Excess Problem for Modular Lattices

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Abstract

In this paper, we show that for every modular lattice L, if its size is at least three times of its excess, then each component of its direct product decomposition is isomorphic to one of a Boolean lattice of rank one B_1, a chain of length two **3**, a diamond M_3, and M_4, where M_n is a modular lattice of rank two which has exactly n atoms.

1 Introduction

Throughout the paper, all lattices are assumed to be finite. We use the terminology on lattices in [22] and [23]. Let L be a lattice. An element $x > 0$ ($x < 1$) is *join-irreducible* (*meet-irreducible*) if for any y, z, $y \vee z = x$ ($y \wedge z = x$) implies $x = y$ or $x = z$, respectively. We denote by $J(L)$ ($M(L)$) the set of join-irreducible (meet-irreducible) elements of L. A lattice L^* is the *dual* of a lattice L if both L and L^* are equal as sets, but $x \leq_L y$ in L if and only if $x \geq_{L^*} y$ in L^* for all x, y. Therefore $J(L) = M(L^*)$ and $M(L) = J(L^*)$ hold. A lattice has a *direct-product decomposition* if it is isomorphic to the product of two nontrivial lattices. We call a lattice *indecomposable* when it has no direct-product decomposition. A lattice K is a *component* of L if L is isomorphic to the product of K and some lattice K' and K is indecomposable.

Suppose that L is nontrivial. A pair of elements $(x, y) \in J(L) \times M(L)$ is called *mismatching* if $x \not\leq y$. Since every nontrivial lattice has at least one mismatching pair, we can define *excess of* L as follows [1]:

$$\mathrm{ex}(L) = |L| - \min\{|V_x| + |I_y| : (x, y) \text{ is mismatching}\},$$

where V_x is the principal filter generated by x, that is, the set of all elements which are greater than or equal to x and I_y is the principal ideal generated by y, that is, the set of all elements which are less than or equal to y. Here we consider two problems relating to excess.

Global Excess Problem *Is there a constant C such that for every nontrivial lattice with* $\mathrm{ex}(L) > 0$, $|L| \leq C\,\mathrm{ex}(L)$?

Local Excess Problem *Let $\mathcal{C}$ be a class of lattices. Then is there a constant $C_{\mathcal{C}}$ such that $|L| \leq C_{\mathcal{C}} \operatorname{ex}(L)$ for all $L \in \mathcal{C}$ with $\operatorname{ex}(L) > 0$ and at least one lattice satisfies the equality? Moreover can we decide all maximal configurations?*

Up to the present, there are the following results:

1. $\operatorname{ex}(L) = 0$ if and only if L is Boolean [1]. Therefore every nonBoolean lattice has positive excess.

2. Suppose that L is indecomposable and nonBoolean.

   (a) If L is distributive, then $|L| \leq 3\operatorname{ex}(L)$, and equality holds if and only if L is isomorphic to a chain of length two **3** [4].

   (b) If L is relatively complemented, then $|L| \leq 5\operatorname{ex}(L)$, and equality holds if and only if L is isomorphic to either a diamond M_3 or the face lattice of a square P_4 [1].

The aim of this paper is to solve the local excess problem for the class of modular lattices.

Theorem 1 *Let L be a nontrivial lattice. If L is modular but not atomistic, then*

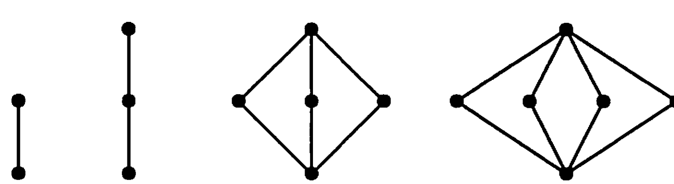
$$|L| \leq 3\operatorname{ex}(L).$$

In addition, suppose that L is indecomposable. Then the following statements are equivalent:

(a) $|L| = 3\operatorname{ex}(L)$;

(b) L is isomorphic to a chain of length two **3**.

