New determinantal identities on Stirling numbers

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Abstract

The Stirling numbers of first kind and Stirling numbers of second kind denoted by s(n,k) and S(n,k) respectively arise in a variety of combinatorial contexts. There are several algebraic and combinatorial relationship between them. Here we state and prove four new identities concerning the determinants of matrices whose entries are *unsigned* Stirling numbers of first kind and Stirling numbers of second kind. We also observe an interrelationship between them based on our identities.

Keywords: determinantal identities
unsigned Stirling numbers of first kind
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1. Introduction:

The *unsigned* Stirling numbers of first kind and Stirling numbers of second kind denoted by s(n,k) and S(n,k) respectively arise in a variety of combinatorial contexts. There are several interrelationship and between the Binomial coefficients. These relationship lead to interesting identities involving sums and products of these numbers. These numbers satisfy recurrence equations and have exponential generating functions [1,3] .

In this paper we deduce four new determinantal identities involving matrices of Stirling numbers. These identities in turn reveal further *hitherto undiscovered* interrelationships.

In this section we first define the Stirling numbers and setup some standard notation. In section 2 and 4 we discuss our identity based on Stirling numbers of first kind and in section 3 and 5 we discuss our identity based on Stirling numbers of second kind.

Given s(n,k), the *unsigned* Stirling numbers of first kind [1,3] and S(n,k), Stirling numbers of Second kind [1,3] ,we have the following identities.

For integers $1 \leq k \leq n$,

$$s(n,k) = s(n-1,k-1)+(n-1)s(n-1,k);$$
$$s(n,1)=(n-1)!\,,\ s(n,n)=1 \qquad (1.1)$$

$$S(n,k) = S(n-1,k-1)+ k\,S(n-1,k);$$
$$S(n,1)= 1,\ S(n,n)= 1 \qquad (1.2)$$

and postulate:

$$s(n,k)= 0,\ S(n,k)= 0 \text{ if } k>n \qquad (1.3)$$

Also the *unsigned* Stirling numbers of first kind, s(n,k), are the coefficients of x^k in the expansion

$$x(x+1)(x+2)\ldots(x+n-1) = \sum_{k=1}^{n} s(n,k)\, x^k$$

The left side is also denoted by the falling factorial as $(x+n-1)^{(n)}$.

The Stirling numbers of second kind, S(n,k), describes the number of ways a set with n elements can be partitioned into k disjoint, non empty subsets

We give below table for the Stirling numbers for $1 \leq k \leq n \leq 6$.

Table 1. *Unsigned* Stirling numbers of first kind, s(n,k)

n/k	**1**	**2**	**3**	**4**	**5**	**6**
1	1					
2	1	1				
3	2	3	1			
4	6	11	6	1		
5	24	50	35	10	1	
6	120	274	225	85	15	1

Table 2. Stirling numbers of second kind, S(n,k)

n\k	**1**	**2**	**3**	**4**	**5**	**6**
1	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

2. Determinantal identity for Stirling numbers of first kind:

In this section we deduce an interesting formula for the determinant of a certain matrix whose entries are unsigned Stirling numbers of first kind.

For any integer $n \geq 1$, let P denote the matrix of the order 'm' whose (i,j)-th entry is s(n+i-1,j) and let det P denote the determinant of the matrix P.

Then we have

Proposition 2:

$$det\ P = ((n-1)!)^{m}$$

Proof: In order to reduce the matrix P to a simpler form, we use the standard combinatorial identities (1.1) and (1.3); and also the following row transformations are performed sequentially on the matrix P.

Let R_α denote the α–th row and $R_\alpha \leftarrow \phi\ R_\alpha$ to mean the multiplication of all elements of α–th row by constant ϕ.

for $\beta = 2, 3, 4, \ldots, m-2, m-1, m$

and $\alpha = m, m-1, m-2, \ldots, \beta$

a) $R_{\alpha-1} \leftarrow [n+\alpha-\beta]\, R_{\alpha-1}$

b) $R_{\alpha} \leftarrow R_{\alpha} - R_{\alpha-1}$

c) $R_{\alpha-1} \leftarrow 1/[n+\alpha-\beta]\, R_{\alpha-1}$

The above row operation transforms the matrix to a simpler form $P = s(n, j-i+1)$ whose elements below the primary diagonal are zero . Hence the determinant of P is obtained as the product of its diagonal elements.

$$\det P = s(n,1)^{m} \tag{2.1}$$

from the identity (1.1) we have

$$s(n,1) = (n-1)!$$

and therefore

$$\det P = ((n-1)!)^{m} \tag{2.2}$$

thus proving the proposition **2**.

