Embeddings of Steiner Quadruple Systems

Chester W.J. Liu
Department of Mathematics, Royal Holloway, University of London, Egham Hill, Egham, Surrey, TW20 0EX, UK
and
Peter R. Wild
Department of Mathematics, Royal Holloway, University of London, Egham Hill, Egham, Surrey, TW20 0EX, UK
p.wild@rhbnc.ac.uk

Abstract

We generalize a construction by Treash of a Steiner triple system on $2v+1$ points that embeds a Steiner triple system on v points. We show that any Steiner quadruple system on $v+1$ points may be embedded in a Steiner quadruple system on $2v+2$ points.

Key words: Steiner Quadruple System, embedding, substructure.

1 Introduction

A *Steiner system* $S(t,k;v)$ is a pair $(P,\mathcal{A})$ where P is a v-set and $\mathcal{A}$ is a collection of k-subsets of P such that every t-subset of P is contained in exactly one member of $\mathcal{A}$. The elements of P are called points and the elements of $\mathcal{A}$ are called blocks. The cases $t=2$, $k=3$ and $t=3$, $k=4$ are called *Steiner triple system* (STS) and *Steiner quadruple system* (SQS) respectively. An STS with $|P|=v$ is said to be of *order* v and is referred to as an STS(v). Similarly an SQS with $|P|=v$ is referred to as an SQS(v). Kirkman [3] proved that there exists an STS(v) if and only if $v \equiv 1$ or $3 \pmod 6$. Hanani [2] proved that there exixts an SQS(v) if and only if $v \equiv 2$ or $4 \pmod 6$.

The internal structure of an SQS, $(P,\mathcal{A})$, at a point $p \in P$ is an incidence structure with point set $P' = P\backslash\{p\}$ and block set $\mathcal{A}'$ consisting of those triples of points of P' which together with p form a block belonging to $\mathcal{A}$. The structure $(P', \mathcal{A}')$ is an STS. An SQS is an extension of an STS if the STS is isomorphic to the internal structure of the SQS at some point of the SQS. An extension of an STS(u) is an SQS($u+1$).

If $(P, \mathcal{A})$ and $(Q, \mathcal{B})$ are two Steiner systems such that $P \subseteq Q$ and $\mathcal{A} \subseteq \mathcal{B}$, then we say $(P, \mathcal{A})$ is embedded in $(Q, \mathcal{B})$ and that $(Q, \mathcal{B})$ contains $(P, \mathcal{A})$ as a substructure. If an STS(u) is embedded in an STS(v) then $v \geq 2u+1$ [1]. It is immediate from this that if an SQS($u+1$) is embedded in an SQS(v) then $v \geq 2u+2$. Treash [4] has shown, by construction, that every STS(u) is embedded in an STS($2u+1$). In this paper, we generalize Treash's construction to show that every SQS($u+1$) is embedded in an SQS($2u+2$).

Treash's construction :
Let $(P, \mathcal{A})$ be an STS(v). Consider the set $Q = (P \times \{1,2\}) \cup \{\infty\}$ where $\infty \notin P \times \{1,2\}$, and define a collection $\mathcal{B} = \mathcal{B}_\infty \cup \mathcal{B}^+$ of triples of Q as follows:

(i) $\mathcal{B}_\infty$ consists of the triples $\{(x,1),(x,2),\infty\}$ for $x \in P$,

(ii) for every block $A = \{x,y,z\} \in \mathcal{A}$, put

$$\begin{aligned}\mathcal{B}_A \;=\; & \{\{(x,1),(y,1),(z,1)\},\ \{(x,1),(y,2),(z,2)\},\\ & \{(x,2),(y,1),(z,2)\},\ \{(x,2),(y,2),(z,1)\}\}\end{aligned}$$

and then $\mathcal{B}^+ = \cup_{A \in \mathcal{A}} \mathcal{B}_A$.

