an odd cycle C_n.

In this paper harmonious labelings of many families of cycle related graphs are given.

2 SHELL GRAPHS

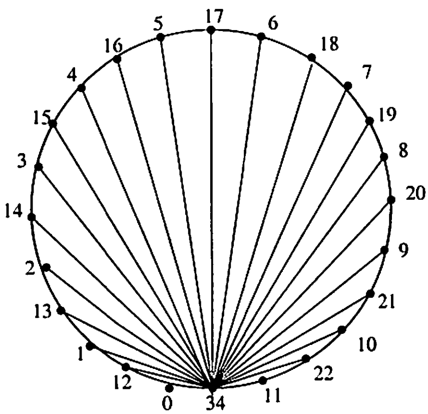
Definition: A shell $S_{n,n-3}$ of width n is a graph obtained by taking $n-3$ concurrent chords in a cycle C_n on n vertices. The vertex at which all the chords are concurrent is called **apex**. The two vertices adjacent to the apex have degree 2, apex has degree $n-1$ and all the other vertices have degree 3.

These graphs, under the name of **fans**, were proved to be harmonious in [6]. We give here a different harmonious labeling for them.

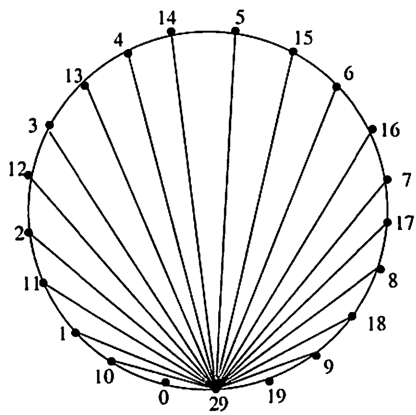
Theorem 1: Shell graphs are harmonious.

Proof: **Case 1:** $S_{n,n-3}$ where n is even. Let $n = 2r + 2$ and hence $m = 4r + 1$. Let the vertices and the edges be as follows: $V(S_{n,n-3}) = \{a_0, b_0, \cdots, a_r, b_r\}$

$E(S_{n,n-3}) = \{a_i b_i, a_i b_{i-1} | 0 \leq i \leq r\} \bigcup \{b_r\ v | v \neq a_0, a_r\}$. Here all the suffixes are taken modulo $r + 1$.



$S_{24,21}$



$S_{21,18}$

The labeling of vertices is as follows:

$\phi(a_i) = i, \quad 0 \leq i \leq r, \qquad \phi(b_i) = r+i+1, \quad 0 \leq i \leq r-1, \qquad \phi(b_r) = 3r+1.$

Case 2: n is odd. Let $n = 2r + 3$ and $m = 4r + 3$. Let the vertices and the edges be as follows: $V(S_{n,n-3}) = \{a_0, b_0, \cdots, a_r, b_r\} \bigcup \{a_{r+1}\}$

$E(S_{n,n-3}) = \{a_i\ b_i, a_i\ b_{i-1} | 1 \leq i \leq r\} \bigcup \{a_{r+1}\ v | v \neq a_0, b_r\} \bigcup \{a_0\ b_0, a_0\ a_{r+1}\}$.

Here all the suffixes are taken modulo $r + 1$. The labeling of vertices is as follows:

$\phi(a_i) = i, \quad 0 \leq i \leq r, \qquad \phi(b_i) = r+1+i, \quad 0 \leq i \leq r, \qquad \phi(a_{r+1}) = 3r+2.$

It can be easily seen that ϕ gives a harmonious labeling in both the cases. Hence $S_{n,n-3}$ is harmonious. □

A graph in which the number of edges is much more than the number of vertices is more likely to be harmonious. This is clearly seen in the shell graphs. In the next section we study many shell type graphs with fewer chords.

3 SHELL TYPE GRAPHS

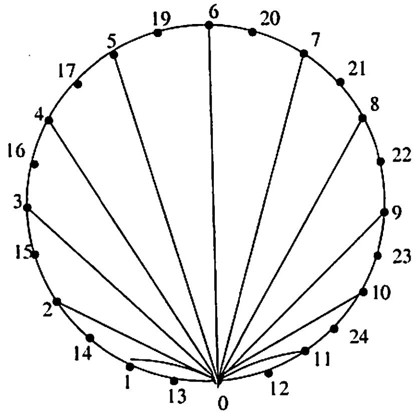
For positive integer $n, k, 1 \leq k \leq n-3$, the family $C(n,k)$ is the family of graphs obtained by taking k concurrent cords in a cycle C_n on n vertices. In general $C(n,k)$ consists of many graphs. The shell graph $S_{n,n-3}$ is the unique member of $C(n, n-3)$. Theta graphs discussed in [4] are members of $C(n,1)$. If we take maximum number of alternate concurrent chords then for $n = 2s$ there is unique such graph. It belongs to $C(2s, s-1)$. For $n = 2s+1$, one has to take some consecutive chords and hence there are many graphs with maximum number alternate concurrent chords on $2s+1$ points. We consider such graphs now.

Theorem 2: The unique graph in $C(2s, s-1)$ is harmonious.