Theorem 2 *For every indecomposable, atomistic, modular lattice L, if $|L| \leq 3\operatorname{ex}(L)$, then L is isomorphic to one of a Boolean lattice of rank one B_1, a diamond M_3, and M_4, where M_n is a modular lattice of rank two which has exactly n atoms. See Figure 1.*



B_1 **3** M_3 M_4

Figure 1.

From the theorems, we can classify modular lattices according to the relation between size and excess.

Theorem 3 (Classification of Modular Lattices) *For every nontrivial modular lattice L, there are five cases:*

(i) if $\mathrm{ex}(L) = 0$, *then* L *is Boolean;*

(ii) if $|L| = 5\,\mathrm{ex}(L)$, *then every component of* L *is isomorphic to either of* B_1 *or* M_3 *and at least one component is isomorphic to* M_3*;*

(iii) if $|L| = 3\,\mathrm{ex}(L)$ *and* L *is atomistic, then every component of* L *is isomorphic to one of* B_1, M_3 *and* M_4 *and at least one component is isomorphic to* M_4*;*

(iv) if $|L| = 3\,\mathrm{ex}(L)$ *and* L *is not atomistic, then every component of* L *is isomorphic to one of* B_1, M_3, M_4 *and* $\mathbf{3}$ *and at least one component is isomorphic to* $\mathbf{3}$*;*

(v) otherwise, $|L| < 3\,\mathrm{ex}(L)$.

Proof. Suppose that $|L| \geq 3\,\mathrm{ex}(L)$. Note that if K_1 and K_2 are nontrivial lattices, then

$$\mathrm{ex}(K_1 \times K_2) \geq \max\{|K_2|\,\mathrm{ex}(K_1), |K_1|\,\mathrm{ex}(K_2)\},$$

and equality holds if they are self-dual. If L is decomposable, then for each component K, $3\,\mathrm{ex}(K) \leq |K|$. By the induction on size, we see that L satisfies one of the four cases. ∎

We argue the relation between the global excess problem and a famous, open problem, which is called Frankl's conjecture or the union-closed sets conjecture. Define

$$\eta(L) = \min\{|V_x| : x \in J(L)\}.$$

The conjecture says that if L is nontrivial then $\eta(L) \leq |L|/2$. Several results for the conjecture appeared in [1–6, 8–17, 19–22, 24, 25]. Since $\mathrm{ex}(L) \leq |L| - (\eta(L) + \eta(L^*))$, the global excess problem implies that

$$\min\{\eta(L), \eta(L^*)\} \leq \frac{C-1}{2C}|L|$$

whenever L is nonBoolean. A lattice is *self-dual* if it and its dual are isomorphic.

Corollary 4 *Let* L *be a modular, nonBoolean, indecomposable lattice. If* L *is self-dual, then* $\eta(L) \leq 2|L|/5$, *and equality holds if and only if* L *is isomorphic to* M_3. *In addition, if* L *is not atomistic, then* $\eta(L) \leq |L|/3$, *and equality holds if and only if* L *is isomorphic to* $\mathbf{3}$.

In sections 2 and 4, we prepare lammas to prove the theorems. Theorem 1 will be proved in Section 3. Finally, Section 5 deals with the proof of Theorem 2.

2 Preliminary Lemmas for Theorem 1

For an element x with $x < 1$ ($x > 0$), we denote by x^+ (x_+) the join of all covers of x (the meet of all cocovers), respectively.

Lemma 5 *Let L be a nontrivial lattice. Then there is a mismatching pair (x, y) such that $x \not\leq 1_+$ and $0^+ \not\leq y$.*

Proof. Take any atom a. Then every maximal element y in L but not in V_a is meet-irreducible. Fix such an element y. On the other hand, let b be a coatom such that $y \leq b$. In similar, every minimal element x in L but not in I_b is join-irreducible. It is clear that the pair (x, y) satisfies the conditions. ∎

Next, we introduce the result of Herrmann [7]. The result is extended by Reuter [18] from modular lattices to balanced lattices. See [23]. Although we can easily generalize preliminary lemmas for the proof of Theorem 1 according to Reuter's result, we only deal with the case of modular lattices. In the rest of the section, we assume that a lattice L is nontrivial and modular.

