Minimal 4-Equitability of $C_{2n}OK_1$

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ABSTRACT

Every labeling of the vertices of a graph with distinct natural numbers induces a natural labeling of its edges: the label of an edge uv is the absolute value of the difference of the labels of u and v. A labeling of the vertices of a graph of order p is *minimally k-equitable* if the vertices are labeled with $1, 2, \cdots, p$ and in the induced labeling of its edges every label either occurs exactly k times or does not occur at all. We prove that the corona graphs $C_{2n}OK_1$ are minimally 4-equitable.

1. INTRODUCTION

A labeling of the vertices of a graph G is an assignment of distinct natural numbers to the vertices of G. Every labeling induces a natural labeling of the edges: the label of an edge uv is the absolute value of the difference of the labels of u and v. Bloom [3] defined a labeling of the vertices of a graph to be k-equitable if in the induced labeling of its edges, every label occurs exactly k-times, if at all. Furthermore, a k-equitable labeling of a graph of order p is said to be *minimal* if the vertices are labeled with $1, 2, \cdots, p$. A graph is *minimally k-equitable* if it has a minimal k-equitable labeling.

Bloom[3] posed the following question: is the condition that k is a proper divisor of p sufficient for the cycle C_p to have a minimal k-equitable labeling? Wojciechowski [5] gave a positive answer to this ques-

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tion. Barrientos, Dejter and Hevia [2] have shown that forests of even size are 2-equitable. They also prove that for $k = 3$ or $k = 4$ a forest F of size kw is k-equitable if and only if the maximum degree of F is at most $2w$ and that if 3 divides the size of the double star $S_{m,n}(1 \leq m \leq n)$, then $S_{m,n}$ is 3-equitable if and only if $q/3 \leq m \leq [(q-1)/2]$. Here $S_{m,n}$ is k_2 with m pendant edges attached at one end and n pendant edges attached at the other end. They discuss the k-equitability of forests for $k \geq 5$ and characterize all caterpillars of diameter 2 that are k-equitable for all possible values of k.

The *corona* $G_1 O G_2$ of two graphs G_1 and G_2 was defined by Frucht and Harary [4] as the graph G obtained by taking one copy of G_1 which has p_1 vertices and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. The authors [1] have recently established minimal 3-equitability of corona graphs $C_{3n} O K_1$.

Here we prove that the corona $C_{2n} O K_1$ is minimally 4-equitable.

2. MAIN RESULT.

The main result is as follows.

Theorem: The graphs $C_{2n} O K_1$ are minimally 4-equitable, $n \geq 2$.

Proof: Here $p(C_{2n} O K_1) = 4n = q(C_{2n} O K_1)$. We show that this graph is minimally 4-equitable with edge-weight set $W = \{1, 2, \cdots, n\}$.

Let $V(C_{2n} O K_1) = \{u_1, u_2, \cdots, u_{2n}; v_1, v_2, \cdots, v_{2n}\}$ where $u_1, u_2, \cdots u_{2n}$ is the cycle C_{2n} and v_i is the pendant vertex adjacent to u_i, $1 \leq i \leq 2n$.

We consider two cases corresponding to $n \equiv 0, 1 (mod 2)$ and in each case we describe a labeling function when $n \geq 5$. We deal with the cases $n = 2, 3, 4$ seperately at the end. In both the cases with $n \geq 5$ the labeling function f is given in four parts, viz., Part I, Part II, Part III and Part IV.

The Case $n \equiv 0(mod 2)$: In this case Part I describes the labeling function for the vertices $u_i, v_i, 1 \leq i \leq \frac{n}{2}$, Part II for u_i, v_i, $\frac{n}{2} + 1 \leq i \leq n$, Part III for $u_i, v_i, n+1 \leq i \leq \frac{3n}{2}$ and Part IV for u_i, v_i, $\frac{3n}{2} + 1 \leq i \leq 2n$.

In this case Part I and Part III are smooth so we describe them first and then Part II and Part IV.

Part I: Here we define

$$f(u_{2i-1}) = n + 3 - 2i \ , \quad 1 \leq i \leq \frac{n}{4} \quad \text{when } n \equiv 0 (mod 4)$$
$$and$$
$$1 \leq i \leq [\frac{n}{4}] + 1 \quad \text{when } n \equiv 2 (mod 4).$$

$$f(v_{2i-1}) = 2i - 1 \ , \quad 1 \leq i \leq \frac{n}{4} \quad \text{when } n \equiv 0 (mod 4)$$
$$and$$
$$1 \leq i \leq [\frac{n}{4}] + 1 \quad \text{when } n \equiv 2 (mod 4).$$

$$f(u_{2i}) = 2i \ , \quad 1 \leq i \leq [\frac{n}{4}].$$

$$f(v_{2i}) = n + 2 - 2i \ , 1 \leq i \leq [\frac{n}{4}].$$

The edge-weight sequene in Part I is smooth in the sense that the edge-weights of the edges $v_1u_1, u_1u_2, u_2v_2, u_2u_3, \cdots, u_{\frac{n}{2}}v_{\frac{n}{2}}$ is $n, n-1, n-2, n-3, \cdots, 2$.

We record here the labels used and the edge-weights covered in Part I.

Labels Used: 1 to $n+1$ except $\frac{n}{2}+1$.

Edge-Weights Covered: 2 to n.

Part III: We define

$$f(u_{n+2i-1}) = 3n - 2 + 2i \ , \quad 1 \leq i \leq \frac{n}{4} \quad \text{when } n \equiv 0 (mod 4)$$
$$and$$
$$1 \leq i \leq [\frac{n}{4}] + 1 \quad \text{when } n \equiv 2 (mod 4).$$

$$f(v_{n+2i-1}) = 4n + 2 - 2i \ , \quad 1 \leq i \leq \frac{n}{4} \quad \text{when } n \equiv 0 (mod 4)$$
$$and$$
$$1 \leq i \leq [\frac{n}{4}] + 1 \quad \text{when } n \equiv 2 (mod 4).$$

$$f(u_{n+2i}) = 4n + 1 - 2i \ , \quad 1 \leq i \leq [\frac{n}{4}].$$

$$f(v_{n+2i}) = 3n - 1 + 2i \ , 1 \leq i \leq [\frac{n}{4}].$$

The labels used and the edge-weights covered in Part III are:

Labels Used: $3n$ to $\frac{7n}{2} - 1$; $\frac{7n}{2} + 1$ to $4n$.

Edge-Weights Covered: 2 to n.