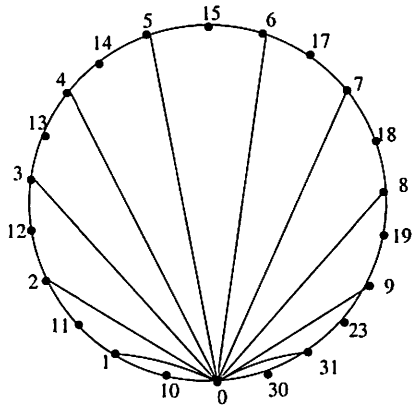
Proof: **Case 1:** $s = 2t$. Let the graph be denoted by $S_{4t,2t-1}$. This graph has $4t$ vertices and $6t-1$ edges. Let the vertices be $\{a_0, b_0, \cdots, a_{2t-1}, b_{2t-1}\}$. These vertices are connected in a cyclic manner and the chords are $a_0a_1, a_0a_2, \cdots, a_0a_{2t-1}$. Label the vertices as follows:

$\phi(a_i) = i,\ \ 0 \le i \le 2t-1, \phi(b_i) = 2t+1+i,\ \ 0 \le i \le t-2,$

$\phi(b_i) = 2t+2+i,\ \ t-1 \le i \le 2t-2,\ \ \phi(b_{2t-1}) = 2t.$



$S_{24,11}$



$S_{22,10}$

Case 2: $s = 2t+1$. Let the graph be denoted by $S_{4t+2,2t}$. This graph has $4t+2$ vertices and $6t+2$ edges. Let the vertices be $\{a_0, b_0, \cdots, a_{2t}, b_{2t}\}$. These vertices are connected in a cyclic manner and the chords are $a_0a_1, a_0a_2, \cdots, a_0a_{2t}$. Label the vertices as follows:

$\phi(a_i) = i,\ \ 0 \le i \le 2t-1,\ \ \ \phi(a_{2t}) = 6t+1,$

$\phi(b_i) = 2t+i,\ \ 0 \le i \le t,\ \ \phi(b_i) = 2t+1+i,\ \ t+1 \le i \le 2t-2,\ \ \phi(b_{2t-1}) = 4t+3,\ \ \phi(b_{2t}) = 6t.$

One can easily check that ϕ gives a harmonious labelings in both the cases. □

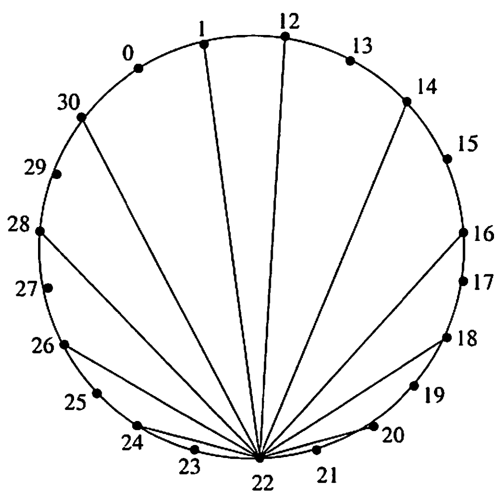
In case of the cycles on odd vertices we cannot take maximum number of alternate chords without taking some consecutive ones. Here we are interested in alternate chords symmetrically placed on two sides of the apex. If $n \equiv 1 \bmod 4$, we have to take two consecutive chords exactly in the middle. This graph is denoted by $S_{4t+1,2t}$. If $n \equiv 3 \bmod 4$, we have to take four consecutive chords in the middle. This graph is denoted by $S_{4t+3,2t+2}$. These graphs are also harmonious as shown by the following Theorem.

Theorem 3: The graphs $S_{4t+1,2t}, S_{4t+3,2t+2}, t \geq 1$ are harmonious.

Proof: **Case 1:** $n = 4t+1$. Consider an odd cycle on the vertices $\{a_0, b_0, \cdots, a_{2t-1}, b_{2t-1}, a_{2t}\}$ with $2t$ chords $\{a_t a_i | i \neq t\}$. Label the vertices as follows:

$\phi(a_i) = 2t + 2 + 2i, \;\; 0 \leq i \leq 2t - 1, \;\; \phi(a_{2t}) = 1,$

$\phi(b_i) = 2t + 3 + 2i \;\; 0 \leq i \leq 2t - 2 \;\; \phi(b_{2t-1}) = 0.$

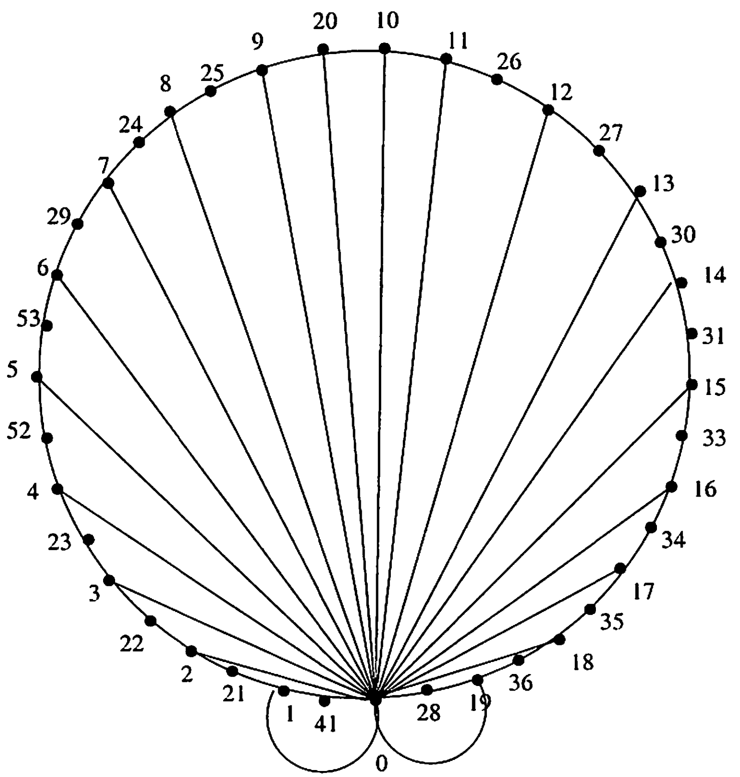


We have given here the labeling of $S_{21,10}$. It is easy to see that the labeling given here is a harmonious labeling. Hence $S_{4t+1,2t}$ is harmonious.

