Bishop Covers of Rectangular Boards

David C. Fisher and Paul Thalos

University of Colorado at Denver

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Abstract

A set of Bishops *cover* a board if they attack all unoccupied squares. *What is the minimum number of Bishops needed to cover a $k \times n$ board?* Yaglom and Yaglom showed that if $k = n$, the answer is n. We extend this result by showing that the minimum is $2\lfloor \frac{n}{2} \rfloor$ if $k < n \leq 2k$. For $n > 2k > 2$, a cover is given with $2\lfloor \frac{k+n}{3} \rfloor$ Bishops. We conjecture that this is the minimum value. This conjecture is verified when $k \leq 3$ or $n \leq 2k + 5$.

1 Introduction

The *Bishop* in "Chess" attacks diagonally on an 8×8 board (see fig. 1).

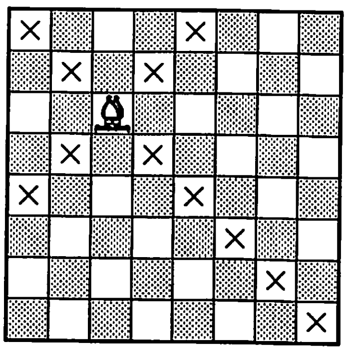


Figure 1 – Bishop Moves. A Bishop attacks squares marked by $\times$.

A Bishop *covers* a square if it attacks it or is on it. Then a set of Bishops *cover* a board if they cover all of its squares. Yaglom and Yaglom [2] asked: *What is the minimum number of Bishops which can cover an $n \times n$ board?* They showed the answer is n (see fig. 2). We try to answer this question for rectangular $k \times n$ boards. Without loss of generality, assume throughout that $k \leq n$.

The Bishop cover problem is a special case of graph domination. A *domination* of a graph G is a subset of its nodes where every node is either

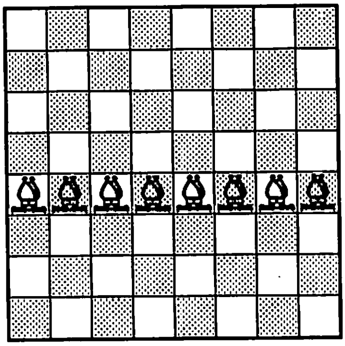


Figure 2 – A Minimum Cover of an 8×8 Board. Yaglom and Yaglom showed that at least n Bishops are needed to cover an $n \times n$ board. This is one of $108^2 = 11664$ ways to cover a Chessboard with 8 Bishops.

in or adjacent to the subset. The *domination number* (notated $\gamma(G)$) is the minimum size of a domination of G. Let the $k \times n$ Bishop graph, $B_{k,n}$, have nodes $\{1,2,\ldots,k\} \times \{1,2,\ldots,n\}$ with an edge between nodes (g,h) and (i,j) if $|i-g| = |j-h| \neq 0$ (see fig. 3). The minimum number of Bishops needed to cover a $k \times n$ board is then the domination number of $B_{k,n}$ and is denoted $\gamma(B_{k,n})$.

Section 2 finds $\gamma(B_{k,n})$ when $k \leq n \leq 2k$, and gives a conjecture when $n > 2k$. Section 3 verifies this conjecture when $k \leq 3$ and when $n \leq 2k+5$.

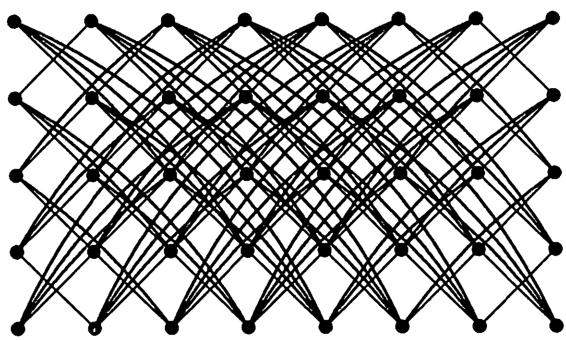
$B_{5,8} =$ 

Figure 3 – The Bishop Graph of a 5×8 board. Nodes represent squares. There is an edge between nodes if their corresponding squares are in the same diagonal or antidiagonal.