Next, we deal with Part II and Part IV. Each of these parts is further divided into four sub-parts. Also, we take $n > 8$ for these two parts and deal with the cases $n = 6, 8$ seperately.

Part II: As mentioned above, Part II is divided into four Sub-parts. We call them S_1, S_2, S_3 and S_4.

Sub-part S_1: This Sub-part contains only two vertices $u_{\frac{n}{2}+1}$ and $v_{\frac{n}{2}+1}$. The labeling function is

$$f\left(u_{\frac{n}{2}+1}\right) = \frac{n}{2} + 1, f\left(v_{\frac{n}{2}+1}\right) = \frac{3n}{2} - 2.$$

Hence the labels used and the edge-weights covered in Sub-part S_1 are:

Labels Used: $\frac{n}{2} + 1, \frac{3n}{2} - 2.$

Edge-Weights Covered: $1, n - 3.$

Here 1 is the weight of the edge $u_{\frac{n}{2}}u_{\frac{n}{2}+1}$ as $u_{\frac{n}{2}}$ has received label $\frac{n}{2}$ when $n \equiv 0(mod4)$ and $\frac{n}{2} + 2$ when $n \equiv 2(mod4)$ in Part I and $u_{\frac{n}{2}+1}$ receives label $\frac{n}{2} + 1$ in Sub-part S_1.

Sub-part S_2: This Sub-part contains six vertices. The labeling functions for $n \equiv 0(mod4)$ and $n \equiv 2(mod4)$ are slightly different.

When $n \equiv 0(mod4)$ it is defined as:

$$\begin{aligned}
f\left(u_{\frac{n}{2}+2}\right) &= \frac{3n}{2} &, f\left(v_{\frac{n}{2}+2}\right) &= \frac{3n}{2} + 1.\\
f\left(u_{\frac{n}{2}+3}\right) &= \frac{3n}{2} - 3 &, f\left(v_{\frac{n}{2}+3}\right) &= \frac{3n}{2} + 2.\\
f\left(u_{\frac{n}{2}+4}\right) &= \frac{3n}{2} + 3 &, f\left(v_{\frac{n}{2}+4}\right) &= \frac{3n}{2} - 1.
\end{aligned}$$

When $n \equiv 2(mod4)$ the labeling function is:

$$\begin{aligned}
f\left(u_{\frac{n}{2}+2}\right) &= \frac{3n}{2} &, f\left(v_{\frac{n}{2}+2}\right) &= \frac{3n}{2} + 1.\\
f\left(u_{\frac{n}{2}+3}\right) &= \frac{3n}{2} + 3 &, f\left(v_{\frac{n}{2}+3}\right) &= \frac{3n}{2} - 1.\\
f\left(u_{\frac{n}{2}+4}\right) &= \frac{3n}{2} - 3 &, f\left(v_{\frac{n}{2}+4}\right) &= \frac{3n}{2} + 2.
\end{aligned}$$

The labels used and the edge-weights covered for both $n \equiv 0(mod4)$ and $n \equiv 2(mod4)$ are same. They are:

Labels Used: $\frac{3n}{2} - 3, \frac{3n}{2} - 1, \frac{3n}{2}$ to $\frac{3n}{2} + 3.$

Edge-Weights Covered: 1, 3 to 6, $n-1$.

Here $n-1$ is the weight of the edge $u_{\frac{n}{2}+1}u_{\frac{n}{2}+2}$ since $u_{\frac{n}{2}+1}$ has received label $\frac{n}{2}+1$ in Sub-part S_1 and $u_{\frac{n}{2}+2}$ receives label $\frac{3n}{2}$ in Sub-part S_2.

Sub-Part S_3: Here we define.

$$\begin{aligned}
f\left(u_{n-(2i-1)}\right) &= n+2i \ . \ 1 \le i \le [\frac{n}{4}]-2. \\
f\left(v_{n-(2i-1)}\right) &= 2n-2i \ .1 \le i \le [\frac{n}{4}]-2. \\
f\left(u_{n-2i}\right) &= 2n-(2i+1) \ . \ 1 \le i \le \frac{n}{4}-3 \text{ when } n \equiv 0(mod4) \\
&\qquad\qquad and \\
&\qquad 1 \le i \le [\frac{n}{4}]-2 \text{ when } n \equiv 2(mod4). \\
f\left(v_{n-2i}\right) &= n+2i+1 \ . \ 1 \le i \le \frac{n}{4}-3 \text{ when } n \equiv 0(mod4) \\
&\qquad\qquad and \\
&\qquad 1 \le i \le [\frac{n}{4}]-2 \text{ when } n \equiv 2(mod4).
\end{aligned}$$

Remark 1: Consider the situation $n \equiv 2(mod4)$. Here, in the Sub-part S_3, for the validity of the range of the parameter i, we need $n \ge 12$. For $8 < n < 12$, we have only one value of n which is 10. When $n = 10$, the set of vertices belonging to S_3 is an empty set. Consider the situation $n \equiv 0(mod4)$. For the validity of the range of the parameter i, we need $n \ge 12$ for the first two steps in the definition of the labeling function f whereas we need $n \ge 16$ for the third and the fourth steps in the definition of the labeling function f. When $n = 12$, S_3 has only two vertices u_{11} and v_{11}.

For the Sub-part S_3 the labels used and the edge-weights covered for both $n \equiv 0(mod4)$ and $n \equiv 2(mod4)$, $n > 8$ are:

Labels Used: $n+2$ to $\frac{3n}{2}-4$; $\frac{3n}{2}+4$ to $2n-2$.

We recall that for $n = 10$, the set S_3 is empty and hence no labels are used.

Edge-Weights Covered: 7 to $n-4$.

When $n = 10$ the set S_3 is empty and so no edge-weights are covered.

Here 7 is the weight of the edge $u_{\frac{n}{2}+1}u_{\frac{n}{2}+5}$ as $u_{\frac{n}{2}+1}$ has received label $\frac{3n}{2}+3$ when $n \equiv 0(mod4)$ and $\frac{3n}{2}-3$ when $n \equiv 2(mod4)$ in Sub-part S_2 and $u_{\frac{n}{2}+5}$ receives label $\frac{3n}{2}-4$ when $n \equiv 0(mod4)$ and $\frac{3n}{2}+4$ when $n \equiv 2(mod4)$ in Sub-part S_3.

Sub-Part S_1: This Sub-part contains only two vertices u_n and v_n. The labeling function is

$$f(u_n) = 2n,\ f(v_n) = 2n + 2.$$

So, the labels used and the edge-weights covered in Sub-part S_1 are:

Labels Used: $2n, 2n + 2$.