Then $(Q, \mathcal{B})$ is an STS($2v+1$).

Note that for each block $\{x,y,z\} \in \mathcal{A}$, $(Q, \mathcal{B})$ contains a substructure isomorphic to $S(2,3;7)$ consisting of the point set

$$\{(x,1),(x,2),(y,1),(y,2),(z,1),(z,2),\infty\}$$

and the blocks of $\mathcal{B}$ contained in this set. We see that $(Q, \mathcal{B})$ is a Steiner triple system which contains an isomorphic copy of $(P, \mathcal{A})$, since $\{x,y,z\} \in \mathcal{A}$ if and only if $\{(x,1),(y,1),(z,1)\} \in \mathcal{B}$. By a change of notation, writing x for $(x,1)$ and x' for $(x,2)$, we may assume that $P \subseteq Q$ and that $(P, \mathcal{A})$ is embedded in $(Q, \mathcal{B})$. We adopt this notation for the remainder of the paper. This construction embeds an STS(v) in an STS($2v+1$).

2 Main Result

Let $(P', \mathcal{A}')$ be an $S(2,3;u)$ and let it be embedded in the $S(2,3;2u+1)$, $(Q', \mathcal{B}')$, by the construction of Treash. Suppose $(P, \mathcal{A})$ is an extension of $(P', \mathcal{A}')$ where $P = P' \cup \{p\}$ and $p \notin P'$. We construct $(Q, \mathcal{B})$, an $S(3,4;2u+2)$, such that $(P, \mathcal{A})$ is embedded in $(Q, \mathcal{B})$, which is also an extension of $(Q', \mathcal{B}')$.

The construction :
$(P', \mathcal{A}')$ has $\frac{u(u-1)}{6}$ blocks. Consider the construction of $(Q', \mathcal{B}')$ from $(P', \mathcal{A}')$ by Treash's construction [4]. We have the collection $\mathcal{B}'$ of triples of Q' as follows:

(i) ${\mathcal{B}'}_\infty = \{\{x, x', \infty\} | x \in P'\}$;

(ii) for $A' = \{x, y, z\} \in \mathcal{A}'$,
${\mathcal{B}'}_{A'} = \{\{x, y, z\}, \{x, y', z'\}, \{x', y, z'\}, \{x', y', z\}\}$;

and

$$\mathcal{B}' = {\mathcal{B}'}_\infty \cup (\cup_{A' \in \mathcal{A}'} {\mathcal{B}'}_{A'}) .$$

Clearly, $(Q', \mathcal{B}')$ contains $(P', \mathcal{A}')$ and there are

$$u + \frac{u(u-1)}{6} \cdot 4 = \frac{(2u+1)u}{3}$$

blocks in $\mathcal{B}'$.

Consider the extension $(P, \mathcal{A})$ of $(P', \mathcal{A}')$. We have $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where $\mathcal{A}_1$ is the set of quadruples containing p, i.e.

$$\mathcal{A}_1 = \{A' \cup \{p\} | A' \in \mathcal{A}'\} ;$$

and $\mathcal{A}_2$ is the set of quadruples not containing p. For each $\{x, y, z, w\} \in \mathcal{A}_2$, we have $\{x, y, z\}$, $\{x, y, w\}$, $\{y, z, w\}$, $\{x, z, w\} \notin \mathcal{A}'$. Since $(P, \mathcal{A})$ has $\frac{(u+1)u(u-1)}{24}$ blocks,

$$|\mathcal{A}_2| = |\mathcal{A}| - |\mathcal{A}'| = \frac{u(u-1)(u-3)}{24} .$$

Consider the set $Q = Q' \cup \{p\}$. Define the following types of blocks,

(a) triples in $\mathcal{B}'$ with p adjoined, i.e.