Case 2: $n = 4t + 3$ where t is odd. For $t = 1, 3, 5, 7$ one can easily verify that $S_{4t+3,2t+2}$ is harmonious. We note that for $t = 1$ it is just the shell graph.

Let $t \geq 9$. Consider vertices $\{a_0, b_0, \cdots, a_{2t}, b_{2t}, a_{2t+1}\}$. The edge set is given by

$$E(G) = \{a_i b_i, b_i a_{i+1}, 0 \leq i \leq 2t\} \bigcup \{a_{2t+1} a_0\} \bigcup \{a_0 a_i, a_0 b_{t-1+i}, 1 \leq i \leq t+1\}.$$



Label the vertices as follows:

$\phi(a_i) = i, \quad 0 \leq i \leq t+1,$

$\phi(a_{t+2}) = 3t - 1, \phi(a_{t+3}) = 3t, \phi(a_{2t+1}) = 3t + 1,$

$\phi(a_i) = 2t - 1 + i, \quad t+4 \leq i \leq (3t+1)/2, \quad \phi(a_i) = 2t + i, \quad (3t+3)/2 \leq i \leq 2t,$

$\phi(b_0) = 4t + 5$,

$\phi(b_i) = 2t + 2 + i, \ \ 1 \le i \le (t-3)/2$,

$\phi(b_{(t-1)/2}) = (11t + 5)/2, \phi(b_{(t+1)/2}) = (11t + 7)/2, \phi(b_{(t+3)/2}) = (7t - 5)/2, \phi(b_t) = 2t + 2$,

$\phi(b_i) = 2t - 1 + i, \ \ (t+5)/2 \le i \le t - 1, \ \ \phi(b_i) = i + 1, \ \ t + 1 \le i \le 2t$.

We have given the labeling of $S_{39,20}$.

We can see that the cyclic edges receive the values $4t+5, 4t+6, 2t+4, 2t+5, \cdots, 3t, 6t+2, 6t+3, 6t+4, 0, 4t-1, 4t, 3t+4, 3t+5, \cdots, 4t-2, 3t+2, 3t+3, 2t+3, 4t+1, 4t+2, 4t+3, 4t+4, 4t+7, \cdots, 5t+1, 5t+3, \cdots, 6t+1, 5t+2$ and $3t+1$ in that order. The chords receive the values $1, 2, \cdots, 2t+2$. This shows that $S_{4t+3,2t+2}$ is harmonious for all odd values of t.

Case 3: $n = 4t + 3$, where t is even. One can easily see that $S_{11,6}$ and $S_{19,10}$ are harmonious.

For $t \ge 6$, consider the vertices and the edges as in case (b). Label the vertices as follows:

$\phi(a_i) = i, \ \ 0 \le i \le t + 1, \ \ \phi(a_i) = 2t + 3 + i, \ \ t + 2 \le i \le 2t + 1$,

$\phi(b_0) = 4t + 5, \ \ \phi(b_i) = 2t + 2 + i, \ \ 1 \le i \le (t-2)/2, \ \ \phi(b_t) = 2t + 2$,

$\phi(b_i) = 2t + 4 + i, \ \ t/2 \le i \le t - 1, \ \ \phi(b_i) = i + 1, \ \ t + 1 \le i \le 2t$.

One can see that this is a harmonious labeling and hence $S_{4t+3,2t+2}$ is harmonious for all even values of t. □

For n odd we can take two consecutive chords in the beginning starting from the apex and then take all the alternate chords. This graph is denoted by $S'_{2s+1,s}$.

Theorem 4: The graph $S'_{2s+1,s}$ is harmonious.

Proof: Consider vertices $\{a_0, b_0, \cdots, a_{s-1}, b_{s-1}, a_s\}$ connected in a cyclic manner in this order. The chords are $\{a_0a_i, 1 \leq i \leq s-1\}$ and a_0b_{s-1}. Label the vertices as follows:

$\phi(a_i) = i,\ \ 0 \leq i \leq s-2,\ \ \phi(a_{s-1}) = s,\ \ \phi(a_s) = 3s-2,$

$\phi(b_i) = s+1+i,\ \ 0 \leq i \leq s-3,\ \ \phi(b_{s-2}) = 2s+1,\ \ \phi(b_{s-1}) = 3s.$

It is easy to see that this is a harmonious labeling. □

3 MULTIPLE SHELLS

We now consider another class of graphs called multiple shells.

Definition: **A multiple shell** $MS\{n_1^{t_1}, \cdots, n_r^{t_r}\}$ is a graph formed by t_i shells of widths n_i, $1 \leq i \leq r$, which have a common apex. This graph has $\sum_{i=1}^{k} t_i(n_i - 1) + 1$ vertices. A multiple shell is said to be **balanced** with width w if it is of the form $MS\{w^t\}$ or $MS\{w^t, (w+1)^s\}$, that is, the cycles involved are more or less of same size. If a multiple shell has in all k shells having a common apex, then it is called k-tuple shell; i.e. double shell if $k = 2$, triple shell if $k = 3$ etc.

Suppose S is a balanced multiple shell on n vertices with k shells having a common apex. If $n = kt$ then $S = MS\{t, (t+1)^{k-1}\}$. On the other hand if $n = kt + r, r \neq 0$, then $S = MS\{(t+1)^{k-r+1}, (t+2)^{r-1}\}$.

We consider the case of balanced double shells.

Theorem 5: The balanced double shells $MS\{p^2\}, p \geq 4$ and $MS\{p, p+1\}$ are harmonious.