Edge-Weights Covered: $2, n - 2, n$.

Here $n - 2$ and n are the weights of the edges $u_{n-1}u_n$ and $u_n u_{n+1}$ respectively as u_{n-1} has received label $n+2$ in Sub-part S_3 and u_n receives label $2n$ in Sub-part S_1 and u_{n+1} has received label $3n$ in Part III.

As S_3 is empty for $n = 10$, u_{n-1} of $C_{20}OK_1$ receives label $n + 2$ in Sub-part S_2.

Part IV: As mentioned earlier we take $n > 8$ and Part IV is also divided into four Sub-parts. We call them T_1, T_2, T_3 and T_4.

Sub-Part T_1: This Sub-part contains two vertices $u_{\frac{3n}{2}+1}$ and $v_{\frac{3n}{2}+1}$. The labeling function is

$$f\left(u_{\frac{3n}{2}+1}\right) = \frac{7n}{2},\ f\left(v_{\frac{3n}{2}+1}\right) = \frac{5n}{2} + 3.$$

Hence, the labels used and the edge-weights covered in Sub-part T_1 are:

Labels Used: $\frac{7n}{2}, \frac{5n}{2} + 3$.

Edge-Weights Covered: $1, n - 3$.

Here 1 is the weight of the edge $u_{\frac{3n}{2}}u_{\frac{3n}{2}+1}$ as $u_{\frac{3n}{2}}$ has received label $\frac{7n}{2} + 1$ when $n \equiv 0(mod4)$ and $\frac{7n}{2} - 1$ when $n \equiv 2(mod4)$ in Part III and $u_{\frac{3n}{2}+1}$ receives label $\frac{7n}{2}$ in Part IV.

Sub-Part T_2: This Sub-part contains six vertices.

The labeling functions for $n \equiv 0(mod4)$ and $n \equiv 2(mod4)$ are slightly different.

When $n \equiv 0(mod4)$ it is defined as:

$$\begin{aligned}
f\left(u_{\frac{3n}{2}+1}\right) &= \frac{5n}{2} - 2,\ f\left(v_{\frac{3n}{2}+1}\right) = \frac{5n}{2} + 2. \\
f\left(u_{\frac{3n}{2}+3}\right) &= \frac{5n}{2} + 4,\ f\left(v_{\frac{3n}{2}+3}\right) = \frac{5n}{2} - 1. \\
f\left(u_{\frac{3n}{2}+2}\right) &= \frac{5n}{2} + 1,\ f\left(v_{\frac{3n}{2}+2}\right) = \frac{5n}{2}.
\end{aligned}$$

When $n \equiv 2(mod4)$ the labeling function is :

$$f\left(u_{\frac{3n}{2}+1}\right) = \frac{5n}{2}+4, f\left(v_{\frac{3n}{2}+1}\right) = \frac{5n}{2}-1.$$
$$f\left(u_{\frac{3n}{2}+3}\right) = \frac{5n}{2}-2, f\left(v_{\frac{3n}{2}+3}\right) = \frac{5n}{2}+2.$$
$$f\left(u_{\frac{3n}{2}+2}\right) = \frac{5n}{2}+1, f\left(v_{\frac{3n}{2}+2}\right) = \frac{5n}{2}.$$

The labels used and the edge-weights covered are same for both $n \equiv 0(mod4)$ and $n \equiv 2(mod4)$. They are:

Labels Used: $\frac{5n}{2}-2$ to $\frac{5n}{2}+2, \frac{5n}{2}+4$.

Edge-Weights Covered: $1, 3$ to $6, n-1$.

Here $n-1$ is the weight of the edge $u_{\frac{3n}{2}+1}u_{\frac{3n}{2}+2}$ as $u_{\frac{3n}{2}+1}$ has received label $\frac{7n}{2}$ in Sub-part T_1 and $u_{\frac{3n}{2}+2}$ receives label $\frac{5n}{2}+1$ in Sub-part T_2.

Sub-Part T_3: Here we define.

$$f(u_{2n+1-2i}) = 3n+1-2i \;,\; 1 \le i \le [\frac{n}{4}]-2.$$
$$f(v_{2n+1-2i}) = 2n+1+2i \;,\; 1 \le i \le [\frac{n}{4}]-2.$$
$$f(u_{2n-2i}) = 2n+2+2i \;,\; 1 \le i \le \frac{n}{4}-3 \text{ when } n \equiv 0(mod4)$$
$$\text{and}$$
$$1 \le i \le [\frac{n}{4}]-2 \text{ when } n \equiv 2(mod4).$$
$$f(v_{2n-2i}) = 3n-2i \;,\; 1 \le i \le \frac{n}{4}-3 \text{ when } n \equiv 0(mod4)$$
$$\text{and}$$
$$1 \le i \le [\frac{n}{4}]-2 \text{ when } n \equiv 2(mod4).$$

Remark 2: Remark 1 which was for the labeling function for S_3 is applicable for T_3 too, the only change being that when $n = 12$ T_3 has only two vertices u_{23} and v_{23} (whereas when $n = 12$ the two vertices of S_3 were u_{11} and v_{11}).

For the Sub-part T_3 the labels used and the edge-weights covered for both $n \equiv 0(mod4)$ and $n \equiv 2(mod4), n > 8$ are:

Labels Used: $\frac{5n}{2}+5$ to $3n-1; 2n+3$ to $\frac{5n}{2}-3$.

We recall that for $n = 10$, the set T_3 is empty and hence no labels are used.

Edge-Weights Covered: 7 to $n-4$.

When $n=10$ the set T_3 is empty and so no edge-weights are covered.

Here 7 is the weight of the edge $u_{\frac{3n}{2}+1}u_{\frac{3n}{2}+5}$ as $u_{\frac{3n}{2}+1}$ has received label $\frac{5n}{2}-2$ when $n \equiv 0(mod4)$ and $\frac{5n}{2}+4$ when $n \equiv 2(mod4)$ in Sub-part T_2 and $u_{\frac{3n}{2}+5}$ receives label $\frac{5n}{2}+5$ when $n \equiv 0(mod4)$ and $\frac{5n}{2}-3$ when $n \equiv 2(mod4)$ in Sub-part T_3.

Sub-Part T_4: This Sub-part contains only two vertices namely u_{2n} and v_{2n}.

For both $n \equiv 0(mod4)$ and $n \equiv 2(mod4)$ we define

$$f(u_{2n}) = 2n+1 \text{ and } f(v_{2n}) = 2n-1.$$

Hence the labels used and the edge-weights covered in T_4 are:

Labels Used: $2n-1, 2n+1$.

