Radio Labelings of Cycles

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Abstract

A radio labeling of a connected graph G is an assignment of distinct positive integers to the vertices of G, with $x \in V(G)$ labeled $c(x)$, such that

$$d(u,v) + |c(u) - c(v)| \geq 1 + \operatorname{diam} G$$

for every two distinct vertices u, v of G, where $\operatorname{diam} G$ is the diameter of G. The radio number $rn(c)$ of a radio labeling c of G is the maximum label assigned to a vertex of G. The radio number $rn(G)$ of G is $\min\{rn(c)\}$ over all radio labelings c of G. Radio numbers of cycles are discussed and upper and lower bounds are presented.

Key Words: radio labeling, radio number.
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1 Introduction

For a vertex v of a connected graph G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* G, and the maximum eccentricity is its *diameter*, *diam* G. A *labeling* of a connected graph is an injection $c : V(G) \to \mathbf{N}$, while a *radio labeling* is a labeling with the added property that

$$d(u,v) + |c(u) - c(v)| \geq 1 + \operatorname{diam} G$$

for every two distinct vertices u, v of G. The *radio number* $rn(c)$ *of a radio labeling* c of G is the maximum label assigned to a vertex of G. The *radio number* $rn(G)$ of G is $\min\{rn(c)\}$ over all radio labelings c of G. A radio labeling c of G is a *minimum radio labeling* if $rn(c) = rn(G)$.

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Let G be a connected graph and k an integer such that $1 \leq k \leq \operatorname{diam} G$. A *radio k-coloring* of G is an assignment of colors (positive integers) to the vertices of G such that

$$d(u, v) + |c(u) - c(v)| \geq 1 + k$$

for every two distinct vertices u, v of G. The minimum of the maximum colors among all radio k-colorings of G is the *radio k-chromatic number* $rc_k(c)$ of G. Let $d = \operatorname{diam} G$. Then $rc_d(G)$ is the radio number, $rc_{d-1}(G)$ is the radio antipodal chromatic number, and $rc_1(G)$ is the classic chromatic number of G. The radio number of a graph has been studied in [1, 2, 6] and the radio antipodal chromatic number has been studied in [3]. We refer to the books [4, 5] for graph theory notation and terminology not described here.

First we make an observation about radio labelings. Let G be a connected graph with $rn(G) = k$. In any radio labeling c of G with $rn(c) = k$, certainly some vertex of G is assigned the label k. Also, some vertex of G is labeled 1, for otherwise the new labeling obtained from c by replacing $c(v)$ by $c(v) - 1$ for each vertex v of G is a radio labeling of G as well, contradicting the fact that $rn(G) = k$. That is, if c is a radio labeling of G with $rn(c) = rn(G)$, then there exist vertices u and v of G with $c(u) = 1$ and $c(v) = k$.

For integers a and b with $a \leq b$, the set $[a..b]$ is defined as $\{x \in \mathbf{Z} \mid a \leq x \leq b\}$. A set S of positive integers is a *radio labeling set* if the elements of S are used in a radio labeling of some graph G and S is a *minimum radio labeling set* if S is a radio labeling set of a minimum radio labeling of some graph G. Thus if S is a minimum radio labeling set for a graph G order n with $rn(G) = k$, then $|S \cap [2..k-1]| = n - 2$.

To illustrate these concepts, consider the graph G of Figure 1(a). Since $\operatorname{diam} G = 3$, it follows that in any radio labeling of G, the labels of every two adjacent vertices must differ by at least 3 and the labels of every two vertices whose distance is 2 must differ by at least 2. Thus the labeling of G given in Figure 1(b) is a radio labeling. Consequently, $rn(G) \leq 8$.

