The spectra for two classes of graph designs[1]

Yanxun Chang

Department of Mathematics
Northern Jiaotong University, 100044, Beijing, China
email: yxchang@center.njtu.edu.cn

Abstract

A $(\lambda K_n, G)$-design is a partition of the edges of λK_n into sub-graphs each of which is isomorphic to G. In this paper we investigate the existence for (K_n, G_{16})-design and (K_n, G_{20})-design, and prove that the necessary conditions for the existence of the two classes of graph designs are also sufficient.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph without isolated vertices. A $(\lambda K_n, G)$-design is a partition of the edges of λK_n into sub-graphs (*G-blocks*) each of which is isomorphic to G. A $(\lambda K_n, K_k)$-design is nothing but a (n, k, λ)-BIBD. If there exists a $(\lambda K_n, G)$-design, then

1. $\lambda n(n-1) \equiv 0 \pmod{2|E(G)|}$, and

2. $\lambda(n-1) \equiv 0 \pmod{d}$, where d is the greatest common divisor of the degrees of the vertices of G.

It was proved in [9] that the necessary conditions 1 and 2 for the existence of a $(\lambda K_n, G)$-design are asymptotically sufficient, that is, there exists an integer $N(G, \lambda)$ such that there is a $(\lambda K_n, G)$-design for $n \geq N(G, \lambda)$ and k, λ satisfying the necessary conditions 1 and 2.

The existence of a $(\lambda K_n, G)$- design for various graphs G has been studied in literature (see [3], [11], [5]). The case where G is a graph with at most four vertices has been solved completely in [2]. If G has no isolated vertices and $|V(G)| = 5$, the known existence of a

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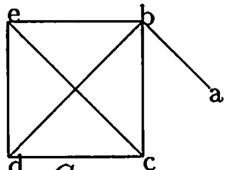
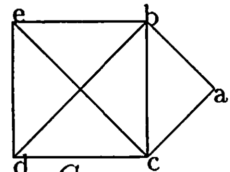
(K_n, G)-design has been very nearly solved in [1], [7], [8], [6], which is also summarized in [3]. There remain several graphs for which there are a few values of n for which it is not known whether or not decompositions of K_n yet. In this article we deal with the two graphs (the notation is borrowed from [1]) as follows:



G_{16}



G_{20}

It is known that in [3]:

Lemma 1 *The necessary condition* $n \equiv 0, 1 \pmod 7$ *and* $n > 8$ *for the existence of a* (K_n, G_{16})*- design is sufficient except the possibly exceptions of* $n = 119, 120, 147, 203, 204$.

Lemma 2 *There exists a* (K_n, G_{20})*-design for* $n \equiv 1 \pmod{16}$ *and* $n \neq 65$*; If a* (K_{65}, G_{20})*- design exists, then there exists a* (K_n, G_{20})*-design for* $n \equiv 0 \pmod{16}$.

2 Constructions for (K_v, G_{16})-design

In this section we will give several direct constructions for (K_v, G_{16})-design by using perfect base, where $G_{16} = (V, E)$ and $V = \{ a, b, c, d, e \}$, $E = \{ab, bc, bd, be, cd, ce, de\}$. We usually denote G_{16} as $\{a; b, c, d, e\}$. Next we introduce the definition of perfect base.

Let G be an Abelian group of order v and G_0 be the set of all elements of order less or equal to 2 in G. A *perfect base* in G is a set of triples $S = \{\{x_{i1}, x_{i2}, x_{i3}\} : 1 \leq i \leq t\}$, $t = \lfloor \frac{v-|G_0|}{6} \rfloor$, with elements x_{ij} from $G \setminus G_0$ satisfying the following properties:

(1) $\pm x_{ij}, 1 \leq j \leq 3, 1 \leq i \leq t$ are different;

(2) $\pm(x_{i1} - x_{i2}), \pm(x_{i1} - x_{i3}), \pm(x_{i2} - x_{i3}) \in G \setminus G_0$, $1 \leq i \leq t$, are different.