2 The Main Results

First we define several subsets of the nodes of $B_{k,n}$ (see fig. 4). As a graph, $B_{k,n}$ has two components which we will call *white* and *black* (which is which will be defined as convenient):

$$\{(i,j) \mid i+j \text{ is even}\} \text{ and } \{(i,j) \mid i+j \text{ is odd}\}.$$

Let the $p^{\underline{\text{th}}}$ diagonal and the $q^{\underline{\text{th}}}$ antidiagonal, respectively, be

$$D_p = \{(i,j) \mid p = k+j-i\} \text{ and } A_q = \{(i,j) \mid q = j+i-1\}.$$

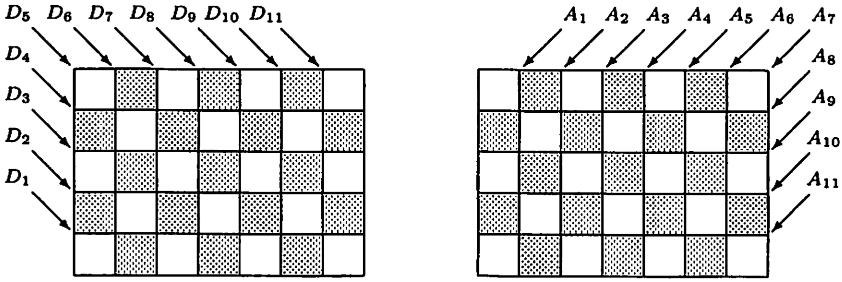


Figure 4 – Diagonals and Antidiagonals. This shows the 11 diagonals and 11 antidiagonals of a 5 × 7 board.

Below is a new proof of a result from Yaglom and Yaglom [**2**]. It is shorter than the proof in [**2**] and an independent proof given in Cockayne, Gamble and Shepard [**1**]. Theorem 2 presents a formula for $\gamma(B_{k,n})$ on boards that are no more than twice as wide as high. Let a diagonal or antidiagonal be *empty* if it does not have a Bishop in it. Key to proving these results is the observation that a Bishop can attack at most one square of an empty diagonal or antidiagonal.

Theorem 1. (Yaglom and Yaglom) *For all $n \geq 1$, we have $\gamma(B_{n,n}) = n$.*

Proof. We can cover $B_{n,n}$ with n Bishops in row $\lceil \frac{n}{2} \rceil$ (see fig. 2).

Suppose a cover of $B_{n,n}$ has fewer than n Bishops. At least n diagonals of $B_{n,n}$ have at least $\lceil \frac{n}{2} \rceil$ squares. So at least one of these "long" diagonals (say it is white) is empty. Therefore there are at least $\lceil \frac{n}{2} \rceil$ white Bishops used to cover this "long" diagonal via antidiagonals leaving fewer than $n - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor$ black Bishops. Since there are at least $\lfloor \frac{n}{2} \rfloor$ "long" black diagonals, at least one is empty. Then at least $\lceil \frac{n}{2} \rceil$ black Bishops are needed to cover this black diagonal via antidiagonals, a contradiction. □

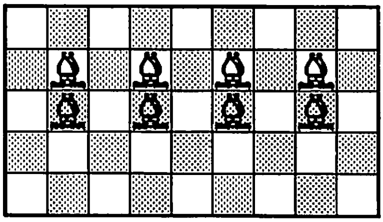


Figure 5 – A Minimum Cover of a 5×9 board with 8 Bishops. Theorem 2 shows that placing Bishops in even columns of two middle-most rows is a minimum cover of a $k \times n$ boards when $k < n \le 2k$.