3. Determinantal identity for Stirling numbers of Second kind:

In this section we deduce an interesting formula for the determinant of a certain matrix whose entries are Stirling numbers of second kind.

For any integer $n \geq 1$, let Q denote the matrix of the order 'm' whose (i,j)-th entry is $S(n+i-1,j)$ and let det Q denote the determinant of the matrix Q.

Then we have

Proposition 3:

$$\det Q = (m!)^{(n-1)}$$

Proof: In order to reduce the matrix Q to a simpler form, we use the standard combinatorial identities (1.2) and (1.3); and also the following row transformation is also performed on the matrix Q. Let R_α denote the α–th row and $R_\alpha \leftarrow \phi\ R_\alpha$ to mean the multiplication of all elements of α–th row by constant ϕ.

for $\alpha = m, m-1, m-2, \ldots, 2$

$R_\alpha \leftarrow R_\alpha - R_{\alpha-1}$

$$Q_{i,j} = \left|\begin{array}{llll} S(n,1) & S(n,2) & \ldots & S(n,m) \\ 0 & S(n,1)+S(n,2) & \ldots & S(n,m-1)+(m-1)S(n,m) \\ 0 & \cdot & & \cdot \\ 0 & \ldots & S(n+i-2,j-1)+(j-1)S(n+j-2,j) & \ldots \\ 0 & \cdot & & \cdot \\ 0 & S(n+m-3,1)+S(n+m-3,2) & .. & S(n+m-3,m-1)+(m-1)S(n+m-3,m) \\ 0 & S(n+m-2,1)+S(n+m-2,2) & .. & S(n+m-2,m-1)+(m-1)S(n+m-1,m) \end{array}\right|$$

since $S(j,1) = 1$ for $j = 1,2,\ldots n$

We reduce the above matrix to a much simpler form by sequentially performing the following row transformations

for $\beta = 2, 3, 4, \ldots, m-2, m-1$

and $\alpha = m, m-1, m-2, \ldots, (\beta+1)$

a) $R_{\alpha-1} \leftarrow \beta\ R_{\alpha-1}$

b) $R_\alpha \leftarrow R_\alpha - R_{\alpha-1}$

c) $R_{\alpha-1} \leftarrow 1/\beta\ R_{\alpha-1}$

and the above row operations transformation the matrix Q with elements $Q_{i,j}$ as

$$Q_{i,j} = \sum_{k=j-i+1}^{j} \binom{k-1}{k-j+i-1} (i-1)^{(k-j+i-1)}\ S(n,k)$$

and thereby its determinant is obtained just by the product of its primary diagonal elements

$$\det Q = \prod_{j=1}^{m} \sum_{k=1}^{j} (j-1)^{(k-1)} S(n,k) \qquad (3.1)$$

$$\det Q = \prod_{j=1}^{m} 1/j \sum_{k=1}^{j} j^{(k)} S(n,k) \qquad (3.2)$$

from the identity

$$\sum_{j=1}^{n} x^{(j)} S(n,j) = x^{n}$$

in the equation (3.2) and we have

$$\det Q = \prod_{j=1}^{m} j^{n-1}$$

and thereby we have

$$\det Q = (m!)^{(n-1)} \qquad (3.3)$$

thus proving the proposition **3.**

4. Determinantal product identity for Stirling numbers of first kind :

In this section we deduce an interesting relationship for the determinant of a certain matrix whose entries are unsigned Stirling numbers of first kind.