Note that the elements in the conditions (1) and (2) are taken from the set $G \setminus G_0$. When $G = Z_v$, the definition here is the same as that in [10], where perfect base is used to construct optical orthogonal codes.

Lemma 3 [4] *If* $v \equiv 1 \pmod 6$ *is a prime power, then there exists a perfect base in* $GF(v)$.

With the aid of the computer we have

Lemma 4 *There exists a perfect base in* Z_v *for* $v =$ *59, 60, 73, 101, 102.*

Proof: There is a perfect base in Z_{73} by Lemma 3. For each of $v = 59, 60, 101, 102$, we list a set of perfect base blocks as follows:

$v =$ 59: $\{1,2,4\}$, $\{3,7,12\}$, $\{5,13,19\}$, $\{6,33,43\}$, $\{8,28,39\}$, $\{9,30,42\}$, $\{10,25,44\}$, $\{11,24,41\}$, $\{14,21,37\}$.

$v =$ 60: $\{1,2,4\}$, $\{3,7,12\}$, $\{5,11,18\}$, $\{6,32,43\}$, $\{8,24,39\}$, $\{9,29,37\}$, $\{10,27,45\}$, $\{13,25,46\}$, $\{16,26,40\}$.

$v =$ 101: $\{5,11,18\}$, $\{6,14,24\}$, $\{8,19,31\}$, $\{9,23,38\}$, $\{15,50,69\}$, $\{7,10,12\}$, $\{13,17,33\}$, $\{16,57,58\}$, $\{20,52,73\}$, $\{21,66,97\}$, $\{22,56,65\}$, $\{25,53,75\}$, $\{27,54,71\}$, $\{29,62,100\}$, $\{34,60,99\}$, $\{37,61,98\}$.

$v =$ 102: $\{6,14,24\}$, $\{10,58,73\}$, $\{3,7,12\}$, $\{15,43,68\}$, $\{9,26,40\}$, $\{17,49,82\}$, $\{8,11,27\}$, $\{13,19,39\}$, $\{16,23,45\}$, $\{18,61,74\}$, $\{21,65,66\}$, $\{22,52,64\}$, $\{25,48,72\}$, $\{31,42,69\}$, $\{32,98,100\}$, $\{35,56,97\}$. □

Theorem 5 *If there exists a perfect base in* Z_v *where* $v = 14t + 3$ *(*$t \geq 2$*), then there exists a* (K_{2v+1}, G_{16})*-design.*

Proof: Let $S = \{B_1, B_2, \cdots, B_s\}$ be a perfect base, where $B_i = \{x_i, y_i, z_i\}$ $(i = 1, 2, \cdots, s)$ and $s = |S| = \lfloor \frac{v-1}{6} \rfloor \geq 2t + 1$.

Let $X = (Z_v \times Z_2) \cup \{\infty\}$. Next we will construct a (K_{2v+1}, G_{16})-design on vertices set X. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be as follows:

$$\mathcal{B}_1 = \{\{0_1, (x_i)_0, (y_i)_0, (z_i)_0\} : \quad i = 1, 2, \cdots, 2t + 1\},$$
$$\mathcal{B}_2 = \{\{0_0, (x_i)_1, (y_i)_1, (z_i)_1\} : \quad i = 1, 2, \cdots, 2t\}.$$

We can assume that

$$R_{00} = (Z_v \setminus \{0\}) \setminus \Delta_{00}(\mathcal{B}_1) = \{\pm a_1, \pm a_2, \cdots, \pm a_{t-2}\},$$
$$R_{11} = (Z_v \setminus \{0\}) \setminus \Delta_{11}(\mathcal{B}_2) = \{\pm b_1, \pm b_2, \cdots, \pm b_{t+1}\},$$
$$R_{01} = (Z_v) \setminus \Delta_{01}(\mathcal{B}_1 \cup \mathcal{B}_2)$$
$$= \{c_1, c_2, \cdots, c_{t-1}\} \cup \{d_1, d_2, \cdots, d_{t+1}\}.$$