Proof: **Case 1:** $MS\{p^2\}$ when p is odd. Bolian and Xiankun in [1] proved that $MS\{3^2\}$ is not harmonious.

Let $p \geq 5$. Let the vertex set be $V(MS\{p^2\}) = \{a_0, b_0, \cdots, a_{p-2}, b_{p-2}, a_{p-1}\}$. The two cycles are $\{a_{(p-1)/2}, a_0, b_0, \cdots, a_{(p-3)/2}, b_{(p-3)/2}, a_{(p-1)/2}\}$ and $\{a_{(p-1)/2}, b_{(p-1)/2}, \cdots, a_{p-2}, b_{p-2}, a_{p-1}, a_{(p-1)/2}\}$. This means $a_{(p-1)/2}$ is the common apex. There are $2p-6$ chords given by $a_{(p-1)/2}a_i, 1 \leq i \leq p-2, i \neq (p-1)/2$, and $a_{(p-1)/2}b_i, 0 \leq i \leq p-2, i \neq (p-3)/2, (p-1)/2$.

We label the vertices as follows:

$\phi(a_0) = 2p-3,\ \ \phi(a_i) = p+i-1,\ 1 \leq i \leq (p-3)/2,\ \ \phi(a_{(p-1)/2}) = 3p-4,$

$\phi(a_{(p+1)/2}) = (p-1)/2,\ \ \phi(a_i) = i,\ (p+3)/2 \leq i \leq p-1,$

$\phi(b_i) = i,\ \ 0 \leq i \leq (p-3)/2,$

$\phi(b_{(p-1)/2}) = 3p-3,\ \ \phi(b_i) = p-2+i,\ (p+1)/2 \leq i \leq p-2.$

Case 2: $MS\{p^2\}$ when p is even. Let the vertex set be as before. The two cycles are $\{b_{(p-2)/2}, a_0, b_0, \cdots, a_{(p-2)/2}, b_{(p-2)/2}\}$ and $\{b_{(p-2)/2}, a_{p/2}, b_{p/2}, \cdots, a_{p-2}, b_{p-2}, a_{p-1}, b_{(p-2)/2}\}$.

This means $b_{(p-2)/2}$ is the common apex. There are $2p-6$ chords given by $b_{(p-2)/2}a_i, 1 \leq i \leq p-2, i \neq (p-2)/2, p/2$ and $b_{(p-2)/2}b_i, 0 \leq i \leq p-2, i \neq (p-2)/2$.

We label the vertices as follows:

$\phi(a_0) = 2p-3,\ \ \phi(a_i) = p+i-2,\ 1 \le i \le (p-2)/2,\ \ \phi(a_{p/2}) = 3p-7,$

$\phi(a_i) = p-3+i,\ (p+2)/2 \le i \le p-1,$

$\phi(b_i) = i,\ \ 0 \le i \le (p-6)/2,\ \ \phi(b_{(p-4)/2}) = (p-2)/2,$

$\phi(b_{(p-2)/2}) = 3p-5,\ \ \phi(b_i) = i,\ p/2 \le i \le p-2.$

Case 3: $MS\{p, p+1\}$ when p is odd. One can easily see that $MS\{3,4\}$ is harmonious.

Let $p \ge 5$. Let the vertex set be $\{a_0, b_0, \cdots, a_{p-1}, b_{p-1}\}$. The two cycles are $\{a_{(p-1)/2}, a_0, b_0, \cdots, a_{(p-3)/2}, b_{(p-3)/2}, a_{(p-1)/2}\}$ and $\{a_{(p-1)/2}, b_{(p-1)/2}, \cdots, a_{p-1}, b_{(p-1)}, a_{(p-1)/2}\}$. This means $a_{(p-1)/2}$ is the common apex. There are $2p-5$ chords given by $a_{(p-1)/2}a_i, 1 \le i \le p-1, i \ne (p-1)/2$ and $a_{(p-1)/2}b_i, 0 \le i \le p-2, i \ne (p-3)/2, (p-1)/2$.

We label the vertices as follows:

$\phi(a_0) = 2p-2,\ \ \phi(a_i) = p+i-1,\ 1 \le i \le (p-3)/2,\ \ \phi(a_{(p-1)/2}) = 3p-3,$

$\phi(a_i) = i,\ (p+1)/2 \le i \le p-1,$

$\phi(b_i) = i,\ \ 0 \le i \le (p-3)/2,\ \ \phi(b_{(p-1)/2}) = 3p-4,\ \ \phi(b_i) = p-2+i,\ (p+1)/2 \le i \le p-1.$

Case 4: $MS\{p, p+1\}$ when p is even. Let the vertex set be as before. The two cycles are $\{b_{(p-2)/2}, a_0, b_0, \cdots, a_{(p-2)/2}, b_{(p-2)/2}\}$ and $\{b_{(p-2)/2}, a_{p/2}, b_{p/2}, \cdots, a_{p-1}, b_{(p-1)}, b_{(p-2)/2}\}$.

This means $b_{(p-2)/2}$ is the common apex. There are $2p-5$ chords given by $b_{(p-2)/2}a_i, 1 \le i \le p-1, i \ne (p-2)/2, p/2$ and $b_{(p-2)/2}b_i, 0 \le i \le p-2, i \ne (p-2)/2$.

We label the vertices as follows:

$\phi(a_0) = 2p-2,\ \ \phi(a_i) = p+i-1,\ 1 \le i \le (p-2)/2,\ \ \phi(a_{p/2}) = 3p-4,$

$\phi(a_i) = p-2+i,\ (p+2)/2 \le i \le p-1,$

$\phi(b_i) = i,\ \ 0 \le i \le (p-4)/2,\ \ \phi b_{(p-2)/2}) = 3p-3,\ \ \phi(b_i) = i,\ p/2 \le i \le p-1.$

One can see that ϕ is a harmonious labeling in all the four cases. □

Next we consider balanced triple shells.