Since there are exactly two vertices of the graph G of Figure 1 whose distance is 3, namely u and x, these are the only vertices that can be labeled with consecutive integers. Thus at most one of the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ can be used as labels in a radio labeling of G. Since the order of G is 5, it follows that $rn(G) \geq 7$. On the other hand, $rn(G) \neq 7$, for assume, to the contrary, that there is a radio labeling c of G with $rn(c) = rn(G) = 7$. Since exactly two of the integers 2, 3, 4, 5, 6 are not used in this labeling, either three consecutive integers in $\{1, 2, \cdots, 7\}$ are labels in c or two pairs of consecutive integers are labels, both of which are impossible. Therefore,

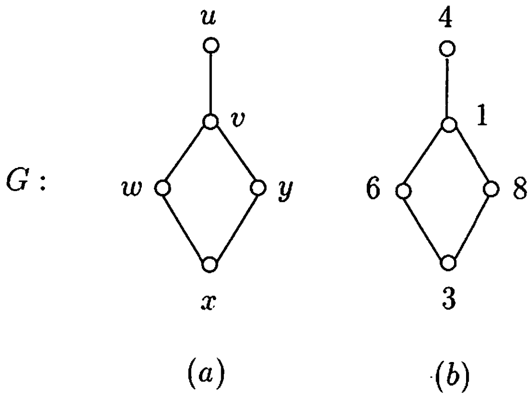

Figure 1: A radio labeling of a graph

$rn(G) = 8$, the labeling given in Figure 1(b) is a minimum radio labeling, and $\{1, 3, 4, 6, 8\}$ is a minimum radio labeling set for G.

Another observation will be useful. Let c be a radio labeling of a graph G with $rn(c) = k$ and let $v \in V(G)$ with $c(v) = k$. For each integer $k' > k$, define a new labeling c' by $c'(v) = k'$ and $c'(u) = c(u)$ for all $u \in V(G) - \{v\}$. Then c' is a radio labeling of G with $rn(c') = k'$. Thus if c is a radio labeling of G with $rn(c) = k$, then for each integer $k' > k$, there exists a a radio labeling c' of G with $rn(c') = k'$.

In [1] the radio numbers of cycles were studied and upper and lower bounds of these numbers were presented. In this paper, we establish improved upper and lower bounds for the radio numbers of cycles in general. Moreover, we will also determine the radio numbers of certain cycles.

2 Bounds for Radio Numbers of Cycles

An upper bound for the radio number of a cycle C_n of order $n \geq 6$ in terms of its diameter was given in [1], which we state below.

Theorem A *Let d be an integer.*

(a) *If $d \geq 3$, then $rn(C_{2d}) \leq d^2 - d + 2$.*

(b) *If $d \geq 2$, then $rn(C_{2d+1}) \leq d^2 + 1$.*

It was shown in [1] that the bounds for radio numbers of C_n in Theorem A are attained for $5 \leq n \leq 8$. Radio labelings of C_n for $3 \leq n \leq 8$ are shown in Figure 2. In fact, all these radio labelings are minimum radio labelings of the respective cycles. Thus, $rn(C_3) = 3$, $rn(C_4) = 5$, $rn(C_5) = 5$,

$rn(C_6) = 8$, $rn(C_7) = 10$, and $rn(C_8) = 14$. On the other hand, the bounds in Theorem A are not sharp for $n \geq 9$. Let $C_n : v_1, v_2, \cdots, v_n, v_1$ be the cycle of order $n \geq 3$. We now present an improved upper bound for $rn(C_n)$ for odd integers $n \geq 9$.