where Δ_{01} denotes the (0,1)-mixed difference and Δ_{ii} ($i \in Z_2$) denotes the (i,i)-pure difference. Let $\mathcal{A}$ consist of the following base G_{16}-blocks:

part 1. $\{\infty; 0_0, (x_1)_1, (y_1)_1, (z_1)_1\}$,

$\{\infty; 0_1, (x_1)_0, (y_1)_0, (z_1)_0\}$;

part 2. $\{(a_i)_0; 0_0, (x_{i+1})_1, (y_{i+1})_1, (z_{i+1})_1\}$, $i = 1, 2, \cdots, t-2$;

part 3. $\{(b_i)_1; 0_1, (x_{i+1})_0, (y_{i+1})_0, (z_{i+1})_0\}$, $i = 1, 2, \cdots, t+1$;

part 4. $\{(c_i)_0; 0_1, (x_{i+t+2})_0, (y_{i+t+2})_0, (z_{i+t+2})_0\}$, $i = 1, 2, \cdots, t-1$;

part 5. $\{(-d_i)_1; 0_0, (x_{i+t-1})_1, (y_{i+t-1})_1, (z_{i+t-1})_1\}$, $i = 1, 2, \cdots, t+1$.

It is readily checked that $\Delta_{00}(\mathcal{A}) = Z_v \setminus \{0\}$, $\Delta_{11}(\mathcal{A}) = Z_v \setminus \{0\}$ and $\Delta_{01}(\mathcal{A}) = Z_v$. Hence, $(X, dev(\mathcal{A}))$ is a (K_{2v+1}, G_{16})-design. □

Theorem 6 *If there exists a perfect base in Z_v where $v = 14t + 4$ ($t \geq 2$), then there exists a (K_{2v}, G_{16})-design.*

Proof: Let $S = \{B_1, B_2, \cdots, B_s\}$ be a perfect base, where $B_i = \{x_i, y_i, z_i\}$ ($i = 1, 2, \cdots, s$) and $s = |S| = \lfloor \frac{v-2}{6} \rfloor \geq 2t + 1$.

Let $X = Z_v \times Z_2$. Next we will construct a (K_{2v}, G_{16})- design on vertices set X. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be as follows:

$$\mathcal{B}_1 = \{\{0_1, (x_i)_0, (y_i)_0, (z_i)_0\} : \quad i = 1, 2, \cdots, 2t + 1\},$$
$$\mathcal{B}_2 = \{\{0_0, (x_i)_1, (y_i)_1, (z_i)_1\} : \quad i = 1, 2, \cdots, 2t\}.$$

We can assume that

$$R_{00} = (Z_v \setminus \{0, \tfrac{v}{2}\}) \setminus \Delta_{00}(\mathcal{B}_1) = \{\pm a_1, \pm a_2, \cdots, \pm a_{t-2}\},$$
$$R_{11} = (Z_v \setminus \{0, \tfrac{v}{2}\}) \setminus \Delta_{11}(\mathcal{B}_2) = \{\pm b_1, \pm b_2, \cdots, \pm b_{t+1}\},$$
$$R_{01} = (Z_v) \setminus \Delta_{01}(\mathcal{B}_1 \cup \mathcal{B}_2)$$
$$= \{c_1, c_2, \cdots, c_t\} \cup \{d_1, d_2, \cdots, d_{t+1}\}.$$

Let $\mathcal{A}$ consist of the following base G_{16}-blocks:

part 1. $\{(a_i)_0; 0_0, (x_i)_1, (y_i)_1, (z_i)_1\}$, $i = 1, 2, \cdots, t-2$;

part 2. $\{(b_i)_1; 0_1, (x_i)_0, (y_i)_0, (z_i)_0\}$, $i = 1, 2, \cdots, t+1$;

part 3. $\{(c_i)_0; 0_1, (x_{i+t+1})_0, (y_{i+t+1})_0, (z_{i+t+1})_0\}$, $i = 1, 2, \cdots, t$;

part 4. $\{(-d_i)_1; 0_0, (x_{i+t-2})_1, (y_{i+t-2})_1, (z_{i+t-2})_1\}$, $i = 1, 2, \cdots, t+1$.