Edge-Weights Covered: $2, n-2, n$.

Here $n-2$ and n are the weights of the edges $u_{2n-1}u_{2n}$ and $u_{2n}u_1$ respectively as u_{2n-1} has received label $3n-1$ in Sub-part T_3. u_1 has received label $n+1$ in Part I and u_{2n} receives label $2n+1$ in T_4.

As T_3 is empty for $n=10$, u_{2n-1} receives label $3n-1$ in Sub-part T_2.

It can be directly verified that the labeling function f is injective and that each edge-weight from 1 to n is repeated exactly four times.

As mentioned earlier we now deal with the Part II and Part IV for $n=6$ and $n=8$. First we define Part II and Part IV for $n=6$, that is, for $C_{12}OK_1$.

Part II of $C_{12}OK_1$: This part has vertices u_i, v_i, $4 \leq i \leq 6$. The labeling function for these six vertices is:

$$f(u_4) = 4, f(v_4) = 8, f(u_5) = 9, f(v_5) = 10, f(u_6) = 12, f(v_6) = 14.$$

The labels used and the edge-weights covered are:

Labels Used: 4, 8, 9, 10, 12, 14.

Edge-Weights Covered: 1 to 6, where 1 is covered twice.

One of the edge-weights 1 is that of the edge u_3u_4 as u_3 has received label 5 in Part I and u_4 receives label 4 in Part II. Also 6 is the weight of the edge u_6u_7 as u_7 has received label 18 in Part III and u_6 receives label 12 in Part II.

Part IV for $C_{12}OK_1$: This part has six vertices $u_i, v_i, 10 \leq i \leq 12$. The labeling function is :

$$f(u_{10}) = 21, f(v_{10}) = 17, f(u_{11}) = 16, f(v_{11}) = 15, f(u_{12}) = 13, f(v_{12}) = 11.$$

The labels used and the edge-weights covered are:

Labels Used: 11, 13, 15, 16, 17, 21.

Edge-Weights Covered: 1 to 6, where 1 is covered twice .

Here 6 is the weight of the edge $u_{12}u_1$ as u_{12} receives the label 13 in Part IV and u_1 has received label 7 in Part I. One of the edge-weights 1 is the weight of the edge u_9u_{10} as u_9 has received label 20 in Part III and u_{10} receives label 21 in Part IV.

Next we define Part II and Part IV for $n = 8$, that is, for $C_{16}OK_1$.

Part II for $C_{16}OK_1$: This part has eight vertices $u_i, v_i, 5 \leq i \leq 8$. The labeling function for these eight vertices is:

$$f(u_5) = 5, f(v_5) = 10, f(u_6) = 12, f(v_6) = 11,$$

$$f(u_7) = 18, f(v_7) = 14, f(u_8) = 16, f(v_8) = 13.$$

The labels used and the edge-weights covered are:

Labels Used: 5, 10 to 14, 16, 18.

Edge-Weights Covered: 1 to 8, where 1 is covered twice.

Here 8 is the weight of the edge u_8u_9 as u_8 receives label 16 in Part II and u_9 has received label 24 in Part III. One of the edge-weights 1 is the weight of edge u_4u_5 as u_4 has received label 4 in Part I and u_5 receives label 5 in Part II.

Part IV for $C_{16}OK_1$: Part IV of $C_{16}OK_1$ has eight vertices u_i, v_i, $13 \leq i \leq 16$. The labeling function for these eight vertices is:

$$f(u_{13}) = 28, f(v_{13}) = 23, f(u_{14}) = 21, f(v_{14}) = 22,$$

$$f(u_{15}) = 15, f(v_{15}) = 19, f(u_{16}) = 17, f(v_{16}) = 20.$$

Labels Used : 15, 17, 19 to 23, 28.

Edge-Weights Covered: 1 to 8, where 1 is covered twice.

Here 8 is the weight of the edge u_1u_{16} as u_1 has received label 9 in Part I and u_{16} receives label 17 in Part IV. One of the edge-weights 1 is the weight of the edge $u_{12}u_{13}$ as u_{12} has received label 29 in Part III and u_{13} receives label 28 in Part IV.

The proof for the case $n \equiv 0(mod2)$ is now over.

The Case $n \equiv 1(mod2)$: In this case Part I describes the labeling function for the vertices $u_i, v_i, 1 \leq i \leq \frac{n+1}{2}$. Part II for $u_i, v_i, \frac{n+3}{2} \leq i \leq n$. Part III for $u_i, v_i, n+1 \leq i \leq \frac{3n+1}{2}$ and Part IV for $u_i, v_i, \frac{3n+3}{2} \leq i \leq 2n$. Similar to the case $n \equiv 0(mod4)$ here also Part I and Part III are smooth so we describe them first and then describe Part II and Part IV.

Part I: Here we define.

$$\begin{aligned}
f(u_{2i-1}) &= n+3-2i \ , \ 1 \leq i \leq [\frac{n}{4}]+1. \\
f(v_{2i-1}) &= 2i-1 \ , 1 \leq i \leq [\frac{n}{4}]+1, \\
f(u_{2i}) &= 2i \ , \ 1 \leq i \leq [\frac{n}{4}] \text{ when } n \equiv 1(mod4) \\
& \qquad \textit{and} \\
& \qquad 1 \leq i \leq [\frac{n}{4}]+1 \text{ when } n \equiv 3(mod4). \\
f(v_{2i}) &= n+2-2i \ , \ 1 \leq i \leq [\frac{n}{4}] \text{ when } n \equiv 1(mod4) \\
& \qquad \textit{and} \\
& \qquad 1 \leq i \leq [\frac{n}{4}]+1 \text{ when } n \equiv 3(mod4).
\end{aligned}$$

The labels used and the edge-weights covered in Part I are same for both $n \equiv 1(mod4)$ and $n \equiv 3(mod4)$. They are:

Labels Used: 1 to $n+1$.

Edge-Weights Covered: 1 to n.

Part III: We define.

$$f(u_{n+2i-1}) = 3n-2+2i \quad . \quad 1 \le i \le [\frac{n}{4}]+1,$$
$$f(v_{n+2i-1}) = 4n+2-2i \quad . \quad 1 \le i \le [\frac{n}{4}]+1.$$
$$f(u_{n+2i}) = 4n-2i+1 \quad . \quad 1 \le i \le [\frac{n}{4}] \text{ when } n \equiv 1(mod4)$$
$$and$$
$$1 \le i \le [\frac{n}{4}]+1 \text{ when } n \equiv 3(mod4).$$
$$f(v_{n+2i}) = 3n+2i-1 \quad . \quad 1 \le i \le [\frac{n}{4}] \text{ when } n \equiv 1(mod4)$$
$$and$$
$$1 \le i \le [\frac{n}{4}]+1 \text{ when } n \equiv 3(mod4).$$

The labels used and the edge-weights covered in Part III are same for both $n \equiv 1(mod4)$ and $n \equiv 3(mod4)$. They are:

Labels Used: $3n$ to $4n$.