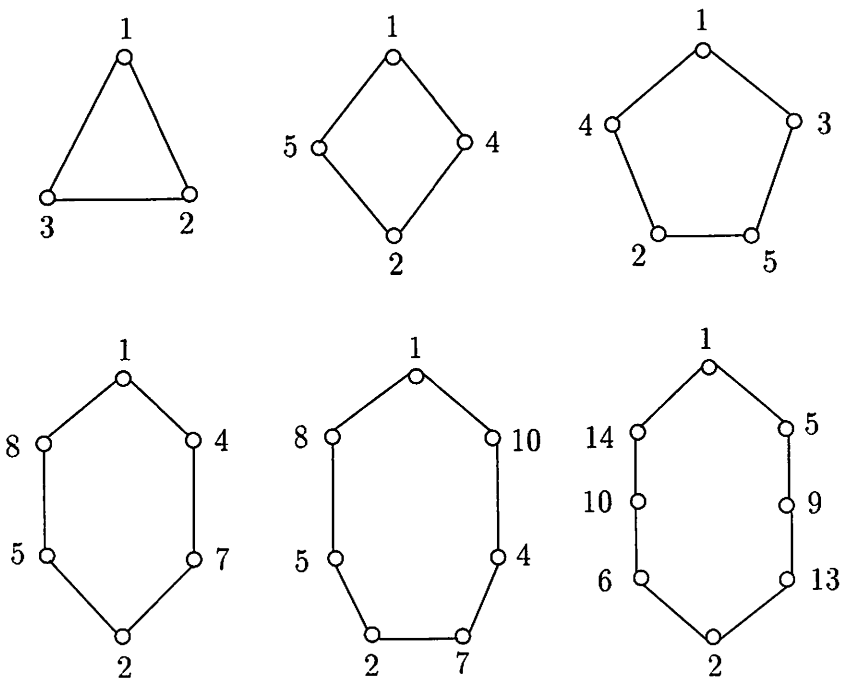

Figure 2: Minimum radio labelings of C_n for $3 \leq n \leq 8$

Proposition 2.1 *Let $d \geq 3$ be an integer. Then*

$$rn(C_{2d+1}) \leq d^2 - \lfloor d/2 \rfloor + 2.$$

Proof. Define a radio labeling $c : V(C_{2d}) \to \mathbf{N}$ by $c(v_i) = (i-1)d+1$ and $c(v_{d+i}) = (i-1)d+2$ for all i with $1 \leq i \leq d$. Now let C_{2d+1} be obtained from the cycle C_{2d} by subdividing the edge $v_{\lfloor d/2 \rfloor} v_{\lfloor d/2 \rfloor + 1}$ with a new vertex v. Since $\operatorname{diam} C_{2d+1} = \operatorname{diam} C_{2d} = d$, we can extend the radio labeling c of C_{2d} described above to a labeling c' of C_{2d+1} by defining $c'(v_i) = c(v_i)$ for all $1 \leq i \leq 2d$ and $c'(v) = d^2 - d + 2 + \lceil d/2 \rceil = d^2 - \lfloor d/2 \rfloor + 2$. To illustrate this labeling, consider the radio labelings of C_n for C_8 and C_9 shown in

Figure 3, where the radio labeling of C_9 is obtained from the radio labeling of C_8 using the described labeling technique.

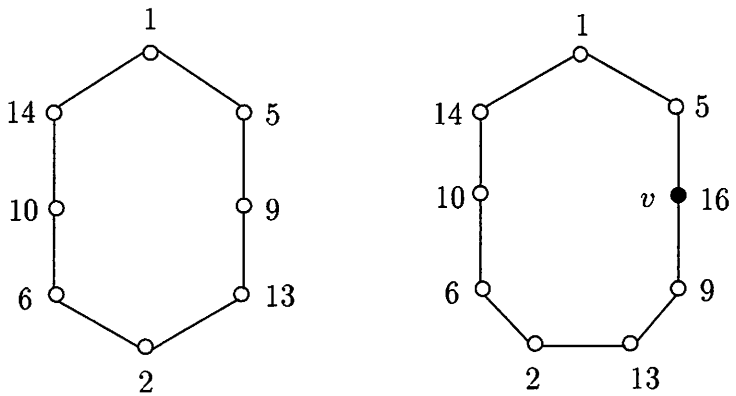

Figure 3: Radio labeling of C_8 and C_9

Next we show that c' is a radio labeling of C_{2d+1}. The only possibilities for a vertex u for which $|c(u) - c(v)| < d$ are $u = v_d$ or $u = v_{2d}$ as $c(v_d) = d^2 - d + 1$ and $c(v_{2d}) = d^2 - d + 2$. If $u = v_d$, then $|c(u) - c(v)| = \lceil d/2 \rceil + 1$ and $d(u, v) = \lfloor d/2 \rfloor$; while if $u = v_{2d}$, then $|c(u) - c(v)| = \lceil d/2 \rceil$. and $d(u, v) = \lceil d/2 \rceil + 1$. Thus, in each case, $d(u, v) + |c(u) - c(v)| \geq d + 1$ and so c' is a radio labeling of C_{2d+1}. Since $rn(c') = c'(v)$, it follows that

$$rn(C_{d+1}) \leq rn(c') = d^2 - \lfloor d/2 \rfloor + 2,$$

as desired. ■

A lower bound for radio numbers of cycles was established in [1], which we state as follows.

Theorem B *For $n \geq 6$,*

$$rn(C_n) \geq 3 \left\lceil \frac{n}{2} - 1 \right\rceil .$$

In order to present an improved lower bound for radio numbers of cycles, we first study some important properties of a radio labeling of a cycle. Observe that three consecutive colors, namely, 1, 2, 3, are used in the radio labelings of C_3 and C_5 shown in Figure 2. This cannot occur for C_n when $n = 4$ or $n \geq 6$, as we show next.