For any integer $n \geq 1$, let P denote the matrix of the order '$m \times (2m-1)$' whose (i,j)-th entry is $s(n+i-1,n-2m+i+j)$ and let A,B & C denote its submatrices of order 'm' whose (i,j)-th entries are

$A_{i,j} = s(n+i-1,n-2m+i+j)$

$B_{i,j} = s(n+i-1,n-m+j)$

$C_{i,j} = s(n+i-1,n-m-1+i+j)$

and let det A, det B and det C denote the determinant of the matrices A, B and C respectively.

Then we have

Proposition 4:

$$\det A = \det B \,.\, \det C$$

Proof: In order to reduce the submatrix A to a simpler form, we use the standard combinatorial identities (1.1) and (1.3); and also the following row transformation is performed on the submatrix A.

for $\beta = 2,3,4,\ldots m\text{-}2, m\text{-}1, m$

and $\alpha = m, m-1, m-2, \ldots, \beta$

$R_\alpha \leftarrow R_\alpha - R_{\alpha-1}$

The above row operation transforms the submatrix A to a simpler form

$$A_{i,j} = \prod_{j=0}^{m-2} (n+j)^{(m-1-j)} \{ s(n\,,\, n-2m+i+j) \}$$

and its determinant is

$$\det A = \prod_{j=0}^{m-2} (n+j)^{(m-1-j)} \{ \det s(n\,,\, n-2m+i+j) \} \qquad (4.1)$$

We observe that the reduced submatrix A is in the form of Hankel matrix, that is one having equal elements along each diagonal line parallel to the secondary diagonal (persymmetry or striped matrix).

In order to reduce the submatrix C to a simpler form, we use the standard combinatorial identities (1.1) and (1.3); and also the following row transformation is performed sequentially on the submatrix C.

for $\beta = 2,3,4,\ldots m-2,m-1,m$

and $\alpha = m,m-1,m-2,\ldots,\beta$

$R_\alpha \leftarrow R_\alpha - R_{\alpha-1}$

The above row operation transforms the submatrix C to a simpler form, whose elements below the secondary diagonal are zero i.e., we obtain an reverse upper triangle Hankel matrix and $C_{i,j}$ becomes

$$C_{i,j} = \prod_{j=0}^{m-2} (n+j)^{(m-1-j)}\{s(n,n-m+i+j-1)\}$$ and its determinant becomes

$$\det C = (-1)\prod_{j=0}^{m-2} (n+j)^{(m-1-j)} \{s(n,n)^m\}$$

but s(n,n)=1 and thereby

$$\det C = (-1)\prod_{j=0}^{m-2} (n+j)^{(m-1-j)} \qquad (4.2)$$

In order to reduce the submatrix B to a simpler form, we use the standard combinatorial identities (1.1) and (1.3); and also the following row transformations are performed sequentially on the submatrix B.

Let R_α denote the α–th row and $R_\alpha \leftarrow \phi R_\alpha$ to mean the multiplication of all elements of α–th row by constant ϕ.

for $\beta = 2,3,4,\ldots m-2,m-1,m$

and $\alpha = m,m-1,m-2,\ldots,\beta$

a) $R_{\alpha-1} \leftarrow (n+\alpha-\beta) R_{\alpha-1}$

b) $R_\alpha \leftarrow R_\alpha - R_{\alpha-1}$

c) $R_{\alpha-1} \leftarrow 1/(n+\alpha-\beta) R_{\alpha-1}$

The above row operation transforms the submatrix B to a simple form and

det B = det s(n,n-m-i+j+1) (4.3)

which has the form of Toeplitz matrix.

It is easy to see that Toeplitz and Hankel matrices are closely related; In (4.1) if the rows of matrix A are reversed in order, it becomes a Toeplitz matrix, the same thing happens if its columns are reversed in order. Conversely, if the row or columns of matrix B in (4.3) are reversed in order than it becomes a Hankel matrix.

We know that det(B) = det(B^T) where det(B^T) denote the determinant of the transposed matrix B of order m. The factor (-1) in (4.2) is nothing but the determinant of the reverse unit matrix. This factor is multiplied with det (B^T) and we have

$$\det B = \det (B^T) \begin{matrix} 0 & 0 & . & . & 1 \\ 0 & . & . & 1 & 0 \\ 0 & 1 & . & . & 0 \\ 1 & 0 & . & . & 0 \end{matrix} = \det s(n,n-2m+i+j) \quad (4.4)$$

thereby

$$\det B \,.\, \det C = \prod_{j=0}^{m-2} (n+j)^{(m-1-j)} \{\det \; s(n,n-2m+i+j)\}$$

but RHS is nothing but det A. Therefore

det A = det B. det C thus proving the proposition 4.