Theorem 2. *For all $k < n \le 2k$, we have $\gamma(B_{k,n}) = 2\lfloor \frac{n}{2} \rfloor$. Further any cover of $B_{k,n}$ contains at least $\lfloor \frac{n}{2} \rfloor$ Bishops of each color.*

Proof. We can cover $B_{k,n}$ with $2\lfloor \frac{n}{2} \rfloor$ Bishops in even columns of rows $\lfloor \frac{k}{2} \rfloor$ and $\lfloor \frac{k}{2} \rfloor + 1$ (see fig. 5).

Suppose a cover of $B_{k,n}$ has fewer than $\lfloor \frac{n}{2} \rfloor$ Bishops of a color, say white. Since $\lfloor \frac{n}{2} \rfloor \le k$, there are not enough white Bishops to cover an empty diagonal of length k. Thus every white diagonal of length k contains a Bishop.

Let $p < k$ be the largest integer where D_p is an empty white diagonal. Then p exists for otherwise all white diagonals D_j with $j \le n$ would have a Bishop, impossible with fewer than $\lfloor \frac{n}{2} \rfloor$ white Bishops. Let $q > n$ be the smallest integer where D_q is an empty white diagonal (q exists for similar reasons). Since $\frac{q-p}{2} - 1$ white diagonals are strictly between D_p and D_q, we have $\frac{q-p}{2} - 1 \le \lfloor \frac{n}{2} \rfloor - 1$ and hence $q - p \le n$ (see fig. 6).

Diagonal D_p is between squares $(k-p+1, 1)$ and (k, p). Diagonal D_q is between squares $(1, q-k+1)$ and $(k+n-q, n)$. So one end of D_p is in antidiagonal A_{k-p+1} and the other is in antidiagonal A_{k+p-1}, and one end of D_q is in antidiagonal A_{q-k+1} and the other is in antidiagonal $A_{k+2n-q-1}$. Since $q-p \le n \le 2k$ and hence $(q-k+1)-(k+p-1) \le 2$, the antidiagonals intersecting D_p are contiguous with the antidiagonals intersecting D_q (there can be overlap, but no "gap"). Since there are $1 + \frac{(k+2n-q-1)-(k-p+1)}{2} = n - \frac{q-p}{2} \ge \frac{n}{2}$ white antidiagonals between A_{k-p+1} and $A_{k+2n-q-1}$ inclusively, covering the squares of both D_p and D_q via antidiagonals requires at least $\frac{n}{2}$ white Bishops, a contradiction. □

What happens for wider boards? For $k \times n$ boards with $n \ge 2k+2$ (and $k > 1$), the cover given in the proof of Theorem 3 has fewer Bishops than the cover in the proof of Theorem 2.