Lemma 6 *These operators x_+ and x^+ are order-preserving.*

Proof. Suppose that $x \leq y < 1$. If a is a cover of x, then either $a \leq y$ or $a \vee y$ covers y. Hence $x^+ \leq y^+$. A similar argument implies that $x_+ \leq y_+$ for all $0 < x \leq y$. ∎

Lemma 7 *For all $x > 0$ and $y < 1$, $x_+ = ((x_+)^+)_+$ and $y^+ = ((y^+)_+)^+$ hold.*

Proof. By the above lemma, $x_+ \leq ((x_+)^+)_+$. Since the interval $[x_+, (x_+)^+]$ is geometric, the statement holds. ∎

Let $S = S(L)$ be the set of all intervals $[x_+, (x_+)^+]$. Define the ordering of S as follows:

$$[x_+, (x_+)^+] \leq_S [y_+, (y_+)^+] \text{ if and only if } x_+ \leq y_+.$$

Note that the latter condition is equivalent to $(x_+)^+ \leq (y_+)^+$. Then S is a lattice by Herremann's result. In fact,

$$[x_+, (x_+)^+] \vee_S [y_+, (y_+)^+] = [x_+ \vee_L y_+, (x_+ \vee_L y_+)^+]$$

and

$$[x_+, (x_+)^+] \wedge_S [y_+, (y_+)^+] = [(x^+ \wedge_L y^+)_+, x^+ \wedge_L y^+].$$

It is easy to show that if $[y_+, (y_+)^+]$ covers $[x_+, (x_+)^+]$ in S, then $x_+ < y_+ \leq (x_+)^+ < (y_+)^+$.

Finally, we show useful lemmas.

Lemma 8 *If an interval $[x_+, (x_+)^+]$ is join-irreducible in S, then it contains a join-irreducible element in L. In particular, these join-irreducible elements are atoms of the interval.*

Proof. Let $[z_+, (z_+)^+]$ be a unique cocover of the interval $[x_+, (x_+)^+]$. Then $x_+ \leq (z_+)^+ < (x_+)^+$. If a cover x_1 of x_+ is not join-irreducible, then $((x_1)_+)^+ < (x_+)^+$, and so $x_1 \leq ((x_1)_+)^+ \leq (z_+)^+$. Since the interval $[x_+, (x_+)^+]$ is geometric and $x_+ \leq (z_+)^+ < (x_+)^+$, we see that there is a cover of x_+ which is join-irreducible. ∎

Lemma 9 *If an interval* $[x_+, (x_+)^+]$ *is meet-irreducible in* S, *then it contains a meet-irreducible element in* L. *In particular, these meet-irreducible elements are coatoms of the interval.*

Proof. To apply Lemmas 7 and 8 to the dual lattice L^*, we see that the statement holds. ∎

Lemma 10 ([13]) *If* L *is geometric, then for every atom* a, $|V_a| \leq |L|/2$.

Lemma 11 ([2]) *For every modular lattice* L, $\eta(L) \leq |L|/2$, *and equality holds if and only if* L *is Boolean.*

3 Proof of Theorem 1

Let $S = S(L)$. Since L is not atomistic, S is not trivial. Define the mapping φ from L to S by sending x to $[x_+, (x_+)^+]$ if $0 < x$ and $\varphi(0) = [0, 0^+]$. Clearly, φ is order-preserving. Applying Lemma 5 to the lattice S, we can take elements x, y in L such that the pair $(\varphi(x), \varphi(y))$ is mismatching in S, and $\varphi(x) \not\leq_S (1_S)_+$, and $(0_S)^+ \not\leq_S \varphi(y)$. Using Lemma 8 and Lemma 9, we have a mismatching pair (a, b) in L such that a is an atom of the interval $[x_+, (x_+)^+]$ and b is a coatom of the interval $[y_+, (y_+)^+]$.