5. Determinantal product identity for Stirling numbers of second kind:

In this section we deduce an interesting relationship for the determinant of a certain matrix whose entries are unsigned Stirling numbers of second kind.

For any integer $n \geq 1$, let Q denote the matrix of the order '(2m-1) x m' whose (i,j)-th entry is S(k+i+j-2, k+j-1) and let A,B & C denote its submatrices of order 'm' whose (i,j)-th entries are

$$A_{i,j} = S(k+i+j-2, k+j-1)$$

$$B_{i,j} = S(k+m+i-2, k+j-1)$$

$$C_{i,j} = S(k+m+i+j-3, k+j-1)$$

and let det A, det B and det C denote the determinant of the matrices A, B and C respectively.

Then we have

Proposition 5:

$$det\ C = det\ A \ .\ det\ B$$

Proof: In order to reduce the submatrix A to a simpler form, we use the standard combinatorial identities (1.1) and (1.3); and also the following row transformation is also performed sequentially on the submatrix A.

Let C_α denote the α–th column and

for $\beta = 2,3,4,\ldots m-2,m-1,m$

and $\alpha = m,m-1,m-2,\ldots,\beta$

$$C_\alpha \leftarrow C_\alpha - C_{\alpha-1}$$

The above column operation transforms the submatrix to a simpler form

$$A_{i,j} = \prod_{j=0}^{m-2} (k+j)^j \{S(k+i-1,k+j-1)\}$$ and its determinant is

$$\det A = \prod_{j=0}^{m-2} (k+j)^j \det S(k+i-1, k+j-1)$$

But det S(k+i-1, k+j-1) is a lower triangular matrix of order 'm' which is equal to $S(k,k)^m$ and therefore

$$\det A = \prod_{j=0}^{m-2} (k+j)^j \det S(k,k)^m$$

but S(k,k)=1 and thereby

$$\det A = \prod_{j=0}^{m-2} (k+j)^j \qquad (5.1)$$

In order to reduce the submatrix C to a simpler form, we use the standard combinatorial identities (1.1) and (1.3); and also the following column transformation is performed sequentially on the submatrix C.

for $\beta = 2,3,4,\ldots m-2,m-1,m$

and $\alpha = m,m-1,m-2,\ldots,\beta$

$C_\alpha \leftarrow C_\alpha - C_{\alpha-1}$

The above column operation transforms the submatrix to a simpler form

$$C_{i,j} = \prod_{j=0}^{m-2} (k+j)^j \{S(k+m+i-2, k+j-1)\}$$

and its determinant is

$$\det C = \prod_{j=0}^{m-2} (k+j)^j \det S(k+m+i-2, k+j-1) \qquad (5.2)$$

but $S(k+m+i-2, k+j-1) = B$ as defined, hence

det $S(k+m+i-2, k+j-1)$= det B and

$$\det A = \prod_{j=0}^{m-2} (k+j)^j$$

thereby in (5.2) we have

det C = det A. det B thus proving the proposition **5.**

Conclusions : We have observed the proof for the determinantal identities of Stirling numbers in factorial notation and the implication of the results (2.2) and (3.3) leads to the conclusion that **Det P = Det Q** *iff* **m = (n-1)**. The other interesting identity is:

$\mathrm{Det}[s(n+i-1,n+j-2)]_{2\times 2} = S(n+1, n-1)$ and

$\mathrm{Det}[S(n+i-1,n+j-2)]_{2\times 2} = s(n+1, n-1)$

which establishes another relationship between the *unsigned* Stirling numbers of first kind and Stirling numbers of second kind. If binomial numbers in the matrices (4) and (5) replaces the Stirling numbers we get similar determinantal identities. On the basis of the above determinantal approach, it would be possible to deduce other matrix identities in the context of several combinatorial identities connecting the Stirling numbers of first kind, Stirling numbers of second kind and Binomial numbers.

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