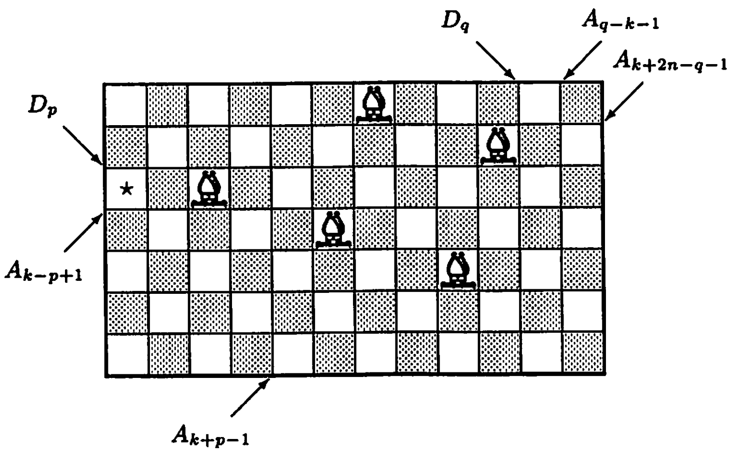


Figure 6 – Trying to Cover the White Squares of a 7 × 12 Board with 5 Bishops. Here $k = 7$ and $n = 12$. Diagonal $D_p = D_5$ is the rightmost empty diagonal in the lower left. It goes from $A_{k-p+1} = A_3$ to $A_{k+p-1} = A_{11}$. Diagonal $D_q = D_{17}$ is the leftmost empty diagonal in the upper right. It goes from $A_{q-k-1} = A_{11}$ to $A_{k+2n-q-1} = A_{13}$. Attacking the squares of both D_p and D_q via antidiagonals requires at least $1 + \frac{(k+2n-q-1)-(k-p+1)}{2} = 6$ Bishops. With only 5 Bishops, some square of D_p or D_q (marked by $\star$) must remain uncovered.

Theorem 3. *For $n > 2k \geq 4$, we have $\gamma(B_{k,n}) \leq 2\left\lfloor \frac{k+n}{3} \right\rfloor$.*

Proof. We will describe a cover of $B_{k,n}$ with $\left\lfloor \frac{k+n}{3} \right\rfloor$ Bishops per color (see fig. 7). Let square $(1,1)$ be white. For white, put a Bishop in square $(2,2)$. Then put $r \equiv \left\lceil \frac{n-2k-1}{3k-3} \right\rceil$ sets of $k-1$ Bishops every $3k-3$ columns alternating between row 1 and row k:

$$\begin{array}{ccccc}
(5,1) & (7,1) & (9,1) & \dots & (2k+1,1) \\
(3k-3+5,k) & (3k-3+7,k) & (3k-3+9,k) & \dots & (3k-3+2k+1,k) \\
(6k-6+5,1) & (6k-6+7,1) & (6k-6+9,1) & \dots & (6k-6+2k+1,1) \\
(9k-9+5,k) & (9k-9+7,k) & (9k-9+9,k) & \dots & (9k-9+2k+1,k) \\
\vdots & \vdots & \vdots & & \vdots
\end{array}$$

Let $s \equiv n-(r-1)(3k-3)-2k-2 = n-3kr+3r+k-5$. Put $\left\lceil \frac{s}{3} \right\rceil$ Bishops in the white squares of either row $\left\lceil \frac{s}{3} \right\rceil + 1$ (if r is even) or row $k - \left\lceil \frac{s}{3} \right\rceil$ (if r is odd) starting with column $r(3k-3)-2k+6+\left\lceil \frac{s}{3} \right\rceil$. If a Bishop is to be in column $n+1$ (happens if $s \equiv 1 \pmod 3$), move it up (if

r is even) or down (if r is odd) one row and left into column n. Then the number of Bishops is

$$1 + r(k-1) + \left\lceil \frac{s}{3} \right\rceil = 1 + r(k-1) + \left\lceil \frac{n - 3kr + 3r + k - 5}{3} \right\rceil = \left\lfloor \frac{k+n}{3} \right\rfloor .$$

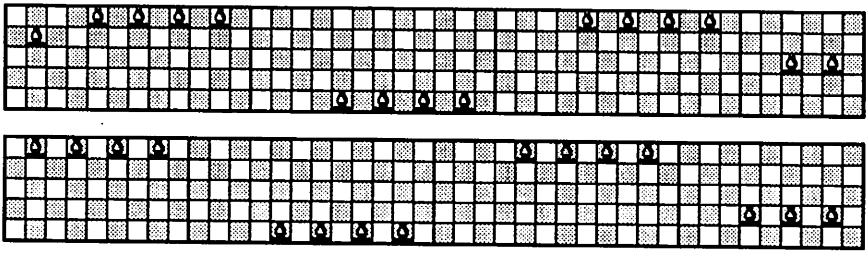


Figure 7 – A Cover of a 5×42 board. This shows the placement of white Bishops (top) and black Bishops (bottom) prescribed by the proof of Theorem 3.