Edge-Weights Covered: 1 to n.

For Part II and Part IV we consider two sub-cases, viz., $n \equiv 1(mod4)$ and $n \equiv 3(mod4)$.

The Sub-Case $n \equiv 1(mod4)$:

Part II: We define.

$$f(u_{n+2-2i}) = 2n+2-2i \quad . \quad f(v_{n+2-2i}) = n+2i.$$
$$f(u_{n+1-2i}) = n+1+2i \quad . \quad f(v_{n+1-2i}) = 2n+1-2i.$$

for $1 \le i \le [\frac{n}{4}]$.

The labels used and the edge-weights covered in Part II are:

Labels Used: $n+2$ to $2n$.

Edge-Weights Covered: 1 to n.

Here $n-1$ and n are the weights of the edges $u_{\frac{n+1}{2}}u_{\frac{n+3}{2}}$ and u_nu_{n+1} respectively as $u_{\frac{n+1}{2}}$ has received label $\frac{n+3}{2}$ when $n \equiv 1(mod2)$ in Part I and $u_{\frac{n+3}{2}}$ receives label $\frac{3n+1}{2}$ in Part II and u_n receives label $2n$ in Part II and u_{n+1} has received label $3n$ in Part III.

Part IV : We define.

$$f\left(u_{2n-2(i-1)}\right) = 2n + 2i - 1 \quad . \quad f\left(v_{2n-2(i-1)}\right) = 3n - (2i - 1).$$

$$f\left(u_{2n-(2i-1)}\right) = 3n - 2i \quad . \quad f\left(v_{2n-(2i-1)}\right) = 2n + 2i.$$

for $1 \leq i \leq \lceil \frac{n}{4} \rceil$.

The labels used and the edge-weights covered are:

Labels Used: $2n + 1$ to $3n - 1$.

Edge-Weights Covered: 1 to n.

Here $n - 1$ and n are the weights of the edges $u_{\frac{3n+1}{2}} u_{\frac{3n+3}{2}}$ and $u_1 u_{2n}$ respectively as $u_{\frac{3n+1}{2}}$ has received label $\frac{7n-1}{2}$ when $n \equiv 1(mod4)$ in Part III and $u_{\frac{3n+3}{2}}$ and u_{2n} receive labels $\frac{5n+1}{2}$ and $2n + 1$ respectively in Part IV and u_1 has received label $n + 1$ in Part I.

The Sub-Case $n \equiv 3(mod4)$:

In this Sub-case each of Part II and Part IV is further divided into two Sub-parts. We call them A_1, A_2 and B_1, B_2 respectively.

Part II:

Sub-Part A_1: In this Sub-part we have six vertices. The labeling function is :

$$f\left(u_{\frac{n+3}{2}}\right) = \frac{3n-1}{2}, \quad f\left(v_{\frac{n+3}{2}}\right) = \frac{3(n+1)}{2}, \quad f\left(u_{\frac{n+5}{2}}\right) = \frac{3(n-1)}{2}.$$

$$f\left(v_{\frac{n+5}{2}}\right) = \frac{3n+5}{2}, \quad f\left(u_{\frac{n+7}{2}}\right) = \frac{3n+7}{2}, \quad f\left(v_{\frac{n+7}{2}}\right) = \frac{3n+1}{2}.$$

The labels used and the edge-weights covered are:

Labels Used: $\frac{3(n-1)}{2}$ to $\frac{3n+7}{2}$.

Edge-Weights Covered: 1 to 5; $n - 1$.

Here $n - 1$ is the weight of the edge $u_{\frac{n+1}{2}} u_{\frac{n+3}{2}}$ as $u_{\frac{n+1}{2}}$ has received label $\frac{n+1}{2}$ when $n \equiv 3(mod4)$ in Part I and $u_{\frac{n+3}{2}}$ receives label $\frac{3n-1}{2}$ in Sub-part A_1.

Sub-Part A_2: Here we define,

$$f\left(u_{n-2(i-1)}\right) = 2n - 2(i-1) \quad . \quad f\left(v_{n-2(i-1)}\right) = n + 2i.$$

$$f\left(u_{n-(2i-1)}\right) = n + 2i + 1 \quad . \quad f\left(v_{n-(2i-1)}\right) = 2n - (2i-1).$$

for $1 \leq i \leq \lfloor \frac{n}{4} \rfloor - 1$.

Remark: For the validity of the range of the parameter i, we need $n \geq 8$. For $5 \leq n \leq 8$, we have only one value of n which is 7. When $n = 7$, the set of vertices belonging to A_2 is an empty set.

For the Sub-part A_2 the labels used and the edge-weights covered are:

Labels Used: $n+2$ to $\frac{3n-5}{2}$; $\frac{3n+9}{2}$ to $2n$.

When $n = 7$, the set A_2 is empty and hence no labels are used.

Edge-Weights Covered: 6 to $n-2$; n.

When $n = 7$, the set A_2 is empty and so no edge-weights are covered.

Here 6 and n are the weights of the edges $u_{\frac{n+7}{2}} u_{\frac{n+9}{2}}$ and $u_n u_{n+1}$ respectively as $u_{\frac{n+7}{2}}$ has received label $\frac{3n+7}{2}$ when $n \equiv 3(mod 4)$ in Sub-part A_1 and $u_{\frac{n+9}{2}}$ receives label $\frac{3n-5}{2}$ in Sub-part A_2 and u_{n+1} has received label $3n$ in Part III and u_n receives label $2n$ in Sub-part A_2.

Remark: Note that for $n = 7$, that is, for $C_{14}OK_1$, the edge-weight 6 is covered in Sub-part A_1. Also, as A_2 is empty for $n = 7$, 7 is the weight of the edge $u_7 u_8$ since u_7 has received label 14 in Sub-part A_1 and u_8 has received label 21 in Part III.