$$\mathcal{B}_p = \{B' \cup \{p\} | B' \in \mathcal{B}'\} ;$$

(b1) for $A' = \{x, y, z\} \in \mathcal{A}'$, put

$$\mathcal{B}^1_{A'} = \{\{x', y', z', \infty\}, \{x, y, z', \infty\}, \{x, y', z, \infty\}, \{x', y, z, \infty\}\};$$

(b2) for $A' = \{x, y, z\} \in \mathcal{A}'$, put

$$\mathcal{B}^2_{A'} = \{\{x, x', y, y'\}, \{x, x', z, z'\}, \{y, y', z, z'\}\};$$

(c) for $A = \{x, y, z, w\} \in \mathcal{A}_2$, put

$$\mathcal{B}_A = \{\{x, y, z, w\}, \{x', y', z', w'\}, \{x', y', z, w\}, \{x', y, z', w\},$$
$$\{x', y, z, w'\}, \{x, y', z', w\}, \{x, y', z, w'\}, \{x, y, z', w'\}\}.$$

Put

$$\mathcal{B} = \mathcal{B}_p \cup (\cup_{A' \in \mathcal{A}'} \mathcal{B}^1_{A'}) \cup (\cup_{A' \in \mathcal{A}'} \mathcal{B}^2_{A'}) \cup (\cup_{A \in \mathcal{A}_2} \mathcal{B}_A).$$

There are

- $\frac{(2u+1)u}{3}$ blocks of type (a);
- $\frac{u(u-1)}{6} \cdot 4 = \frac{2u(u-1)}{3}$ blocks of type (b1);
- $\frac{u(u-1)}{6} \cdot 3 = \frac{u(u-1)}{2}$ blocks of type (b2);
- $\frac{u(u-1)(u-3)}{24} \cdot 8 = \frac{u(u-1)(u-3)}{3}$ blocks of type (c).

Thus we have the required number, $\frac{(u+1)(2u+1)u}{6}$, of blocks as the total number of blocks of $(Q, \mathcal{B})$. We claim that $(Q, \mathcal{B})$ is the required SQS$(2u+2)$.

Theorem 2.1 Let $(P, \mathcal{A})$ be an $S(3, 4; u+1)$ and let $p \in P$. Let $(P', \mathcal{A}')$ be the internal structure of $(P, \mathcal{A})$ at p. If $(Q', \mathcal{B}')$ is obtained from $(P', \mathcal{A}')$ by Treash's construction, then $(Q', \mathcal{B}')$ is extendable to an $S(3, 4; 2u+2)$ $(Q, \mathcal{B})$ such that $(P, \mathcal{A})$ is embedded in $(Q, \mathcal{B})$.

Proof Let $(Q, \mathcal{B})$ arise from the construction described above. We partition the triples of points in Q into the following classes:

(i) triples that contain p;

(ii) triples that contain ∞ but not p;

(iii) triples from $Q\backslash\{\infty, p\}$.

We show that each triple belongs to a unique block of $\mathcal{B}$:

(i) a triple containing p is on the unique block in $\mathcal{B}_p$ corresponding to the unique block $B' \in \mathcal{B}'$ of the STS$(2u+1)$ $(Q', \mathcal{B}')$ that contains the two points of the triple other than p;

(ii) triples containing ∞ of the form $\{x, y, \infty\}$, $\{x', y', \infty\}$, $\{x, y', \infty\}$ where $x, y \in P'$ are only on the corresponding block of type (b1) in $\mathcal{B}^1_{A'}$ where $A' \in \mathcal{A}'$ is the unique block containing x and y, and clearly the remaining possible form $\{x, x', \infty\}$ of a triple containing ∞ is on the block $\{x, x', \infty, p\}$ of type (a);

(iii) let $\{x, y, z\}$ be a triple where $x, y, z \in P'$:

- if x, y, z are not on a line in $(P', \mathcal{A}')$, then $\{x, y, z\}$ is on the block $\{x, y, z, w\}$ of type (c) where $\{x, y, z, w\}$ is the unique quadruple of $\mathcal{A}_2$ that contains x, y, z;
- if x, y, z are on a line in $(P', \mathcal{A}')$, then $\{x, y, z\}$ is on the block $\{x, y, z, p\}$ of type (a);

let $\{x', y', z'\}$ be a triple where $x, y, z \in P'$:

- if x, y, z are not on a line in $(P', \mathcal{A}')$, then $\{x', y', z'\}$ is on the block $\{x', y', z', w'\}$ of type (c) where $\{x, y, z, w\}$ is the unique quadruple of $\mathcal{A}_2$ that contains x, y, z;
- if x, y, z are on a line in $(P', \mathcal{A}')$, then $\{x', y', z'\}$ is on the block $\{x', y', z', \infty\}$ of type (b1);

let $\{x, y, z'\}$ be a triple where $x, y, z \in P'$:

- if x, y, z are not on a line in $(P', \mathcal{A}')$, then $\{x, y, z'\}$ is on the block $\{x, y, z', w'\}$ of type (c) where $\{x, y, z, w\}$ is the unique quadruple of $\mathcal{A}_2$ that contains x, y, z;
- if x, y, z are on a line in $(P', \mathcal{A}')$, then $\{x, y, z'\}$ is on the block $\{x, y, z', \infty\}$ of type (b1);

let $\{x, y', z'\}$ be a triple where $x, y, z \in P'$:

- if x, y, z are not on a line in $(P', \mathcal{A}')$, then $\{x, y', z'\}$ is on the block $\{x, y', z', w\}$ of type (c) where $\{x, y, z, w\}$ is the unique quadruple of $\mathcal{A}_2$ that contains x, y, z;
- if x, y, z are on a line in $(P', \mathcal{A}')$, then $\{x, y', z'\}$ is on the block $\{x, y', z', p\}$ of type (a);

the triples of the form $\{x, x', y\}$ and $\{x, x', y'\}$ where $x, y \in P'$ are contained in the block $\{x, x', y, y'\}$ in $\mathcal{B}^2_{A'}$ where $A' = \{x, y, z\}$ is the unique block of $\mathcal{A}'$ containing x, y.

We have listed the blocks containing the various triples of points and shown that each triple is contained in exactly one block. Thus $(Q, \mathcal{B})$ is an $S(3, 4; 2u + 2)$. It contains $(P, \mathcal{A})$ since the blocks in $\mathcal{A}_1$ are of type (a) as $\mathcal{A}' \subseteq \mathcal{B}'$ and the blocks in $\mathcal{A}_2$ are of type (c). Also, clearly, by the construction of blocks of type (a), $(Q, \mathcal{B})$ is an extension of $(Q', \mathcal{B}')$.

■

3 Conclusion

Our construction has other similarities to that of Treash, apart from the embedding of one Steiner system in another of minimal size. For each quadruple $\{x, y, z, p\}$ of $(P, \mathcal{A})$ that contains p, $(Q, \mathcal{B})$ contains an SQS(8) as a substructure. The points of this substructure are $\{x, x', y, y', z, z', \infty, p\}$. The blocks are the 7 blocks of type (a) corresponding to the blocks of the STS(7) which is the substructure of $(Q', \mathcal{B}')$ arising from $\{x, y, z\} \in \mathcal{A}'$ and the 4 blocks of type (b1) and 3 blocks of type (b2) arising from $\{x, y, z\}$. Similarly, for each quadruple $\{x, y, z, w\}$ of $(P, \mathcal{A})$ that does not contain p, $(Q, \mathcal{B})$ contains an SQS(8) as a substructure. The points of this substructure are $\{x, x', y, y', z, z', w, w'\}$. The blocks are the 8 blocks of type (c) arising from the block $\{x, y, z, w\} \in \mathcal{A}_2$ and 6 blocks of type (b2) of the form $\{\alpha, \alpha', \beta, \beta'\}$ where α, β are distinct elements of $\{x, y, z, w\}$.

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