Lemma 2.2 *If c is a radio labeling of a cycle C_n, where $n = 4$ or $n \geq 6$, then c cannot use three consecutive labels from* $\mathbf{N}$.

Proof. Assume, to the contrary, that there is a radio labeling c of C_n, where $n = 4$ or $n \geq 6$, such that $a, a+1, a+2$ are in the label set C of c for some positive integer a. If $n = 4$, then there exist three consecutive vertices, say v_1, v_2, and v_3, in C_4 such that $\{c(v_1), c(v_2), c(v_3)\} = \{a, a+1, a+2\}$. This implies that there exist at least two consecutive vertices in C_4 assigned two consecutive labels, which is a contradiction.

Now assume that $n \geq 6$ and, without loss of generality, that $c(v_1) = a+1$. If $n = 2k$ for some integer $k \geq 3$, then v_k is the only vertex of C_n that can be labeled by a or $a+2$, which is impossible. If $n = 2k+1$ for some integer $k \geq 3$, then $\{c(v_k), c(v_{k+1})\} = \{a, a+2\}$. Then $d(v_k, v_{k+1}) + |c(v_k) - c(v_{k+1})| = 3 < 1+k$ for $k \geq 3$, which is a contradiction. ∎

Theorem 2.3 *Let $k \geq 2$ be an integer and let c be a radio labeling of a cycle C_n.*

(i) If n is even and $n \geq 4k+2$, then for every positive integer a at most two vertices of C_n are labeled with elements of the set $[a \,..\, a+k+1]$.

(ii) If n is odd and $n \geq 4k-1$, then for every positive integer a at most two vertices of C_n are labeled with elements of the set $[a \,..\, a+k]$.

Proof. We proceed by induction on k. First, we verify (i). Let $k = 2$. Assume, to the contrary, that there is a radio labeling c of C_n, where $n = 2d \geq 4k+2 = 10$ for some integer d, such that at least three elements from $\{a, a+1, a+2, a+3\}$ are in the label set C of c for some positive integer a. It then follows by Lemma 2.2 that exactly three elements from $\{a, a+1, a+2, a+3\}$ are in C and these elements are not consecutive, say $a, a+1, a+3$ are in C. Assume, without loss of generality, that $c(v_1) = a$ and $c(v_{d+1}) = a+1$. Also, we may assume that $c(v_i) = a+3$ for some i with $1 \leq i \leq d$. Since $c(v_i) - c(v_{d+1}) = 2$ and c is a radio labeling, $d(v_i, v_{d+1}) = d+1-i \geq 1+d-2$, implying that $i = 2$. However, then, $|c(v_1) - c(v_2)| + d(v_1, v_2) = 4 < d$, which is a contradiction.

Assume now that the statement is true for $k-1$, where $k \geq 3$, that is, we assume that if c is a radio labeling of C_n, where n is even and $n \geq 4(k-1)+2$, then for every positive integer a, at most two vertices of C_n are labeled with elements of the set $[a \,..\, a+k]$. Let c be a radio labeling of a cycle C_n, where n is even, $n \geq 4k+2$, and $k \geq 3$. Let $n = 2d$ and so $d \geq 2k+1$. Assume, to the contrary, that there exist three vertices u, v, w in C_n such that $c(u), c(v), c(w) \in [a \,..\, a+k+1]$ for some positive integer a. By the induction hypothesis, we may assume that

$c(u) = a$ and $c(v) = a + k + 1$. Furthermore, let $u = v_1$. Since $c(v_1) = a$, it follows that $v \in \{v_{d+1-k}, v_{d+2-k}, \cdots, v_{d+k+1}\}$. Let $r = \lceil (k+1)/2 \rceil$ and we may assume that $c(w) \in [a+1 .. a+r]$, say $c(w) = a + i$ for some i with $1 \le i \le r$. This implies that $w \in \{v_{d+1-i}, v_{d+2-i}, \cdots, v_{d+i+1}\}$. Since $c(v) = a + k + 1$, it follows that $|c(v) - c(w)| = k + 1 - i$. Note that $v \in \{v_{d+1-k}, v_{d+2-k}, \cdots, v_{d+k+1}\}$ and $w \in \{v_{d+1-i}, v_{d+2-i}, \cdots, v_{d+i+1}\}$. This implies that $d(v, w) \le d(v_{d+1-k}, v_{d+i+1}) = (d+i+1) - (d+1-k) = i+k$. Hence

$$d(v, w) + |c(v) - c(w)| \le (k + 1 - i) + (i + k) = 2k + 1 \le d,$$

which is a contradiction and so (i) holds.