For black, put $r \equiv \left\lfloor \frac{n-2k+2}{3k-3} \right\rfloor$ sets of $k-1$ Bishops every $3k-3$ columns alternating between row 1 and row k:

$$\begin{array}{ccccc} (2,1) & (4,1) & (6,1) & \dots & (2k-2,1) \\ (3k-3+2,k) & (3k-3+4,k) & (3k-3+6,k) & \dots & (3k-3+2k-2,k) \\ (6k-6+2,1) & (6k-6+4,1) & (6k-6+6,1) & \dots & (6k-6+2k-2,1) \\ (9k-9+2,k) & (9k-9+4,k) & (9k-9+6,k) & \dots & (9k-9+2k-2,k) \\ \vdots & \vdots & \vdots & & \vdots \end{array}$$

Let $s \equiv n - (r-1)(3k-3) - 2k + 1 = n - 3kr + 3r + k - 2$. Place $\lceil \frac{s}{3} \rceil$ Bishops in the black squares of either row $\lceil \frac{s}{3} \rceil + 1$ (if r is even) or row $k - \lceil \frac{s}{3} \rceil$ (if r is odd) starting in column $r(3k-3) - k + 2 + \lceil \frac{s}{3} \rceil$. If a Bishop is in column $n+1$ (happens if $s \equiv 1 \pmod 3$), move it up (if r is even) or down (if r is odd) one row and left into column n. Then the number of black Bishops is

$$r(k-1) + \left\lceil \frac{s}{3} \right\rceil = r(k-1) + \left\lceil \frac{n - 3kr + 3r + k - 2}{3} \right\rceil = \left\lfloor \frac{k+n}{3} \right\rfloor . \square$$

We believe that the cover given in the proof of Theorem 3 is minimum.

Conjecture. *For all $n \geq 2k > 2$, we have $\gamma(B_{k,n}) = 2 \left\lfloor \frac{k+n}{3} \right\rfloor$.*

3 Partial Verifications of the Conjecture

We now verify the conjecture in several cases. Theorems 4 and 5 verify the conjecture when $n \leq 2k+5$. The proof of Theorem 4 and the first part of the proof of Theorem 5 are similar to the proof of Theorem 2. Theorems 6 and 7 verify the conjecture for $2 \times n$ and $3 \times n$ boards, respectively.

Theorem 4. *For all $k > 1$, we have $\gamma(B_{k,2k+1}) = \gamma(B_{k,2k+2}) = 2k$. Further any cover of $B_{k,2k+1}$ or $B_{k,2k+2}$ contains at least k Bishops of each color.*

Proof. Let $n = 2k+1$ or $2k+2$, and let $\ell = n - 2k$. Theorem 3 shows

$$\gamma(B_{k,2k+\ell}) \leq 2\left\lfloor \frac{k+n}{3} \right\rfloor = 2\left\lfloor \frac{k+2k+\ell}{3} \right\rfloor = 2\left\lfloor k + \frac{\ell}{3} \right\rfloor \leq 2\left\lfloor k + \frac{2}{3} \right\rfloor = 2k.$$

Suppose a cover of $B_{k,2k+\ell}$ with $\ell = 1,2$ has fewer than k Bishops of a color, say white. Let $p < k$ be the largest integer where D_p is an empty white diagonal, and let $q > n$ be the smallest integer where D_q is also an empty white diagonal. Since $\frac{q-p}{2} - 1$ white diagonals are strictly between D_p and D_q, we have $\frac{q-p}{2} - 1 \leq k - 1$ and hence $q - p \leq 2k$. As in the proof of Theorem 2, the number of white Bishops needed to cover the squares of diagonals D_p and D_q via antidiagonals is $n - \frac{1}{2}(q-p) \geq n - k = k + \ell \geq k+1$, a contradiction. □

Theorem 5. *For all $k > 1$, we have $\gamma(B_{k,2k+3}) = \gamma(B_{k,2k+4}) = \gamma(B_{k,2k+5}) = 2k+2$. Further any cover of $B_{k,2k+3}$, $B_{k,2k+4}$, or $B_{k,2k+5}$ contains at least $k+1$ Bishops of each color.*

Proof. Let $n = 2k+3$, $2k+4$ or $2k+5$, and let $\ell = n - 2k$. Theorem 3 shows

$$\gamma(B_{k,2k+\ell}) \leq 2\left\lfloor \frac{k+n}{3} \right\rfloor = 2\left\lfloor k + \frac{\ell}{3} \right\rfloor \leq 2\left\lfloor k + \frac{5}{3} \right\rfloor = 2k+2.$$

Suppose a cover of $B_{k,2k+\ell}$ for $\ell = 3,4,5$ has fewer than $k+1$ Bishops for one color, say white.