Claim 1 $|V_a| \leq |V_{x_+}|/2$ *and* $|I_b| \leq |I_{(y_+)^+}|/2$.

Proof. It is enough to show the first inequality. If $\varphi(x)$ is the maximum element of S, then V_x is geometric. Hence Lemma 10 implies that $|V_a| \leq |V_{x_+}|/2$. Suppose that $\varphi(x)$ is not maximum. Since $\varphi(x) \not\leq_S (1_S)_+$, there is an element $z \in L$ such that $\varphi(z)$ is a coatom of S and $\varphi(x) \not\leq_S \varphi(z)$. On the other hand, $\varphi(x) \leq_S \varphi(1)$ implies that $x_+ \leq 1_+$. If $a \leq 1_+$, then $a \leq 1_+ \leq (z_+)^+$ because $\varphi(z)$ is covered by $\varphi(1)$. Hence $\varphi(x) = \varphi(a) \leq_S \varphi(z)$. This is a contradiction. Therefore we can take a coatom c of L with $a \not\leq c$. Then two intervals $[a \wedge c, c] = [x_+, c]$ and $[a, a \vee c] = [a, 1]$ are isomorphic (Theorem 2.1.4 in [23]) and disjoint, and so $|V_a| \leq |V_{x_+}|/2$. ∎

Since $a_+ = x_+$ and $b^+ = (y_+)^+$, neither $a \leq (y_+)^+$ nor $x_+ \leq b$ holds. Using Claim 1, we have that

$$\begin{aligned} |L| - (|V_a| + |I_b|) &\geq \max\{|V_{x_+} \setminus V_a|, |I_{(y_+)^+} \setminus I_b|\} \\ &\geq \max\{|V_a|, |I_b|\} \geq (|V_a| + |I_b|)/2 \end{aligned}$$

or

$$3\,\mathrm{ex}(L) \geq 3(|L| - (|V_a| + |I_b|)) \geq |L|.$$

Finally, suppose that L is indecomposable. We will show that two conditions (a) and (b) are equivalent. It is enough to show that (a) implies (b). By the above argument, each inequality in Claim 1 must be equality and $L \setminus (V_a \cup I_b) = V_{x_+} \setminus V_a = I_{(y_+)^+} \setminus I_b$. Hence $L \setminus (V_a \cup I_b) = [x_+, (y_+)^+]$. On the other hand, three intervals V_a, I_b, and $[x_+, (y_+)^+]$ are isomorphic. Hence L is isomorphic to $3 \times I_b$. In fact, we can define the isomorphism form $\{0, x_+, a\} \times I_b$ to L by sending (s,t) to $s \vee t$. From the assumption on L, the condition (b) holds. This completes the proof.

4 Preliminary Lemmas for Theorem 2

A proper element s in a lattice L is *central* if it satisfies the following conditions:

1. s has a complement, that is, there is an element t such that $s \vee t = 1$ and $s \wedge t = 0$;
2. for each $x \in L$, $x = (x \wedge s) \vee (x \wedge t)$;
3. for all $x \leq s$ and $y \leq t$, $(x \vee y) \wedge s = x$ and $(x \vee y) \wedge t = y$ hold.

It is known that a lattice has a central element if and only if it is not indecomposable. Note that under modularity, the third condition is implied by the first condition.

Now let L be a nontrivial lattice. We will define the bipartite graph $G = G(L) = (V, E)$ associated with L. Its vertex set V is the disjoint union of $J(L)$ and $M(L)$. When x is doubly irreducible element, that is, $x \in J(L) \cap M(L)$, we distinguish x as a join-irreducible element from x as a meet-irreducible element. We define that $\{x, y\} \in E$ if and only if either (x, y) or (y, x) is a mismatching pair in L.