Next we verify (ii). By Lemma 2.2, the statement is true for $k = 2$. Assume now that the statement is true for $k - 1$, where $k \ge 3$, that is, we assume that if c is a radio labeling of C_n, where n is odd and $n \ge 4(k-1)-1$ for some integer $k \ge 3$, then for every positive integer a, at most two vertices of C_n are labeled with elements of the set $[a .. a + k - 1]$. Let c be radio labeling of C_n, where $n = 2d + 1$ is odd, $n \ge 4k - 1$, and $k \ge 3$. So $d \ge 2k - 1$. Assume, to the contrary, that there exist three vertices u, v, w in C_n such that $c(u), c(v), c(w) \in [a .. a + k]$ for some positive integer a. By the induction hypothesis, we may assume that $c(u) = a$ and $c(v) = a + k$. Furthermore, let $u = v_1$. Since $c(v_1) = a$, it follows that $v \in \{v_{d+2-k}, v_{d+3-k}, \cdots, v_{d+1+k}\}$. Let $r = \lceil (k+1)/2 \rceil$ and we may assume that $c(w) \in [a .. a + r]$, say $c(w) = a + i$ for some i with $1 \le i \le r$. Thus $w \in \{v_{d+2-i}, v_{d+3-i}, \cdots, v_{d+1+i}\}$. Since $c(v) = a + k$, it follows that $|c(v) - c(w)| = k - i$. Moreover, since $v \in \{v_{d+2-k}, v_{d+3-k}, \cdots, v_{d+1+k}\}$ and $w \in \{v_{d+2-i}, v_{d+3-i}, \cdots, v_{d+1+i}\}$, we have $d(v, w) \le k + i - 1$. Hence

$$d(v, w) + |c(v) - c(w)| \le (k - i) + (k + i - 1) = 2k - 1 \le d,$$

which is a contradiction and so (ii) holds. ■

As a consequence of Theorem 2.3, we are now in a position to establish a lower bound for $rn(C_n)$ for $n \ge 9$.

Corollary 2.4 *Let $k \ge 2$ be an integer.*

(*i*) $rn(C_{4k+1}) \ge 2k^2 + 2k + 1$,

(*ii*) $rn(C_{4k+2}) \ge 2k^2 + 5k + 2$,

(*iii*) $rn(C_{4k+3}) \ge 2k^2 + 5k + 3$

(*iv*) $rn(C_{4k+4}) \ge 2k^2 + 6k + 4$.

Proof. First, assume that c is a radio labeling of $G = C_{4k+1}$ with $rn(c) = rn(G)$, where $k \ge 2$. Note that c must use the labels 1 and $rn(G)$. Since

$n = 4k+1$ is odd, by Theorem 2.3, the labeling c uses at most two elements from each of the sets $[(i-1)(k+1)+1 \mathinner{..} i(k+1)]$ for all i with $1 \leq i \leq 2k$. This implies that the elements from the set $[1 \mathinner{..} 2k(k+1)]$ can be used to label at most $4k$ vertices of C_{4k+1}. Since all labels of c are distinct,

$$rn(G) = rn(c) \geq (2k)(k+1)+1 = 2k^2+2k+1$$

and so (i) holds.

Assume that c is a radio labeling of $G = C_{4k+2}$ with $rn(c) = rn(G)$, where $k \geq 2$. Since $n = 4k+2$ is even, by Theorem 2.3, the labeling c uses at most two elements from each of the sets $[(i-1)(k+2)+1 \mathinner{..} i(k+2)]$ for all i with $1 \leq i \leq 2k+1$. This implies that

$$rn(G) = rn(c) \geq (2k+1)(k+2) = 2k^2+5k+2,$$

which establishes (ii).