First assume no diagonals of length k are empty. Let $p < k$ be the largest integer where D_p is an empty white diagonal, and let $q > n$ be the smallest integer where D_q is also a white empty diagonal. Since there are $\frac{q-p}{2} - 1$ white diagonals strictly between D_p and D_q, we have $\frac{q-p}{2} - 1 \leq k$ and hence $q - p \leq 2k+2$. As in Theorem 2, the number of Bishops needed to cover the squares of both diagonals D_p and D_q via antidiagonals is $n - \frac{1}{2}(q-p) \geq n - (k+1) = k + \ell - 1 \geq k+2$ which is more than the number of white Bishops, a contradiction.

Otherwise, let D_r be an empty white diagonal of length k. It takes k white Bishops to attack the squares of D_r via antidiagonals. So there must be a Bishop is in every white antidiagonal between A_{r-k+1} and A_{r+k-1}, inclusively, and all other antidiagonals are empty (see fig. 8). Since square $(k, 1)$, or squares $(k-1, 1)$ and $(k, 2)$ must be covered, a Bishop is in A_{k-1}, A_k, or A_{k+1}. So $r - k + 1 \leq k + 1$ and hence $r \leq 2k$. Since square $(1, n)$, or squares $(1, n-1)$ and $(2, n)$ are also covered, there must be a Bishop is in one of antidiagonals A_{n-1}, A_n, or A_{n+1}. So $r + k - 1 \geq n - 1$ and hence $r \geq n - k$. Since $n - k \leq r \leq 2k$, antidiagonal A_{r-k-1} exists (because $r - k - 1 \geq (n - k) - k - 1 = \ell - 1 \geq 2$) and has length $r - k - 1$ (since $r - k - 1 \leq 2k - k - 1 = k - 1$). Further, antidiagonal A_{r+k+1} exists (because $r + k + 1 \leq 2k + k + 1 = n - \ell + k + 1 \leq n + k - 2$) has length $k + n - (r + k + 1) = n - r - 1$ (since $r + k + 1 \geq n - k + k + 1 = n + 1$). So A_{r-k-1} and A_{r+k+1} are empty and the set of diagonals intersecting them are disjoint (D_r separate the sets). Thus the number of white Bishops needed to attack the squares of both A_{r-k-1} and A_{r+k+1} via diagonals is the sum of their lengths: $r - k - 1 + n - r - 1 = n - k - 2 = k + \ell - 2 \geq k + 1$, a contradiction.