Theorem 6: All the balanced triple shells are harmonious.

Proof: Let G be a balanced triple shell on n vertices.

Case I: $n \equiv 0 \bmod 3$. Let $n = 3p, p \ge 3$. In this case we must have one shell of size p and two shells of size $p+1$.

Case I-a: Let p be odd. It is easy to check that $MS\{3, 4^2\}$ is harmonious. Let $p \ge 5$. Let $V(G) = \{a_0, b_0, \cdots, a_{(3p-3)/2}, b_{(3p-3)/2}, a_{(3p-1)/2}\}$.

The three cycles are

$$C_1 = \{a_{(p-1)/2}, a_0, b_0, \cdots, a_{(p-3)/2}, b_{(p-3)/2}, a_{(p-1)/2}\},$$

$$C_2 = \{a_{(p-1)/2}, b_{(p-1)/2}, a_{(p+1)/2}, \cdots, b_{p-1}, a_{(p-1)/2}\},$$

$$C_3 = \{a_{(p-1)/2}, a_p, b_p, \cdots, a_{(3p-1)/2}, a_{(p-1)/2}\}.$$

This means that $a_{(p-1)/2}$ is the common apex. The $3p-7$ chords are

$$\{a_{(p-1)/2}\, a_i \mid 1 \le i \le (3p-3)/2, i \ne p, (p-1)/2, \}$$

$$\{a_{(p-1)/2}\, b_i \mid 0 \le i \le (3p-3)/2, i \ne (p-1), (p-1)/2, (p-3)/2.\}$$

We label the vertices as follows:

$\phi(a_i) = i, \;\; 0 \le i \le (p-3)/2,$

$\phi(a_{(p-1)/2}) = (9p-7)/2, \;\; \phi(a_p) = 5p-5,$

$\phi(a_i) = i-1, \;\; (p+1)/2 \le i \le p-1,$

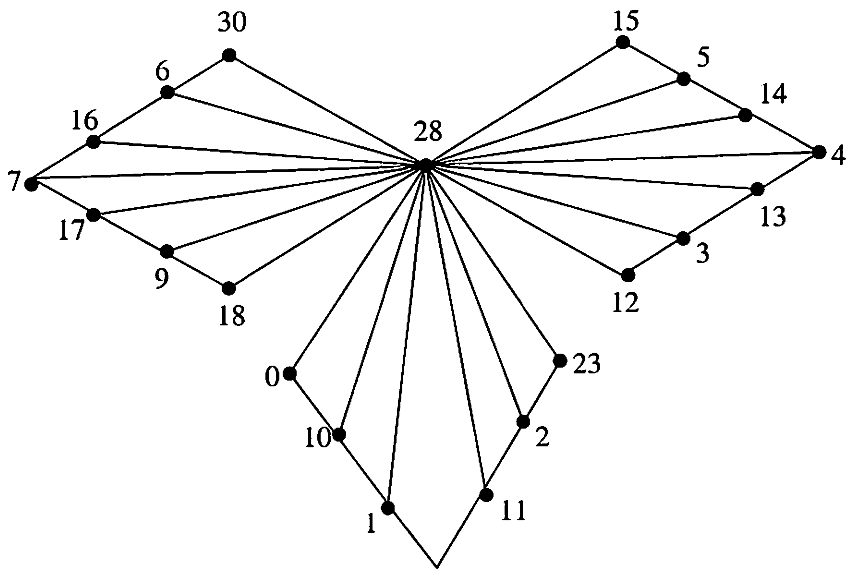
$\phi(a_i) = (3p-5)/2 + i, \;\; p+1 \le i \le (3p-1)/2,$

$\phi(b_i) = (3p-1)/2 + i, \;\; 0 \le i \le (p-5)/2,$

$\phi(b_{(p-3)/2}) = 4p-5, \;\; \phi(b_{(3p-3)/2}) = (3p-3)/2,$

$\phi(b_i) = (3p-3)/2 + i, \;\; (p-1)/2 \le i \le p-1,$

$\phi(b_i) = i-1, \;\; p \le i \le (3p-5)/2,$



$\mathrm{MS}\{7, 8^2\}$

Case I-b: Let p be even. It is easy to check that $MS\{4, 5^2\}$ is harmonious. Let $p \ge 6$. Let $V(G) = \{a_0, b_0, \cdots, a_{(3p-2)/2}, b_{(3p-2)/2}\}$. The three cycles are

$$C_1 = \{b_{(p-2)/2}, a_0, b_0, \cdots, a_{(p-2)/2}, b_{(p-2)/2}\},$$

$$C_2 = \{b_{(p-2)/2}, a_{p/2}, b_{p/2}, \cdots, a_{p-1}, b_{p-1}, b_{(p-2)/2}\},$$

$$C_3 = \{b_{(p-2)/2}, a_p, b_p, \cdots, b_{(3p-2)/2}, b_{(p-2)/2}\}.$$

This means that $b_{(p-2)/2}$ is the common apex. The $3p-7$ chords are

$$\{b_{(p-2)/2}\ a_i \mid 1 \le i \le (3p-2)/2, i \ne p, (p-2)/2, p/2\}$$

$$\{b_{(p-2)/2}\ b_i \mid 0 \le i \le (3p-4)/2, i \ne (p-1), (p-2)/2\}.$$

We label the vertices as follows:

$$\phi(a_i) = (3p)/2 + i, \ \ 0 \le i \le (p-4)/2,$$

$$\phi(a_{(p-2)/2}) = (4p-5), \ \ \phi(a_p) = 5p-5,$$

$$\phi(a_i) = (3p-2)/2 + i, \ \ p/2 \le i \le p-1,$$

$$\phi(a_i) = (3p-4)/2 + i, \ \ (p+1) \le i \le (3p-2)/2,$$

$$\phi(b_i) = i, \ \ 0 \le i \le (p-4)/2,$$

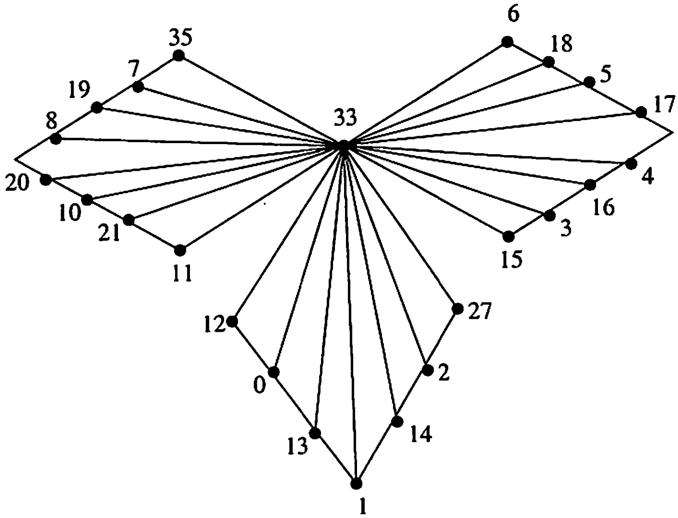
$$\phi(b_{(p-2)/2}) = \frac{9p-6}{2}, \ \ \phi(b_{(3p-4)/2}) = \frac{3p-4}{2}, \ \ \phi(b_{(3p-2)/2}) = \frac{3p-2}{2},$$

$$\phi(b_i) = i-1, \ \ p/2 \le i \le (3p-6)/2,$$

Case II: $n \equiv 1 \bmod 3$. Let $n = 3p+1, p \ge 3$. In this case we must have three shells of size $p+1$.

Case II-a: Let p be odd. It is easy to check that $MS\{4^3\}$ is harmonious. Let $p \ge 5$. Let $V(G) = \{a_0, b_0, \cdots, a_{(3p-1)/2}, b_{(3p-1)/2}\}$. The three cycles are

$$C_1 = \{b_{(p-1)/2}, a_0, b_0, \cdots, a_{(p-1)/2}, b_{(p-1)/2},\},$$



$\mathrm{MS}\{8, 9^{2}\}$

$$C_2 = \{b_{(p-1)/2}, a_{(p+1)/2}, b_{(p+1)/2}, \cdots, a_p, b_{(p-1)/2}\},$$

$$C_3 = \{b_{(p-1)/2}, b_p, a_{p+1}, \cdots, b_{(3p-1)/2}, b_{(p-1)/2}\}.$$

This means that $b_{(p-1)/2}$ is the common apex. The $3p-6$ chords are

$$\{b_{(p-1)/2}\, a_i \mid 1 \le i \le (3p-1)/2, i \ne p, (p-1)/2, (p+1)/2\}$$

$$\{b_{(p-1)/2}\, b_i \mid 0 \le i \le (3p-3)/2, i \ne (p-1)/2, p\}.$$

We label the vertices as follows:

$$\phi(a_i) = \frac{3p-1}{2} + i, \;\; 0 \le i \le (p-3)/2,$$

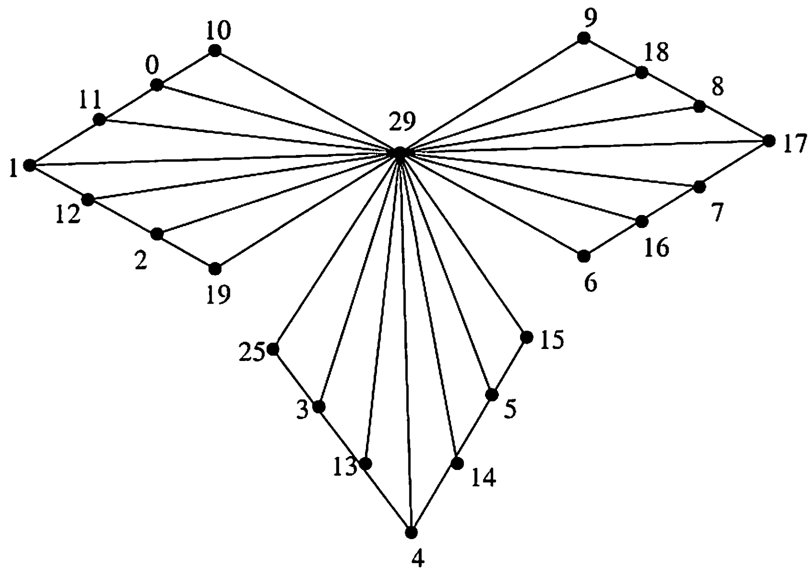
$$\phi(a_{(p-1)/2}) = 3p-2, \;\; \phi(a_{(p+1)/2}) = 4p-3,$$

$$\phi(a_i) = \tfrac{3p-5}{2} + i, \;\; (p+3)/2 \le i \le \tfrac{3p-1}{2},$$

$$\phi(b_i) = i, \;\; 0 \le i \le \frac{p-3}{2},$$

$$\phi(b_{(p-1)/2}) = \frac{9p-5}{2},$$

$\phi(b_i) = i - 1, \;\; (p+1)/2 \leq i \leq (3p-1)/2,$



$\mathrm{MS}\{8^3\}$

Case II-b: Let p be even. It is easy to check that $MS\{3^3\}$ is harmonious. Let $p \geq 4$. Let $V(G) = \{a_0, b_0, \cdots, a_{(3p-2)/2}, b_{(3p-2)/2}, a_{(3p)/2}\}$. The three cycles are