Then $\Delta_{00}(\mathcal{A}) = Z_v \backslash \{0, \frac{v}{2}\}$, $\Delta_{11}(\mathcal{A}) = Z_v \backslash \{0, \frac{v}{2}, \pm(x_{2t} - y_{2t}), \pm(x_{2t} - z_{2t}), \pm(y_{2t} - z_{2t})\}$ and $\Delta_{01}(\mathcal{A}) = Z_v \setminus \{x_{2t}, y_{2t}, z_{2t}\}$.

Let $\mathcal{B}$ consist of the following base G_{16}- blocks:

(1) $\{(\frac{v}{2} + k)_0; (0 + k)_0, (x_{2t} + k)_1, (y_{2t} + k)_1, (z_{2t} + k)_1\}$, where $k = 0, 1, 2, \cdots, \frac{v}{2} - 1$.

(2) $\{(x_{2t}+k)_1; (x_{2t}+\frac{v}{2}+k)_1, (\frac{v}{2}+k)_0, (y_{2t}+\frac{v}{2}+k)_1, (z_{2t}+\frac{v}{2}+k)_1\}$, where $k = 0, 1, 2, \cdots, \frac{v}{2} - 1$.

Firstly, the five elements in each block of (2) are different. (If $x_{2t} = y_{2t} + \frac{v}{2}$, then change the blocks in (2) by the blocks in (2') as follows:

(2') $\{(z_{2t}+k)_1; (z_{2t}+\frac{v}{2}+k)_1, (\frac{v}{2}+k)_0, (x_{2t}+\frac{v}{2}+k)_1, (y_{2t}+\frac{v}{2}+k)_1\}$, where $k = 0, 1, 2, \cdots, \frac{v}{2} - 1$.

Then the five elements in each block of (2') are different. Otherwise, if $z_{2t} = x_{2t} + \frac{v}{2}$ or $z_{2t} = y_{2t} + \frac{v}{2}$, it implies that $z_{2t} = y_{2t}$ or $z_{2t} = x_{2t}$ (since $x_{2t} = y_{2t} + \frac{v}{2}$). It is impossible by the definition of perfect base.)

It is readily checked that $(X, dev(\mathcal{A} \cup \mathcal{B}))$ is a (K_{2v}, G_{16})-design. □

Theorem 7 *The necessary condition* $n \equiv 0, 1 \pmod 7$, $n > 8$ *for the existence of a* (K_n, G_{16})*- design is also sufficient.*

Proof: It follows by Lemma 1, Theorem 5 and Theorem 6. □

3 A direct construction for (K_{65}, G_{20})-design

In this section we will give a direct construction for (K_{65}, G_{20})-design, where $G_{20} = (V, E)$, $V = \{a, b, c, d, e\}$ and $E = \{$ *ab*, *ac*, *bc*, *bd*, *be*, *cd*, *ce*, *de* $\}$. We usually denote G_{20} as $(a : b, c, d, e)$.

Lemma 8 *There exists a* (K_{65}, G_{20})*-design.*

Proof: Let $\mathcal{B}$ consist of the following base G_{20}-blocks on Z_{65}:

$(0:57,7,2,27)$; $(0:49,26,22,28)$;

$(0:24,52,21,33)$; $(0:17,64,28,50)$.

It is readily checked that $(Z_{65}, dev(\mathcal{B}))$ is a (K_{65}, G_{20})- design. □

Theorem 9 *The necessary condition* $n \equiv 0,1 \pmod{16}$ *for the existence of a* (K_n, G_{20})*- design is also sufficient.*

Proof: It follows immediately by Lemma 2 and Lemma 8. □

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