Either way, $k + 1$ Bishops are needed of each color. □

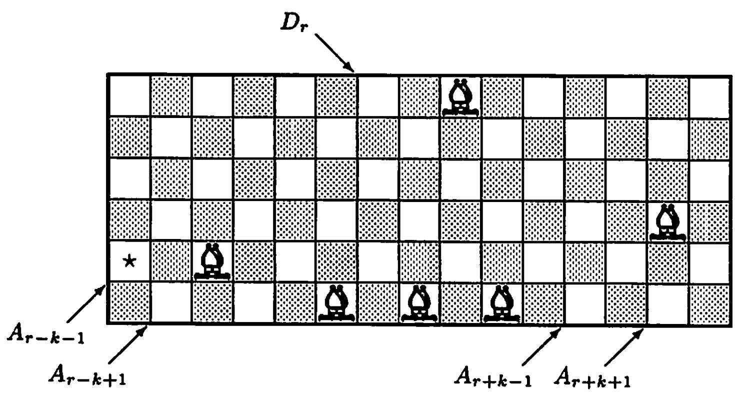


Figure 8 – Trying to Cover the White Squares of a 6 × 15 Board with 6 Bishops. Here $k = 6$ and $n = 15$. Diagonal $D_r = D_{12}$ is an empty diagonal of length $k = 6$. It goes from $A_{r-k+1} = A_7$ to $A_{r+k-1} = A_{17}$. Attacking the squares of D_r via antidiagonals requires all $k = 6$ Bishops. So antidiagonals $A_{r-k-1} = A_5$ and $A_{r+k+1} = A_{19}$ are empty. Since A_{r-k-1} has $r - k - 1 = 5$ squares, and A_{r+k+1} has $n - r - 1 = 2$ squares, at least $5 + 2 = 7$ Bishops are needed to cover these two antidiagonals via diagonals. With only 6 Bishops, some square of A_{r-k-1} and A_{r+k+1} (marked by ⋆) must not be covered.

Theorem 6. *For all n, we have $\gamma(B_{2,n}) = 2\left\lceil \frac{n}{3} \right\rceil$.*

Proof. Let P_n a path of length n. Then $B_{2,n} = P_n \cup P_n$ is disjoint union of two paths of length n. It is well known that $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$. So $\gamma(B_{2,n}) = \gamma(P_n \cup P_n) = 2\gamma(P_n) = 2\left\lceil \frac{n}{3} \right\rceil$. □

Theorem 7. *For all $n \geq 4$, we have $\gamma(B_{3,n}) = 2\left\lfloor \frac{n+3}{3} \right\rfloor$. Further, any cover of $B_{3,n}$ contains at least $\left\lfloor \frac{n+3}{3} \right\rfloor$ Bishops of each color.*

Proof. Theorem 2 shows $\gamma(B_{3,4}) = 2\left\lfloor \frac{4}{2} \right\rfloor = 4$, $\gamma(B_{3,5}) = 2\left\lfloor \frac{5}{2} \right\rfloor = 4$, and $\gamma(B_{3,6}) = 2\left\lfloor \frac{6}{2} \right\rfloor = 6$. Theorem 4 gives $\gamma(B_{3,7}) = \gamma(B_{3,8}) = 2 \cdot 3 = 6$. Theorem 5 shows $\gamma(B_{3,9}) = 2(3+1) = 8$. Further, these minimum cover equally divide the Bishops between the two colors. For $n = 4, 5, 6, 7, 8,$ and 9, we have $2\left\lfloor \frac{n+3}{3} \right\rfloor = 4, 4, 6, 6, 6,$ and 8, respectively. So the result holds for $4 \leq n \leq 9$.

For $n > 9$, Theorem 3 shows $\gamma(B_{3,n}) \leq 2\left\lfloor \frac{n+3}{3} \right\rfloor$. Assume (for induction) that for all m with $4 \leq m < n$, a cover of $B_{3,m}$ contains at least $\left\lfloor \frac{m+3}{3} \right\rfloor$ Bishops of each color. Consider a cover of one color, say white, of $B_{3,n}$.

First assume square $(1, n)$ is white. To cover square $(1, n)$, a Bishop must be on either square $(1, n)$, $(2, n-1)$ or $(3, n-2)$. If a Bishop on square $(1, n)$, it can be moved to square $(2, n-1)$ and we still have a cover.