$$C_1 = \{a_{p/2}, a_0, b_0, \cdots, b_{(p-2)/2}, a_{p/2}\},$$

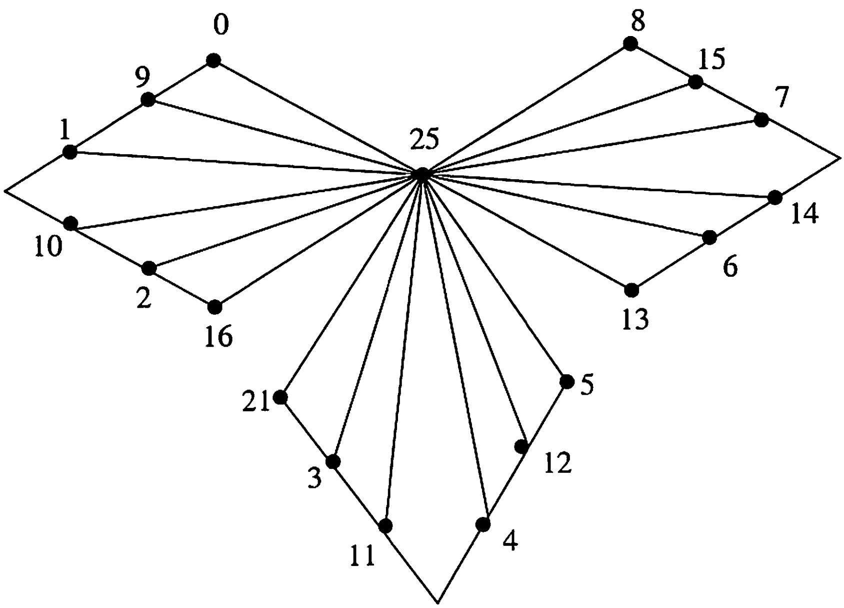
$$C_2 = \{a_{p/2}, b_{p/2}, a_{(p+2)/2}, \cdots, a_p, a_{p/2}\},$$

$$C_3 = \{a_{p/2}, b_p, a_{p+1}, \cdots, a_{3p/2}, a_{p/2}\}.$$

This means that $a_{p/2}$ is the common apex. The $3p-6$ chords are

$$\{a_{p/2}\, a_i \;\mid\; 1 \leq i \leq (3p-2)/2, i \neq p, p/2, \}$$

$\{a_{p/2}\, b_i \mid 0 \le i \le (3p-2)/2, i \ne p/2, (p-2)/2, p\}$.



$$MS\{7^3\}$$

We label the vertices as follows:

$\phi(a_i) = i,\ \ 0 \le i \le (p-2)/2,$

$\phi(a_{p/2}) = (9p-4)/2,$

$\phi(a_i) = i-1,\ \ (p+2)/2 \le i \le 3p/2,$

$\phi(b_i) = 3p/2 + i,\ \ 0 \le i \le (p-4)/2,$

$\phi(b_{(p-2)/2}) = 3p-2,\ \ \phi(b_{p/2}) = 4p-3,$

$\phi(b_i) = (3p-4)/2 + i,\ \ (p+2)/2 \le i \le (3p-2)/2,$

<u>Case III:</u> $n \equiv 2 \bmod 3$. Let $n = 3p+2, p \ge 3$. In this case we must have

two shells of size $p+1$ and one shell of size $p+2$.

Case III-a: Let p be odd. Let $V(G) = \{a_0, b_0, \cdots, a_{(3p-1)/2}, b_{(3p-1)/2}, a_{(3p+1)/2}$

The three cycles are

$$C_1 = \{b_{(p-1)/2}, a_0, b_0, \cdots, a_{(p-1)/2}, b_{(p-1)/2}, \},$$

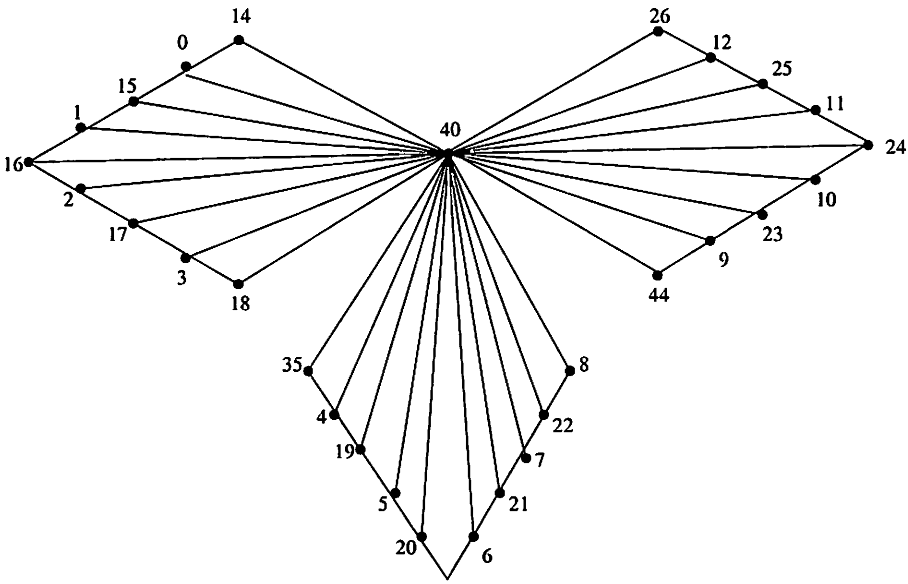
$$C_2 = \{b_{(p-1)/2}, a_{(p+1)/2}, b_{(p+1)/2}, \cdots, b_p, b_{(p-1)/2}\},$$