Next, assume that c is a radio labeling of $G = C_{4k+3}$ with $rn(c) = rn(G)$, where $k \geq 2$. Since $n = 4k+3 = 4(k+1)-1$, by Theorem 2.3, the labeling c uses at most two elements from each of the sets $[(i-1)(k+2)+1 \mathinner{..} i(k+2)]$ for all i with $1 \leq i \leq 2k+1$. This implies that the elements from the set $[1 \mathinner{..} (2k+1)(k+2)]$ can be used to label at most $4k+2$ vertices of C_{4k+3}. Thus

$$rn(G) = rn(c) \geq (2k+1)(k+2)+1 = 2k^2+5k+3,$$

which establishes (iii).

Finally, assume that c is a radio labeling of $G = C_{4k+4}$ with $rn(c) = rn(G)$, where $k \geq 2$. Since $n = 4k+4 \geq 4k+2$, it then follows by Theorem 2.3 that the labeling c uses at most two elements from each of the sets $[(i-1)(k+2)+1 \mathinner{..} i(k+2)]$ for all i with $1 \leq i \leq 2k+2$. This implies that

$$rn(G) = rn(c) \geq (2k+2)(k+2) = 2k^2+6k+4,$$

establishing (iv). ∎

3 Radio numbers of Certain Cycles

By Corollary 2.4, $rn(C_{10}) \geq 20$ and $rn(C_{11}) \geq 21$. On the other hand, in Figure 4, the radio labeling of C_{10} has the radio number 20 and the radio labeling of C_{11} has the radio number 21. Thus $rn(C_{10}) = 20$ and $rn(C_{11}) = 21$. In this section, we determine the radio numbers of certain cycles. First we show that, for all even $k \geq 2$, the inequality $rn(C_{4k+2}) \geq 2k^2+5k+2$ in Corollary 2.4 is, in fact, equality and, for odd $k \geq 3$, the numbers $rn(C_{4k+2})$ and $2k^2+5k+2$ differ by at most 1.

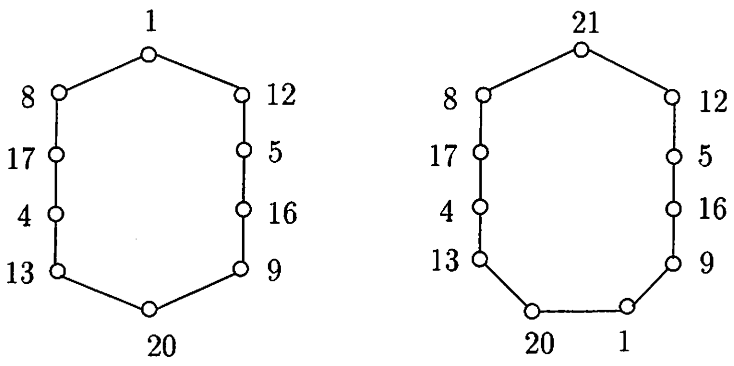

Figure 4: Minimum radio labelings of C_{10} and C_{11}

Theorem 3.1 *Let $k \geq 2$ be an integer.*

(i) If k is even, then $rn(C_{4k+2}) = 2k^2 + 5k + 2$;

(ii) If k is odd, then there is a radio labeling c of C_{4k+2} with $rn(c) = 2k^2 + 5k + 3$.