Lemma 12 *The graph $G = G(L)$ is connected if and only if L is indecomposable.*

Proof. It is enough to show the sufficiency. Suppose on the contrary that G is not connected. Take a connected component C in G. Clearly C intersects both $J(L)$ and $M(L)$. Let s be the join of all elements in $C \cap J(L)$ and let t be the meet of all elements in $C \cap M(L)$. We will show that s is central. Note that for all $x \in J(L) \setminus C$ and $y \in M(L) \setminus C$, $x \leq t$ and $s \leq y$ because C is a connected component. Since $V \neq C$, $0 < s < 1$.

Suppose that $s \vee t < 1$. Then there is a join-irreducible element x such that $x \not\leq s \vee t$, and so $x \not\leq t$. We have that $\{x, y\}$ is an edge for some $y \in C$. Hence $x \leq s$. This is a contradiction. In similar, $s \wedge t = 0$.

Let z be an element in L. We will show that for any join-irreducible element x with $x \leq z$, $x \leq (z \wedge s) \vee (z \wedge t)$. If $x \in C$, then $x \leq z \wedge s$. Otherwise, $x \leq z \wedge t$. Hence $z = (z \wedge s) \vee (z \wedge t)$.

Finally, take $x \leq s$ and $y \leq t$. Let b be a meet-irreducible element with $x \leq b$. If $b \in C$, then $t \leq b$. Otherwise, $s \leq b$. Hence $(x \vee y) \wedge s \leq b$. Since $(x \vee y) \wedge s \geq x$, we have that $(x \vee y) \wedge s = x$. A similar argument implies $(x \vee y) \wedge t = y$.

Therefore L is not indecomposable. ∎

Lemma 13 *If an atomistic, modular lattice L is indecomposable, then for all atoms a and coatoms b, V_a and I_b are isomorphic. Therefore $\eta(L) = \eta(L^*)$ and $\mathrm{ex}(L) = |L| - 2\eta(L)$.*

Proof. For all mismatching pair (a,b), the modularity implies that $V_a = [a, a \vee b]$ and $I_b = [a \wedge b, b]$ are isomorphic. Since $G(L)$ is connected, we see that the statement holds. ∎

5 Proof of Theorem 2

Suppose that L is isomorphic to neither B_1 nor M_3. We have already seen that $3\,\mathrm{ex}(L) \leq |L| < 5\,\mathrm{ex}(L)$. By Lemma 13, this is equivalent to that $\mathrm{ex}(L) \leq \eta(L) < 2\,\mathrm{ex}(L)$. Let (a,b) be a mismatching pair and let $Z = L \setminus (V_a \cup I_b)$. Note that $Z \cap J(L)$ and $Z \cap M(L)$ are nonempty. We may assume that $|Z \cap J(L)| \leq |Z \cap M(L)|$. Suppose that $|Z \cap M(L)| = 1$. Then there are an atom c and a coatom d such that $Z = [c,d]$. Since intervals $[0, b \wedge d]$, $[c,d]$, and $[c \vee a, 1]$ are isomorphic, Lemma 10 implies that

$$|L| = |I_b| + |V_a| + |Z| \geq 2|[0, b \wedge d]| + 2|[c \vee a, 1]| + |Z| \geq 5|Z| = 5\,\mathrm{ex}(L).$$

This is a contradiction. Hence $|Z \cap M(L)| \geq 2$.

Next suppose that I_b is not indecomposable. Let s be a central element of I_b and let t be a complement of s in I_b. Note that t is also central. Set $s' = s \vee a$ and $t' = t \vee a$. Since V_a is isomorphic to I_b, both s' and t' are central in V_a.