$$C_3 = \{b_{(p-1)/2}, a_{p+1}, b_{p+1}, \cdots, a_{(3p+1)/2}, b_{(p-1)/2}\}.$$

This means that $b_{(p-1)/2}$ is the common apex. The $3p-5$ chords are

$$\{b_{(p-1)/2}\, a_i \mid 1 \le i \le (3p-1)/2, i \ne p+1, (p-1)/2, (p+1)/2\}$$

$$\{b_{(p-1)/2}\, b_i \mid 0 \le i \le (3p-1)/2, i \ne (p-1)/2, p\}.$$



$MS\{10^2, 11\}$

We label the vertices as follows:

$$\phi(a_i) = \frac{3p+1}{2} + i, \;\; 0 \le i \le (p-1)/2,$$

$$\phi(a_{(p+1)/2}) = 4p - 1, \;\; \phi(a_{p+1}) = 5p - 1,$$

$$\phi(a_i) = \tfrac{3p-1}{2} + i, \;\; (p+3)/2 \le i \le p,$$

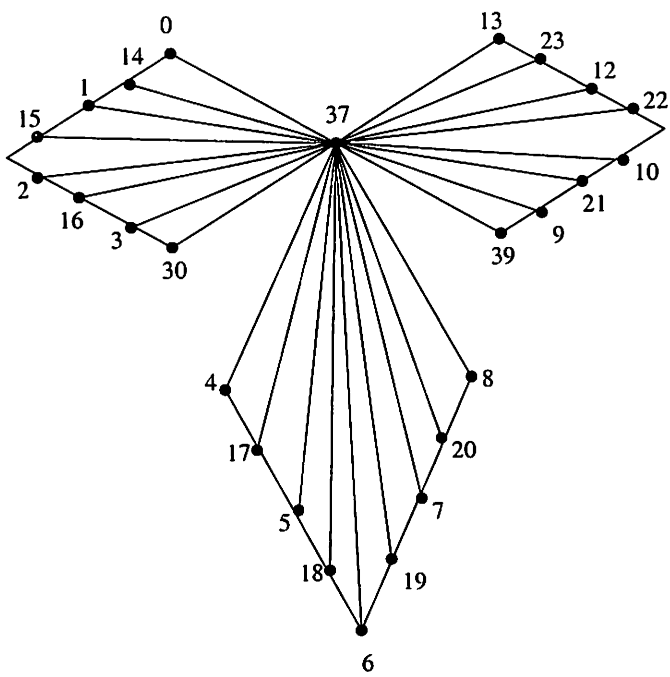
$$\phi(a_i) = \tfrac{3p-3}{2} + i, \;\; p+2 \le i \le (3p+1)/2,$$

$$\phi(b_i) = i, \;\; 0 \le i \le \frac{p-3}{2},$$

$$\phi(b_{(p-1)/2}) = \frac{9p-1}{2},$$

$$\phi(b_i) = i - 1, \;\; (p+1)/2 \le i \le (3p-1)/2,$$

Case III-b: Let p be even. It is easy to check that $MS\{3^2, 4\}$ and $MS\{5^2, 6\}$ are harmonious.



$MS\{9^2, 10\}$

Let $p \geq 6$. Let $V(G) = \{a_0, b_0, \cdots, a_{(3p-2)/2}, b_{(3p-2)/2}, a_{(3p)/2}, b_{(3p)/2}\}$. The three cycles are

$$C_1 = \{a_{p/2}, a_0, b_0, \cdots, b_{(p-2)/2}, a_{p/2}\},$$

$$C_2 = \{a_{p/2}, b_{p/2}, a_{(p+2)/2}, \cdots, b_p, a_{p/2}\},$$

$$C_3 = \{a_{p/2}, a_{p+1}, b_{p+1}, \cdots, b_{3p/2}, a_{p/2}\}.$$

This means that $a_{p/2}$ is the common apex. The $3p-5$ chords are

$$\{a_{p/2}\, a_i \mid 1 \leq i \leq (3p)/2, i \neq p+1, p/2,\}$$

$$\{a_{p/2}\, b_i \mid 0 \leq i \leq (3p-2)/2, i \neq p/2, (p-2)/2, p.\}$$

We label the vertices as follows:

$$\phi(a_i) = i, \;\; 0 \leq i \leq (p-2)/2,$$

$$\phi(a_{p/2}) = (9p+2)/2, \;\; \phi(a_{p+1}) = 5p-1,$$

$$\phi(a_i) = (3p)/2 + i, \;\; (p+2)/2 \leq i \leq p,$$

$$\phi(a_i) = (3p-2)/2 + i, \;\; (p+2) \leq i \leq (3p)/2,$$

$$\phi(b_i) = (3p+4)/2 + i, \;\; 0 \leq i \leq (p-4)/2,$$

$$\phi(b_{(p-2)/2}) = 4p-2, \;\; \phi(b_{(3p-2)/2}) = (3p)/2, \;\; \phi(b_{3p/2}) = (3p)/2 + 1,$$

$$\phi(b_i) = i, \;\; p/2 \leq i \leq (3p-4)/2,$$

One can easily see that the map ϕ is a harmonious labeling in all the cases. □

Remark: We have found harmonious labelings of some more multiple shells, some of which are not balanced. Hence we conjecture **All multiple shells are harmonious.**

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