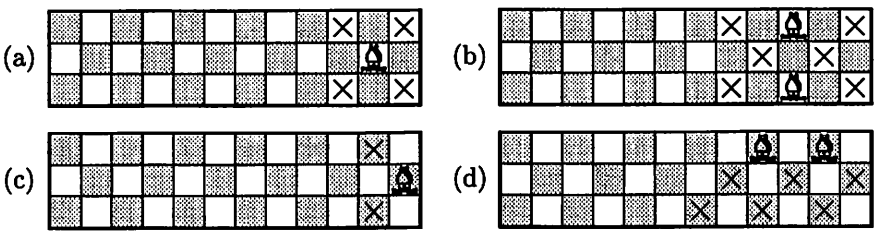


Figure 9 – Covering the White Squares of a $3 \times n$ Board. If square $(1, n)$ is white, then without loss of generality, either a Bishop is at square $(2, n-1)$ or Bishops are at squares $(1, n-2)$ and $(3, n-2)$. For (a), we must cover the white squares of a $3 \times (n-3)$ board. For (b), we must cover the white squares of a $3 \times (n-5)$ board. If square $(1, n)$ is black, either a Bishop is at square $(2, n)$ or, without loss of generality, Bishops are at squares $(1, n-1)$ and $(3, n-1)$. For (c), we must cover the white squares of a $3 \times (n-2)$ board. For (d), we must cover the white squares of a $3 \times (n-6)$ board.

If a Bishop is on square $(2, n-1)$ (see fig. 9(a)), it only covers squares in the right 3 columns. So the left $n-3$ columns have at least $\left\lfloor \frac{n}{3} \right\rfloor$ Bishops (any other white Bishops in the right 3 columns can be moved to square

$(2, n-4)$ and we still have a cover of $B_{3,n-3}$). Thus this cover contains at least $1 + \lfloor \frac{n}{3} \rfloor \geq \lfloor \frac{n+3}{3} \rfloor$ white Bishops.

Otherwise a Bishop is on square $(3, n-2)$ (see fig. 9(b)). Then another Bishop must cover square $(3, n)$. Among squares where this Bishop could be, one at square $(1, n-2)$ covers a superset of the uncovered squares. Then these two Bishops cover only squares in the right 5 columns. So the left $n-5$ columns have at least $\lfloor \frac{n-2}{3} \rfloor$ Bishops (other white Bishops in the right 5 columns can be moved to $(2, n-5)$ and we still have a cover of $B_{3,n}$). Thus the cover contains at least $2 + \lfloor \frac{n-2}{3} \rfloor \geq \lfloor \frac{n+3}{3} \rfloor$ white Bishops.

Now assume square $(1, n)$ is black. To cover square $(2, n)$, a Bishop must be on either square $(2, n)$, $(1, n-1)$, or $(3, n-1)$.

If a Bishop is on square $(2, n)$ (see fig. 9(c)), it only covers squares in the right 2 columns. So the left $n-2$ columns have at least $\lfloor \frac{n+1}{3} \rfloor$ white Bishops (a Bishop on square $(1, n-1)$ or $(3, n-1)$ can be moved to square $(2, n-2)$ and we still have a cover of $B_{3,n}$). Thus this cover contains at least $1 + \lfloor \frac{n-2}{3} \rfloor + 1 \geq \lfloor \frac{n+3}{3} \rfloor$ white Bishops.

Otherwise without loss of generality, a Bishop is on square $(1, n-1)$ (see fig. 9(d)). Then another Bishop must cover square $(3, n-1)$. Among the squares where this Bishop could be, square $(1, n-3)$ covers a superset of the uncovered squares. These two Bishops only cover squares in the right 6 columns. So the left $n-6$ columns have at least $\lfloor \frac{n-3}{3} \rfloor$ white Bishops (other white Bishops in the right 6 columns can be moved to $(2, n-6)$ and we still have a cover of $B_{3,n}$). Thus the cover contains at least $2 + \lfloor \frac{n-3}{3} \rfloor = \lfloor \frac{n+3}{3} \rfloor$ white Bishops. □

We also tried to numerically find counterexamples to the conjecture. In particular, we used Lindo (a commercial optimization system) to solve integer programming formulations for the minimum number in a Bishop cover of a $k \times n$ board for all $k \leq 9$ and $n \leq 110$. The answers were consistent with the conjecture. While not showing the conjecture is true even for these values, these calculations increase our confidence in the correctness of the conjecture.

Reference

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