Proof. We first verify (i). By Corollary 2.4, $rn(C_{4k+2}) \geq 2k^2 + 5k + 2$. Thus it suffices to show that, for each even integer $k \geq 2$, there exists a radio labeling c of C_{4k+2} with $rn(c) = 2k^2 + 5k + 2$. According to the proof of Corollary 2.4, such a radio labeling c of C_{4k+2} uses exactly two labels from $[1 + i(k+2) .. (i+1)(k+2)]$ for every integer i with $0 \leq i \leq 2k$. For each i with $0 \leq i \leq 2k$, we assign the vertex v_{1+ki} the label $1 + (k+2)i$, where the operations in the subscript $1 + ki$ for v_{1+ki} are computed modulo $4k+2$ and expressed as an element of $[1 .. 4k+2]$. In particular, for $i = 2k$, $v_{1+ki} = v_{1+2k^2} = v_{3k+3}$ since $2k^2 + 1 \equiv 3k + 3 \pmod{4k+2}$. Thus v_{3k+3} is assigned the label $2k^2 + 4k + 1$. We have now labeled all vertices v_j for j odd with $j \in [1 .. 4k+2]$. We illustrate this labeling of C_{18} in Figure 5, where the solid vertices of C_{18} have odd subscripts. Also, notice that, for $n = 18 = 4 \cdot 4 + 2$, we have $k = 4$. Thus, in this labeling of C_{18}, we have $c(v_1) = 1$ and $c(v_{15}) = c(v_{3k+3}) = 49 = 2k^2 + 4k + 1$.
Letting $c(v_{2k+2}) = 2k^2 + 5k + 2$ and $c(v_{j+2k+1}) = c(v_j) - 1$ for all such odd $j \geq 3$, that is, assign the vertex antipodal to v_1 the label $2k^2 + 5k + 2$ and the vertex antipodal to v_j, where j is odd and $j \in [3 .. 4k+2]$, the label $c(v_j) - 1$, we obtain a radio labeling c of C_{4k+2} with $rn(c) = 2k^2 + 5k + 2$. Moreover, the label set of c is $\{1 + i(k+2), (i+1)(k+2) \mid 0 \leq i \leq 2k\}$. This radio labeling c of C_{18} is shown in Figure 5 with $rn(c) = 54$.

Next we verify (ii). We show that for each odd integer $k \geq 3$, there exists a radio labeling c of C_{4k+2} with $rn(c) = 2k^2 + 5k + 3$. Let $k = 2\ell + 1$

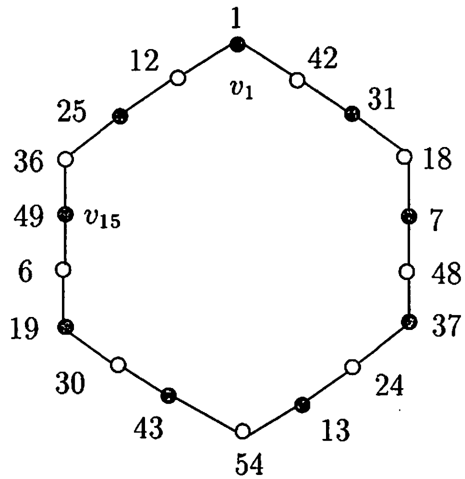

Figure 5: A minimum radio labeling c of C_{18} with $rn(c) = 54$

for some integer $\ell \geq 1$. First we consider a radio labeling of C_{4k+2}, where $k = 2\ell + 1$ for some integer $\ell \geq 1$. Hence $4k + 2 = 8\ell + 6$. Observe that $2\ell + 1$ and $8\ell + 6$ are relatively prime and so k is a generator of the cyclic group $\mathbf{Z}_{4k+2}$. Hence $\{v_{1+ki} : 0 \leq k \leq 4k+1\} = V(C_{4k+2})$. Also, $1 + k(2k+1) \equiv 2k+2 \pmod{4k+2}$, so $v_{1+k(2k+1)} = v_{2k+2}$. We now assign the vertex v_{1+ki} the label $1+(k+2)i$ for every integer i with $0 \leq i \leq 4k+1$, where the operations in the subscript $1+ki$ for v_{1+ki} are computed modulo $4k+2$ and expressed as an element of $[1 \,..\, 4k+2]$, while the operations in the color $1 + (k+2)i$ are computed modulo $2k^2 + 5k + 3$ and expressed as an element of $[1 \,..\, 2k^2 + 5k + 3]$. We illustrate the radio labelings of C_{14} and C_{22} are shown in Figure 6.
By the observations made above, this is a radio labeling c of C_{4k+2} with $rn(c) = 2k^2 + 5k + 3$ in which the antipodal vertex v_{2k+2} of v_1 is labeled $2k^2 + 5k + 3$ and $|c(v_i) - c(v_{i+2k+1})| = 1$ for all $i \in [2 \,..\, 4k+2]$. ■

By Theorem 3.1 and Corollary 2.4, we see that if $k \geq 3$ is odd, then

$$2k^2 + 5k + 2 \leq rn(C_{4k+2}) \leq 2k^2 + 5k + 3.$$

Using the radio labeling of C_{4k+2} described in Theorem 3.1, we can show that the inequality $rn(C_{4k+3}) \geq 2k^2 + 5k + 3$ in Corollary 2.4 is equality for all $k \geq 2$.