Claim 2 *There is an element $d \in Z \cap M(L)$ such that $s \leq d$.*

Proof. Suppose on the contrary that there is no element $d \in Z \cap M(L)$ with $s \leq d$. Then $t \leq d$ for all $d \in Z \cap M(L)$. We will show that t is central in L. First we have $s' \vee t = s \vee a \vee t = b \vee a = 1$ and $s' \wedge t = s' \wedge t \wedge b = s \wedge t = 0$. Next take an element z in L. If $z \leq b$, then $z = (z \wedge s) \vee (z \wedge t)$. Since $z \wedge s' = z \wedge s$, we have that $z = (z \wedge s') \vee (z \wedge t)$. If $z \geq a$, then $z = (z \wedge s') \vee (z \wedge t')$. The modularity implies that $z \wedge s' = (z \wedge s) \vee a$. Since $a \leq z \wedge t'$, $z = (z \wedge s') \vee (z \wedge t)$ holds. If $z \in Z$, then there is a coatom $d \in Z$ such that $z \leq d$. Since $a \wedge d = 0$,

$$\begin{aligned} z = (z \vee a) \wedge d &= (((z \vee a) \wedge s') \vee ((z \vee a) \wedge t')) \wedge d \\ &= ((z \wedge s') \vee (z \wedge t') \vee a) \wedge d \\ &= ((z \wedge s') \vee (z \wedge t')) \wedge d \\ &= (z \wedge s') \vee (z \wedge t' \wedge d) = (z \wedge s') \vee (z \wedge t). \end{aligned}$$

Hence t is central in L. This is a contradiction. ∎

Here we may assume that $[0,t]$ is indecomposable. In addition, we may assume that $\eta(I_b) = |[0,s]|\eta([0,t])$. Then Claim 2 implies that $s \leq d$ for some $d \in Z \cap M(L)$, and so $|I_b \cap I_d| = \eta(I_b)$. On the other hand, applying Claim 2 to the dual lattice L^*, we have an element $c \in Z \cap J(L)$ such that $c \leq s'$. Suppose that $c \leq d$. Since s' covers s and (a,d) is mismatching, $d \wedge s' = s$. However, this implies $c \leq d \wedge s' = s \leq b$, a contradiction. Therefore (c,d) is mismatching. We have that

$$|I_b| \geq |Z| \geq |I_d \setminus I_b| + |V_c \setminus V_a| \geq |I_b| - \eta(I_b) + |I_b|/2,$$

and so $\eta(I_b) \geq |I_b|/2$. By Lemma 11, I_b is Boolean. Since I_b is not indecomposable, its rank is at least two, and so the rank of L is at least three. On the other hand, since L is modular but not Boolean, there is an interval $[x,y]$ of length two which contains M_3 as a sublattice (see Corollary 7.2.18 in [23]). By Lemma 13, I_b is not Boolean. This is a contradiction.

Suppose that I_b is indecomposable. If the rank of L is two, then L is isomorphic to M_4. Otherwise, take two elements $d_1, d_2 \in Z \cap M(L)$. Then

$$|I_b| \geq |Z| \geq |I_{d_1} \setminus I_b| + |I_{d_2} \setminus I_b| - |(I_{d_1} \cap I_{d_2}) \setminus I_b| \geq 2(|I_b| - \eta(I_b)) - \eta(I_b),$$

and so

$$3\,\mathrm{ex}(I_b) \leq |I_b|.$$

By induction, we have that I_b is isomorphic to either M_3 or M_4. We consider the former case. Then the set $I_b \cup I_{d_1} \cup I_{d_2} \cup V_a$ contains at least seven atoms. Here we use Dilworth's covering theorem for modular lattices (see Theorem 6.1.9 in [23]): the number of atoms equals the number of coatoms. We have that $|L| \geq 2 + 7 \times 2 = 16$. However, $|I_b| = 5 < 6 \leq |L| - (|I_b| + |V_a|) = \mathrm{ex}(L)$. In similar, the latter case also does not take place. This completes the proof.

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