Part IV:

Sub-Part B_1: B_1 contains six vertices. The labeling function is:

$$f\left(u_{\frac{3n+7}{2}}\right) = \frac{5(n-1)}{2}, \quad f\left(v_{\frac{3n+7}{2}}\right) = \frac{5n+1}{2}, \quad f\left(u_{\frac{3n+5}{2}}\right) = \frac{5(n+1)}{2}.$$

$$f\left(v_{\frac{3n+5}{2}}\right) = \frac{5n-3}{2}, \quad f\left(u_{\frac{3(n+1)}{2}}\right) = \frac{5n+3}{2}, \quad f\left(v_{\frac{3(n+1)}{2}}\right) = \frac{5n-1}{2}.$$

The labels used and the edge-weights covered in Sub-part B_1 are:

Labels used: $\frac{5(n-1)}{2}$ to $\frac{5(n+1)}{2}$.

Edge-Weights Covered: 1 to 5, $n-1$.

Here $n-1$ is the weight of the edge $u_{\frac{3n+1}{2}} u_{\frac{3n+3}{2}}$ as $u_{\frac{3n+1}{2}}$ has received lable $\frac{7n+1}{2}$ when
$n \equiv 3(mod4)$ in Part III and $u_{\frac{3(n+1)}{2}}$ receives label $\frac{5n+3}{2}$ in Sub-part B_1.

Sub-Part B_2: Here we define.

$$f\left(u_{2n-2(i-1)}\right) = 2n + (2i-1), \quad f\left(v_{2n-2(i-1)}\right) = 3n - (2i-1).$$

$$f\left(u_{2n-(2i-1)}\right) = 3n - 2i, \quad f\left(v_{2n-(2i-1)}\right) = 2n + 2i.$$

for $1 \leq i \leq \lfloor \frac{n}{4} \rfloor - 1$.

Remark: For the validity of the range of the parameter i, we need $n \geq 8$. For $5 \leq n \leq 8$, we have only one value of n which is 7. When $n = 7$, the set of vertices belonging to B_2 is empty.

For the Sub-part B_2 the labels used and the edge-weights covered are:

Labels Used: $2n+1$ to $\frac{5n-7}{2}$; $\frac{5n+7}{2}$ to $3n-1$.

When $n = 7$, the set B_2 is empty and hence no labels are used.

Edge-Weights Covered: 6 to $n-2$; n.

When $n = 7$, the set B_2 is empty and hence no edge-weights covered.

Here 6 and n are the weights of the edges $u_{\frac{3n+7}{2}} u_{\frac{3n+9}{2}}$ and $u_1 u_{2n}$ as $u_{\frac{3n+7}{2}}$ has received label $\frac{5(n-1)}{2}$ when $n \equiv 3(mod4)$ in Sub-part B_1 and u_1 has received label $n+1$ in Part I and $u_{\frac{3n+9}{2}}$ and u_{2n} receive labels $\frac{5n+7}{2}$ and $2n+1$ respectively in Sub-part B_2.

Remark: Note that for $n = 7$, that is for $C_{14}OK_1$, the edge-weight 6 is covered in Sub-part B_1. Also as B_2 is empty for $n = 7$, 7 is the weight of the edge $u_1 u_{14}$ as u_1 has received label 8 in Part I and u_{14} has received label 15 in Sub-part B_1.

It can be directly varified that the labeling function f is injective and that each edge-weight from 1 to n is repeated exactly four times.

The proof for the case $n \equiv 1(mod2)$ is now complete.

As mentioned earlier we give actual labelings for $C_{2n}OK_1$ when $n = 2, 3, 4$. We do not draw the edges but give only the labels of u_i, v_i where the labels in the first row are those of v_i and the labels in the second row are those of u_i. The labels of v_i, u_i are given one below the other.

Minimal 4-equitable labeling of C_4OK_1:

The graph C_4OK_1 is not minimally 4-equitable with edge-weights $\{1, 2\}$

but it is minimally 4-equitable with edge-weights $\{1, 3\}$. The labeling is :

$$\begin{matrix} 1 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 \end{matrix}$$

Minimal 4-equitable labeling of C_6OK_1:

Here we give minimal 4-equitable labeling of C_6OK_1 with edge-weights $\{2, 3, 4\}$ and not with edge-weights $\{1, 2, 3\}$.

$$\begin{matrix} 1 & 2 & 11 & 10 & 5 & 4 \\ 3 & 6 & 8 & 12 & 9 & 7 \end{matrix}$$

Minimal 4-equitable labeling of C_8OK_1:

The labeling displayed below gives a minimal 4-equitable labeling of C_8OK_1 with edge-weights $\{1, 2, 3, 4\}$.

$$\begin{matrix} 1 & 3 & 10 & 13 & 16 & 14 & 6 & 2 \\ 5 & 7 & 9 & 12 & 15 & 11 & 8 & 4. \end{matrix}$$

Proof of the theorem is now complete.

Illustrations

We apply the labeling functions used in the proof of the main theorem and give actual labelings for $2 \le n \le 16$. For convenience we cut the graphs into parts corresponding to Part I, Part II, Part III and Part IV. We do not draw the edges but give only the labels of u_i, v_i where the labels in the first row are those of v_i and the labels in the second row are those of u_i. For the sake of clarity we mention the vertices of the first and the last pendant edges for each part.

Our illustrations will cover at least two illustrations for each subcase and will also cover all the cases when certain vertex subsets are empty.

The actual labelings for $n = 2, 3, 4$ are given at the end of the proof of the theorem. We now cover $n = 5, 7, 9, 11, 13$ and 15 which correspond to $n \equiv 1(mod2)$. We first deal with $n = 5, 9, 13$ which correspond to the subcase $n \equiv 1(mod4)$ and then deal with $n = 7, 11, 15$ which correspond to the subcase $n \equiv 3(mod4)$.

Minimal 4-equitable labelng of $C_{10} O K_1$:

Part I

(v_1)	1	5	3	(v_3)
(u_1)	6	2	4	(u_3)

Part II

(v_1)	9	7	(v_5)
(u_1)	8	10	(u_5)

Part III

(v_6)	20	16	18	(v_8)
(u_6)	15	19	17	(u_8)

Part IV

(v_9)	12	14	(v_{10})
(u_9)	13	11	(u_{10})

Minimal 4-equitable labeling of $C_{18}OK_1$:

Part I

(v_1)	1	9	3	7	5	(v_5)
(u_1)	10	2	8	4	6	(u_5)

Part II

(v_6)	15	13	17	11	(v_9)
(u_6)	14	16	12	18	(u_9)

Part III

(v_{10})	36	28	34	30	32	(v_{14})
(u_{10})	27	35	29	33	31	(u_{14})

Part IV

(v_{15})	22	24	20	26	(v_{18})
(u_{15})	23	21	25	19	(u_{18})

Minimal 4-equitable labeling of $C_{26}OK_1$:

Part I

(v_1)	1	13	3	11	5	9	7	(v_7)
(u_1)	14	2	12	4	10	6	8	(u_7)

Part II

(v_8)	21	19	23	17	25	15	(v_{13})
(u_8)	20	22	18	24	16	26	(u_{13})

Part III

(v_{14})	52	40	50	42	48	44	46	(v_{20})
(u_{14})	39	51	41	49	43	47	45	(u_{20})

Part IV

(v_{21})	32	34	30	36	28	38	(v_{26})
(u_{21})	33	31	35	29	37	27	(u_{26})

Minimal 4-equitable labeling of $C_{14}OK_1$:

Part I

(v_1)	1	7	3	5	(v_4)
(u_1)	8	2	6	4	(u_4)

Part II

A_1 :	(v_5)	12	13	11	(v_7)
	(u_5)	10	9	14	(u_7)

A_2 : For $n = 7$. A_2 *is empty.*

Part III

(v_8)	28	22	26	24	(v_{11})
(u_8)	21	27	23	25	(u_{11})

Part IV

B_1 :	(v_{12})	17	16	18	(v_{14})
	(u_{12})	19	20	15	(u_{14})

B_2 : For $n = 7$. B_2 is empty.

Minimal 4-equitable labeling of $C_{22}OK_1$:

Part I

(v_1)	1	11	3	9	5	7	(v_6)
(u_1)	12	2	10	4	8	6	(u_6)

Part II

A_1	:	(v_7)	18	19	17	(v_9)
		(u_7)	16	15	20	(u_9)

A_2	:	(v_{10})	21	13	(v_{11})
		(u_{10})	14	22	(u_{11})

Part III

(v_{12})	44	34	42	36	40	38	(v_{17})
(u_{12})	33	43	35	41	37	39	(u_{17})

Part IV

B_1	:	(v_{18})	27	26	28	(v_{20})
		(u_{18})	29	30	25	(u_{20})

B_2	:	(v_{21})	24	32	(v_{22})
		(u_{21})	31	23	(u_{22})

Minimal 4-equitable labeling of $C_{30}OK_1$:

Part I

(v_1)	1	15	3	13	5	11	7	9	(v_8)
(u_1)	16	2	14	4	12	6	10	8	(u_8)

Part II

A_1	:	(v_9)	24	25	23	(v_{11})
		(u_9)	22	21	26	(u_{11})

A_2	:	(v_{12})	27	19	29	17	(v_{15})
		(u_{12})	20	28	18	30	(u_{15})

Part III

(v_{16})	60	46	58	48	56	50	54	52	(v_{23})
(u_{16})	45	59	47	57	49	55	51	53	(u_{23})

Part IV

B_1	:	(v_{21})	37	36	38	(v_{26})	
		(u_{21})	39	40	35	(u_{26})	
B_2	:	(v_{27})	34	42	32	44	(v_{30})
		(u_{27})	41	33	43	31	(u_{30}) .

Next we deal with $n = 6, 8, 10, 12, 14$ and 16. They correspond to $n \equiv 0(mod2)$. We first handle $n = 6, 10, 14$ which correspond to the subcase $n \equiv 2(mod4)$ and then $n = 8, 12, 16$ which correspond to the subcase $n \equiv 0(mod4)$.

Minimal 4-equitable labeling of $C_{12}OK_1$:

Note that for $n = 6$, that is, for $C_{12}OK_1$, our labeling function given in the proof of the theorem is slightly different from that of the labeling function for all the other values of $n \equiv 2(mod4)$.

Here we give the actual labeling.

Part I

(v_1)	1	6	3	(v_3)
(u_1)	7	2	5	(u_3)

Part II

(v_4)	8	10	14	(v_6)
(u_4)	4	9	12	(u_6)

Part III

(v_7)	24	19	22	(v_9)
(u_7)	18	23	20	(u_9)

Part IV

(v_{10})	17	15	11	(v_{12})
(u_{10})	21	16	13	(u_{12})

Minimal 4-equitable labeling of $C_{20}OK_1$:

Part I

(v_1)	1	10	3	8	5	(v_5)
(u_1)	11	2	9	4	7	(u_5)

Part II

S_1 :	(v_6)	13				
	(u_6)	6				

S_2 :	(v_7)	16	14	17	(v_9)	
	(u_7)	15	18	12	(u_9)	

S_3 : For $n = 10$. S_3 is empty.

S_4 :	(v_{10})	22
	(u_{10})	20

Part III

(v_{11})	40	31	38	33	36	(v_{15})
(u_{11})	30	39	32	37	34	(u_{15})

Part IV

T_1 :	(v_{16})	28
	(u_{16})	35

T_2 :	(v_{17})	25	27	24	(v_{19})
	(u_{17})	26	23	29	(u_{19})

T_3 : For $n = 10$. T_3 is empty.

T_1 :	(v_{20})	19
	(u_{20})	21

Minimal 4-equitable labeling of $C_{28}OK_1$:

Part I

(v_1)	1	14	3	12	5	10	7	(v_7)
(u_1)	15	2	13	4	11	6	9	(u_7)

Part II

S_1	:	(v_8)	19			
		(u_8)	8			
S_2	:	(v_9)	22	20	23	(v_{11})
		(u_9)	21	24	18	(u_{11})
S_3	:	(v_{12})	17	26	(v_{13})	
	:	(u_{12})	25	16	(u_{13})	
S_4	:	(v_{14})	30			
	:	(u_{14})	28			

Part III

(v_{15})	56	43	54	45	52	47	50	(v_{21})
(u_{15})	42	55	44	53	46	51	48	(u_{21})

Part IV

T_1	:	(v_{22})	38			
		(u_{22})	49			
T_2	:	(v_{23})	35	37	34	(v_{25})
		(u_{23})	36	33	39	(u_{25})
T_3	:	(v_{26})	40	31	(v_{27})	
		(u_{26})	32	41	(u_{27})	
T_4	:	(v_{28})	27			
		(u_{28})	29			

Minimal 4-equitable labeling of $C_{16}OK_1$:

Note that for $n = 8$, that is, for $C_{16}OK_1$ our labeling function given in the proof of the theorem is slightly different from that of the labeling function for all the other values of $n \equiv 0(mod4)$. Here we give the actual labeling.