Theorem 3.2 *For every integer $k \geq 2$,*

$$rn(C_{4k+3}) = 2k^2 + 5k + 3.$$

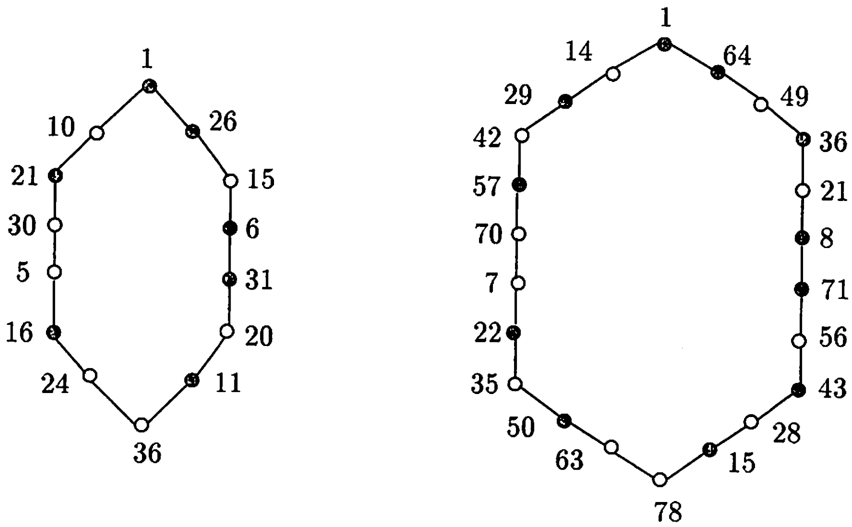

Figure 6: Radio labelings of C_{14} and C_{22}

Proof. By Corollary 2.4, $rn(C_{4k+3}) \geq 2k^2+5k+3$. Thus it suffices to show that there is a radio labeling c of C_{4k+3} with $rn(C_{4k+3}) = 2k^2+5k+3$. Let C_{4k+3} be obtained from the cycle C_{4k+2} by subdividing the edge $v_{2k+1}v_{2k+2}$ by introducing a new vertex v. Note that $\operatorname{diam} C_{4k+3} = \operatorname{diam} C_{4k+2} = 2k+1$. Let c be the radio labeling of C_{4k+2} described in the proof of Theorem 3.1. We obtain a radio labeling c' of C_{4k+3} from c by defining $c'(v_1) = 2k^2+5k+3$, $c'(v) = 1$ and $c'(v_{2k+2}) = 2k^2+5k+2$, and $c'(v_i) = c(v_i)$ for all $i \neq 1, 2k+2$. We illustrate the radio labelings of C_{15} and C_{19} described in Theorem 3.2 in Figure 7.

It is then routine to verify that c' is a radio labeling of C_{4k+3} with $rn(c') = 2k^2+5k+3$. ∎

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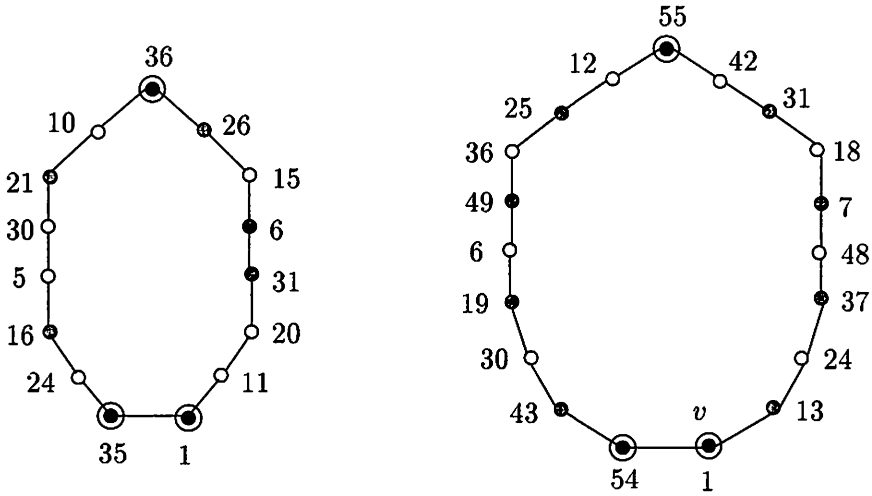

Figure 7: Minimum radio labelings of C_{15} and C_{19}

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