Part I

(v_1)	1	8	3	6	(v_4)
(u_1)	9	2	7	4	(u_4)

Part II

(v_5)	10	11	14	13	(v_8)
(u_5)	5	12	18	16	(u_8)

Part III

(v_9)	32	25	30	27	(v_{12})
(u_9)	24	31	26	29	(u_{12})

Part IV

(v_{13})	23	22	19	20	(v_{16})
(u_{13})	28	21	15	17	(u_{16})

Minimal 4-equitable labeling of $C_{24}OK_1$:

Part I

(v_1)	1	12	3	10	5	8	(v_6)
(u_1)	13	2	11	4	9	6	(u_6)

Part II

S_1 :	(v_7)	16				
	(u_7)	7				
S_2 :	(v_8)	19	20	17	(v_{10})	
	(u_8)	18	15	21	(u_{10})	
S_3 :	(v_{11})	22				
:	(u_{11})	14				
S_1 :	(v_{12})	26				
:	(u_{12})	24				

Part III

(v_{13})	48	37	46	39	44	41	(v_{18})
(u_{13})	36	47	38	45	40	43	(u_{18})

Part IV

T_1 :	(v_{19})	33			
	(u_{19})	42			
T_2 :	(v_{20})	30	29	32	(v_{22})
	(u_{20})	31	34	28	(u_{22})
T_3 :	(v_{23})	27			
	(u_{23})	35			
T_1 :	(v_{24})	23			
	(u_{24})	25			

Minimal 4-equitable labeling of $C_{32}OK_1$:

Part I

(v_1)	1	16	3	14	5	12	7	10	(v_8)
(u_1)	17	2	15	4	13	6	11	8	(u_8)

Part II

S_1	:	(v_9)	22			
		(u_9)	9			
S_2	:	(v_{10})	25	26	23	(v_{12})
		(u_{10})	24	21	27	(u_{12})
S_3	:	(v_{13})	28	19	30	(v_{15})
	:	(u_{13})	20	29	18	(u_{15})
S_4	:	(v_{16})	34			
	:	(u_{16})	32			

Part III

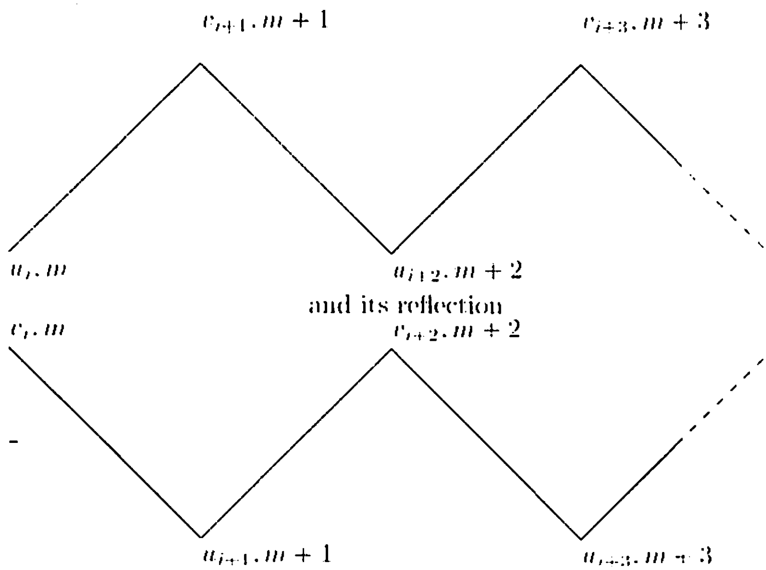
(v_{17})	64	49	62	51	60	53	58	55	(v_{24})
(u_{17})	48	63	50	61	52	59	54	57	(u_{24})

Part IV

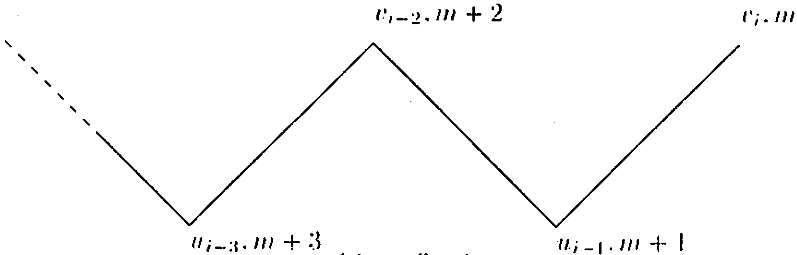
T_1	:	(v_{25})	43			
		(u_{25})	56			
T_2	:	(v_{26})	40	39	42	(v_{28})
		(u_{26})	41	44	38	(u_{28})
T_3	:	(v_{29})	37	46	35	(v_{31})
		(u_{29})	45	36	47	(u_{31})
T_4	:	(v_{32})	31			
		(u_{32})	33			

An Important Remark.

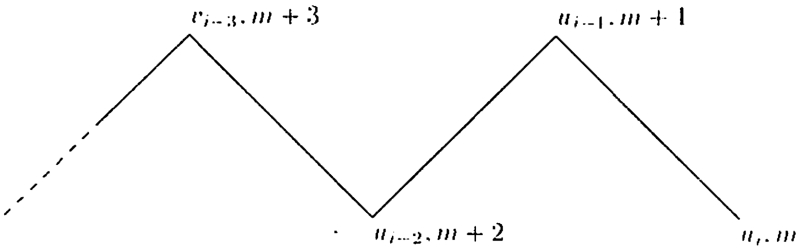
When it comes to actually labeling the vertices of $C_{2n}OK_1$ for a particular value of n using the labeling function f described for different parts of $C_{2n}OK_1$ the following remarks make the job easy. Carefully observe the actual labelings that are described in the Illustrations given above, in particular observe them for $C_{26}OK_1$, $C_{28}OK_1$ and $C_{30}OK_1$ which correspond to $n \equiv 1(mod4)$, $n \equiv 0(mod2)$ and $n \equiv 3(mod4)$ respectively. Except for the subparts which have only a few vertices, their number being independent of n, the labeling in all the other parts and subparts basicaly follows one of the following patterns. In the diagrams showing the patterns we have placed the integer labels $m, m+1$, etc. next to the vertex names u_i, v_i etc. when a continuous string of labels $m, m+1, m+2, m+3, \ldots$ is used.



OR



and its reflection



In fact, the pattern for part III is always a reflection of the pattern for Part I, the pattern for B_2 is a reflection of the pattern for A_2 and so on . In view of such observations, one should allot the labels of the two ends of any Part or subparts other than those for which the number of vertices involved is independent of n, and then fill the labels for the in-between vertices as per the particular pattern that is followed for the particular case to which n belongs. For those Parts in which the number of vertices involved is independent of n give the labels as per the function f for that Part. Luckily, such Parts are few and further, the number of vertices involved in any one of them is at most six.

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